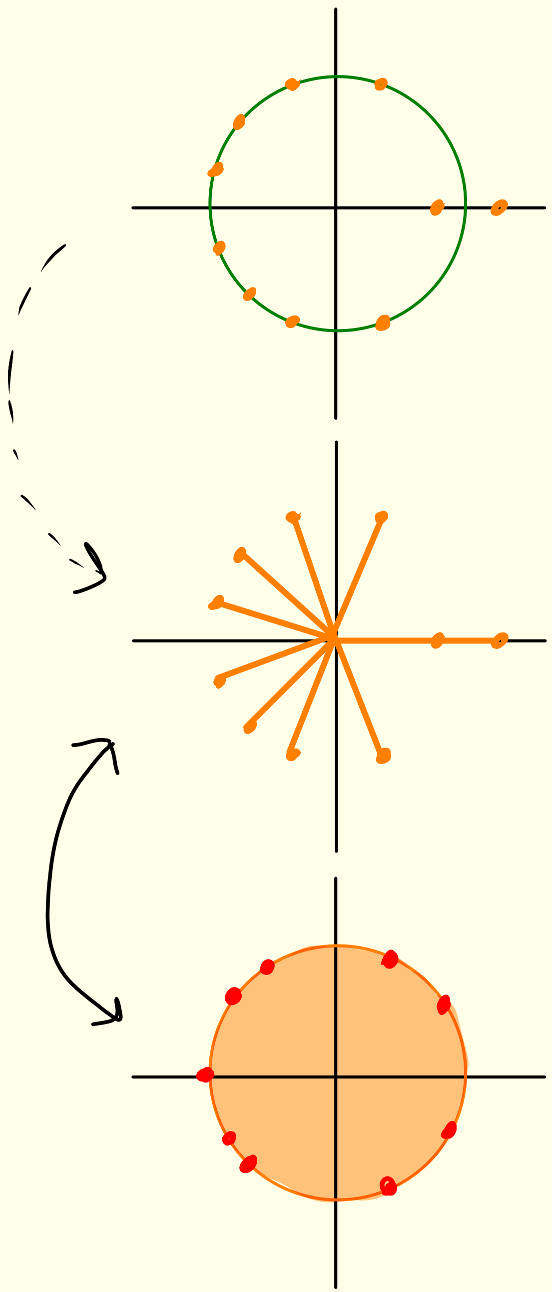

# Hedgehogs in Lehmer's problem

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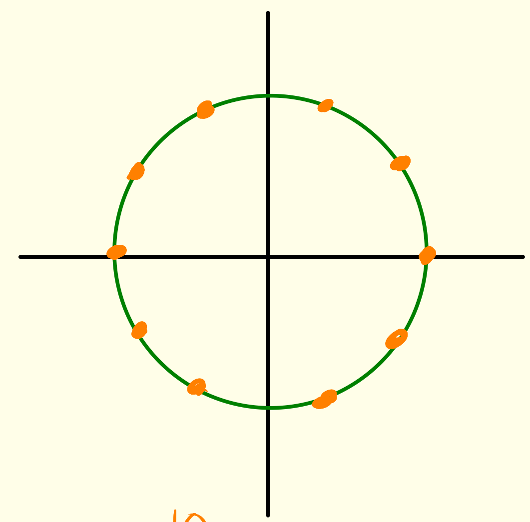
# Lehmer's conjecture on the Mahler measure

**Defn** Given $f(x) = a_n \prod_{i=1}^{n} (x - \alpha_i) \in \mathbb{Z}[x] \setminus \{0\}$, the Mahler measure of $f$ is given by

$$M(f) := |a_n| \prod_{i=1}^{n} \max(|\alpha_i|, 1).$$



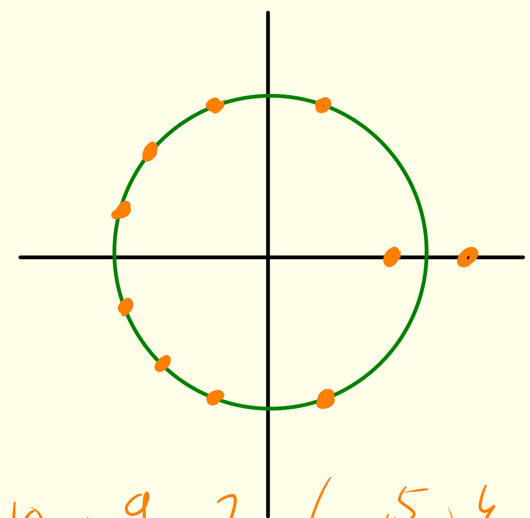
$x^{10} - 1$

**Note** $M(f) \geq 1$ by construction.

**Thm** (Kronecker) $M(f) = 1 \Longleftrightarrow f$ cyclotomic

**Conj** (Lehmer) $\exists \mu > 1$ s.t. for $f \in \mathbb{Z}[x] \setminus \{0\}$

$$M(f) = 1 \quad \text{or} \quad M(f) \geq \mu$$



$x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$

Assume w.l.o.g. $a_n = 1$ & $f$ irreducible

# The Mahler measure and the slash action

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PGL_2(\mathbb{Q})$, write

$$f|_\gamma(z) = (cz+d)^n f\left(\frac{az+b}{cz+d}\right) \quad \in \mathbb{Z}[z]$$

$\pm f(-z)$

Note $M(f) = M\left(f|_{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}\right)$

Thm (Smyth, '71) $f = \pm f|_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$ or $M(f) \geq M(x^3 - x - 1) \approx 1.32$

reciprocal

Idea Study $f / f|_{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$ as an analytic function on the unit disc

# The Mahler measure and the slash action

**Thm** (Zhang, '92 / Zagier, '93)

$z,\ z-1,\ z^2-z+1$ $\qquad\qquad$ $f(1-z)$

$$M(f)\,M\left(f\middle|\begin{pmatrix}-1&1\\0&1\end{pmatrix}\right)=0 \quad\text{or}\quad M(f)\,M\left(f\middle|\begin{pmatrix}-1&1\\0&1\end{pmatrix}\right)\geq\left(\frac{1+\sqrt{5}}{2}\right)^{n/2}$$

**Thm** (vI) For all finite $\Gamma\leq PGL_2(\mathbb{Q})$ s.t. $\Gamma S^1\neq S^1$ $\exists\,\mu>1$ s.t.

$$\prod_{\gamma\in\Gamma}M(f|\gamma)=1 \quad\text{or}\quad \prod_{\gamma\in\Gamma}M(f|\gamma)\geq\mu^n$$

finitely many cyclotomic $f$

often explicitly computable

Recall: $M(f)>\mu$ in Lehmer's conjecture

**Idea** Use harmonic functions to show ineq. as

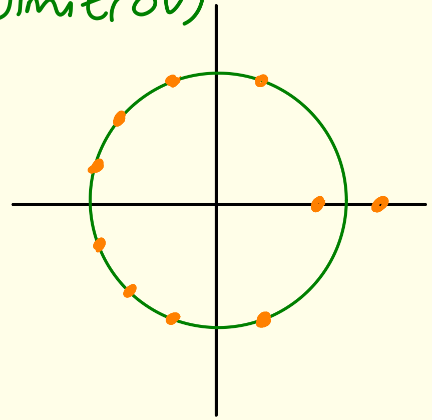
$$\max(0,\log|z|)+\max(0,\log|1-z|)\geq\frac{\sqrt{5}-1}{2\sqrt{5}}\log|z^2-z|+\frac{1}{2\sqrt{5}}\log|z^2-z+1|+\frac{1}{2}\log\frac{\sqrt{5}+1}{2}$$

**Upshot** We have a result on $\sum_{\gamma\in\Gamma}M(f|\gamma)$ for all finite $\Gamma$ except $|\Gamma|=1$

(Lehmer's conjecture)

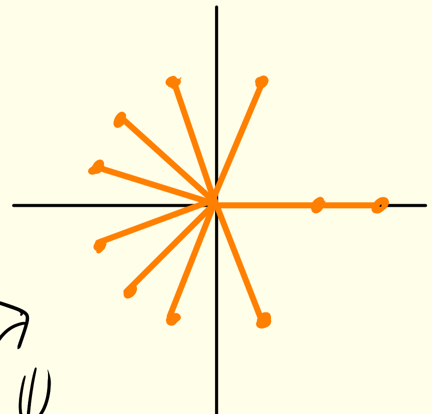
# Hedgehogs

(Dimitrov)



Roots of $f$

Kronecker: $M(f)=1 \iff$ all roots on unit disk



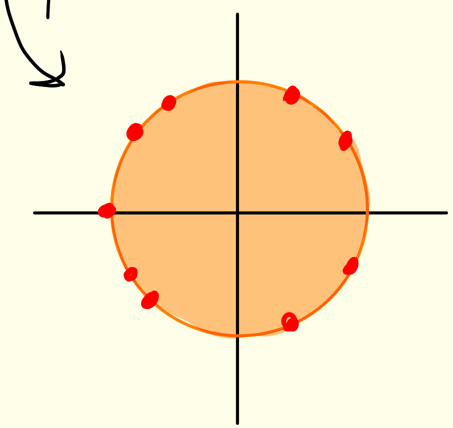
$F = \sum \frac{a_i}{f^i} \in \mathbb{Z}[[x]]$

holomorphic outside hedgehog

$M(f)=1 \iff F$ rational

Riemann mapping thm

$\psi$

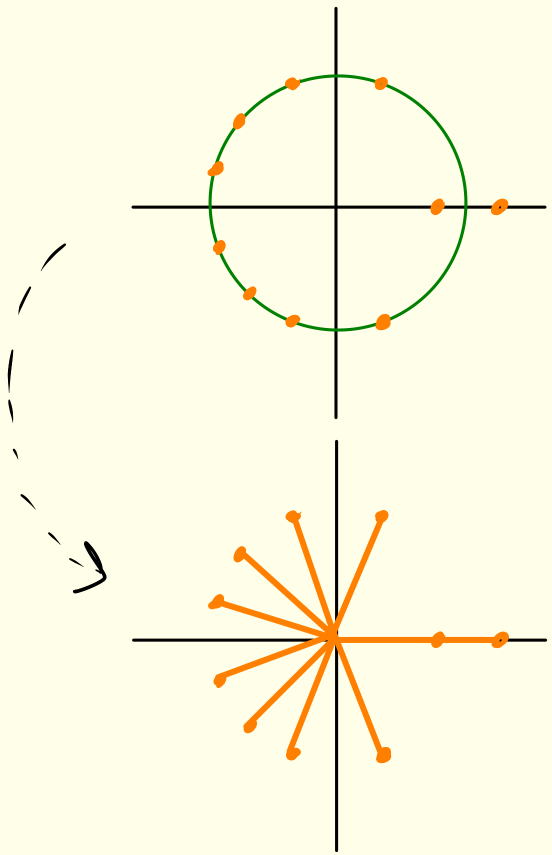


$\tilde{F}$ holomorphic on $\hat{\mathbb{C}} \setminus D^1$

$M(f)=1 \iff$ Riemann mapping $\psi \sim cz$ as $z \to \infty$ with $c \geq 1$

* assume $f$ and $f_2$ are irreducible

## Dimitrov's construction*



Write $f_k = \prod (x - \alpha_i^k)$ and $F := \sqrt{f_2 f_4}$

Then

- $F$ is analytic outside hedgehog with spines $[0, \alpha_i^2]$ and $[0, \alpha_i^4]$.
- $F$ is rational $\longleftrightarrow$ $f$ cyclotomic $(M(f) = 1)$

**Proof** $f$ cyclotomic, then $f = f_2 = f_4$

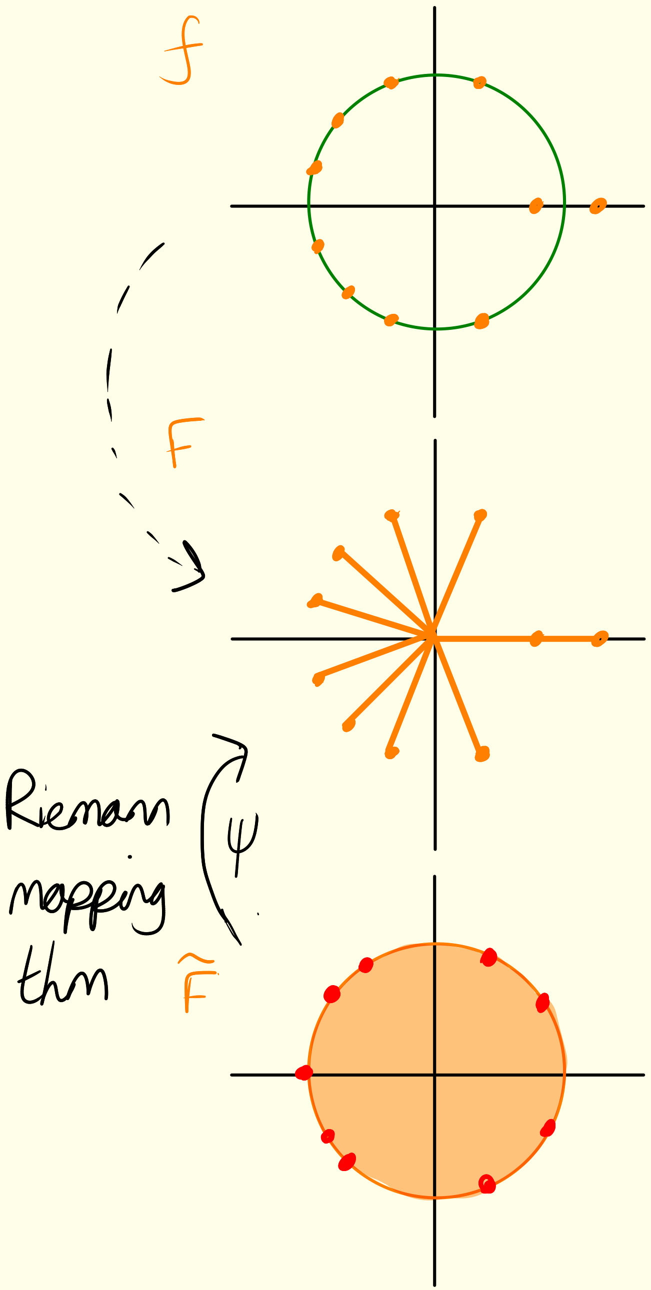
$F$ rational. If $f_2(\alpha) = 0$, then $f_4(\alpha) = 0$. Hence $\alpha = \sigma(\alpha^2)$ for $\sigma \in Gal(f_2)$. Therefore $M(f_2) = M(f_4) = M(f_2)^2$. $\square$

- Moreover, $F \in \mathbb{Z}[[x]]$

**Proof** One has $f_2 \equiv f_4 \mod 4\mathbb{Z}[x]$. Hence, $\sqrt{f_2 f_4} = \sqrt{f_2^2 + 4Q} = f_2 \sqrt{f_2 + 4\frac{Q}{f_2}}$

$$= \sum_{i=0}^{\infty} \underbrace{\binom{1/2}{i} 4^i}_{\in \mathbb{Z}} \underbrace{(Q/f_2)^i}_{\in \mathbb{Z}[x]}$$

# Proof of Schinzel–Zassenhaus conjecture




Consider $\lceil f \rceil = \max_i |\alpha_i|$. Note

$$\lceil f \rceil = 1 \iff M(f) = 1 \iff F \text{ rational}$$

**Conj** $\exists\, c > 0$ s.t. $\lceil f \rceil \geq 1 + c/n$ for all non-cyclotomic $f$.

**Ex** $\lceil x^n - 2 \rceil = 2^{1/n} = 1 + \log 2/n + \mathcal{O}(1/n^2)$.

Explicitly, the Riemann mapping is given by (Lev–Schmidt)

$$\psi(z) = c \prod_{k=1}^{\hat{}} (z - z_k)(z^{-1} - z_k^{-1})^{r_k}$$

Bertrandias theorem ✓ $(c \geq 1)$

for certain $z_k \in S^1$, and with $\frac{r_k}{2\pi}$ the angles between the spines.

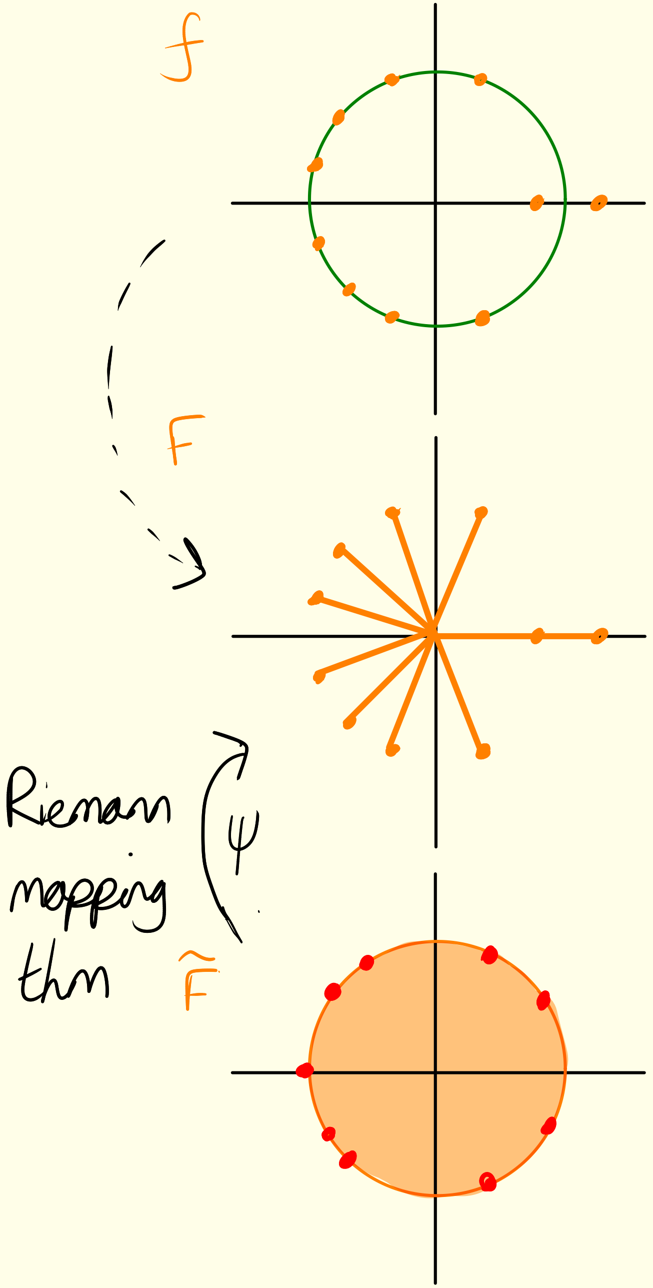
Hence,

$$|\alpha_i|^4 \geq \max_{z \in [z_{j-1}, z_j]} \prod_i |z - z_i|^{2r_k}$$

and

$$\lceil f \rceil^4 \geq \max_{z \in S^1} \prod_i |z - z_i|^{2r_k}.$$

# Proof of Schinzel–Zassenhaus conjecture



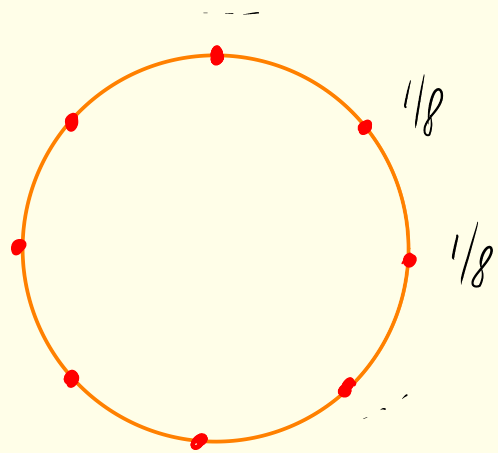

$$\lceil f \rceil^4 \ge \max_{z \in S^1} \prod_i |z - z_i|^{2 r_k}$$

Thm (Dubinin)

For $z_1, \dots, z_n \in S^1$, $\sum r_i = 1$ this quantity is minimal $4^{1/k}$, with equality for

$z_i$ uniform

$r_i = \frac{1}{k}$




Hence,

$$\lceil f \rceil \ge \left(4^{1/k}\right)^{1/4} \ge 2^{1/4n} \ge 1 + \frac{\log 2}{4n}. \qquad \square$$

# Proof of Lehmer's conjecture?

$$M(f)^{4+2} \geq \prod_{i=1}^{n} \max\left(1, \max_{z\in[z_{j-1},z_j]} \prod_i |z-z_i|^{2r_k}\right) \qquad (*)$$

How small can $(*)$ be given $z_1, \dots, z_k$ and $r_1, \dots, r_k$ with $r_1 + \dots + r_k = 1$?

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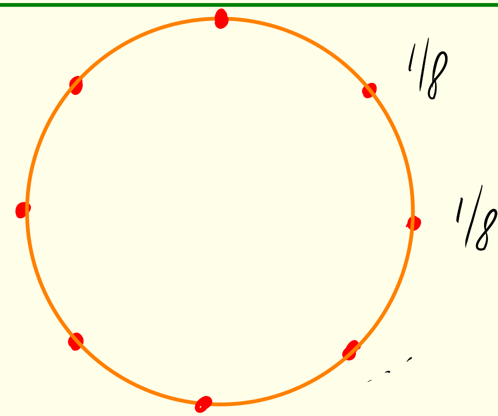
$z_i$ uniform

$r_i = \frac{1}{k}$




(uniform distribution)

$(*) = 4$

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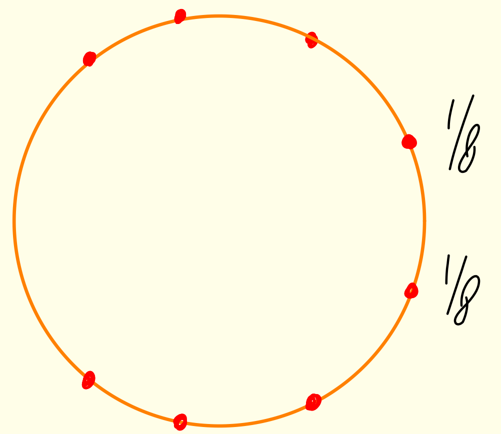
$z_i$ s.t.

$$\max_{z\in[z_{j-1},z_j]} \prod_i |z-z_i|^{2r_k} = 1,$$

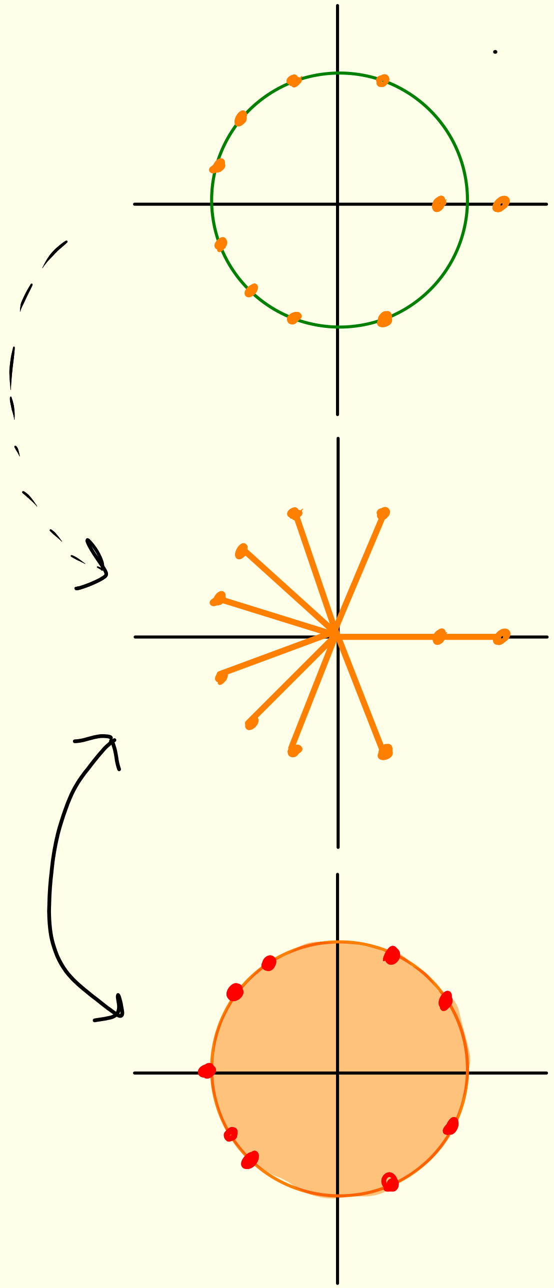
$r_i = \frac{1}{k}$




(vI-Ringeling-Zudilin)

$$(*) = 1 + \frac{\log 4}{n} + \mathcal{O}\left(\frac{1}{n^2}\right)$$

want: $(*) \geq \mu > 1$



Thank you!