Integer partitions detect the primes

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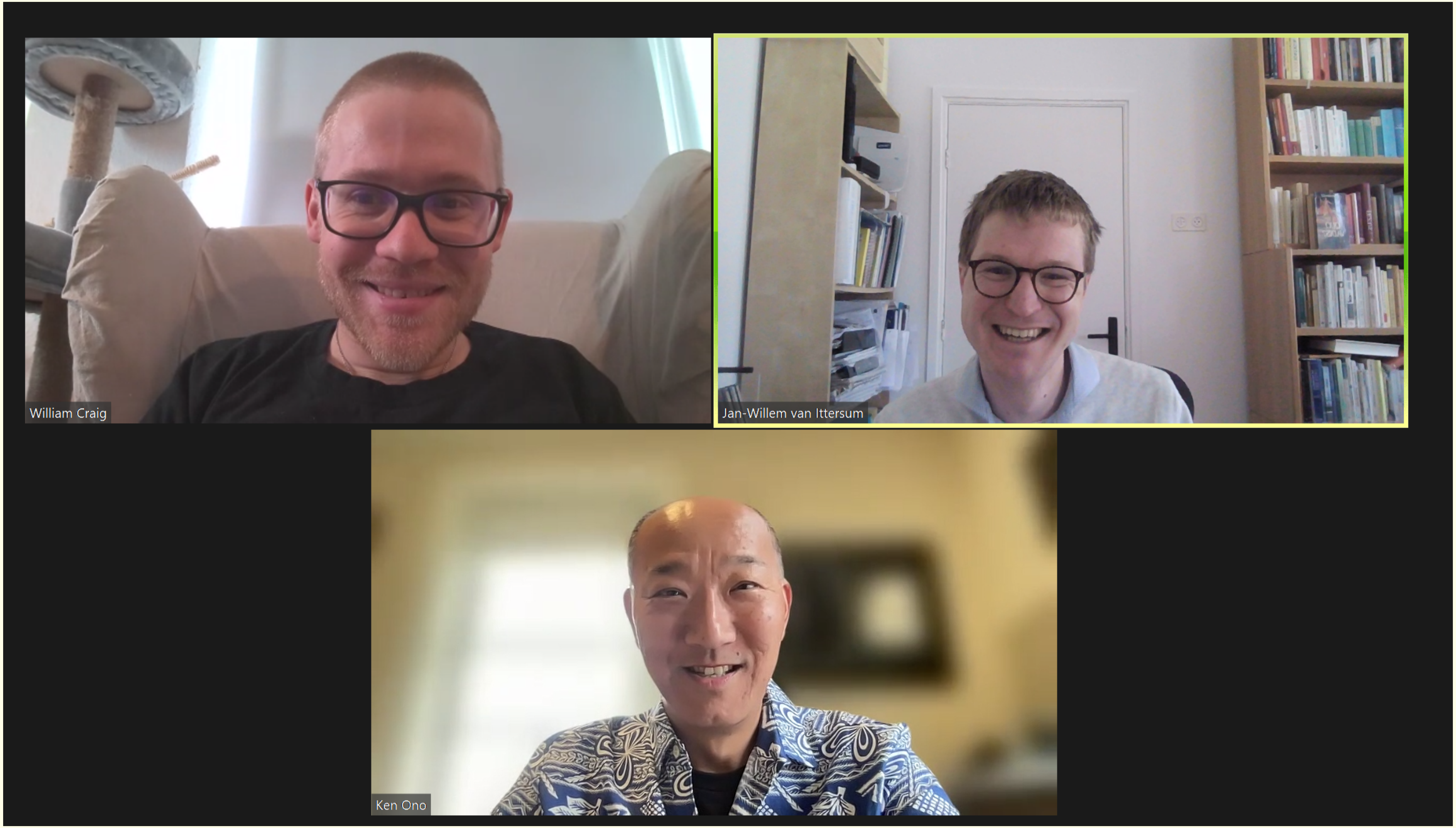
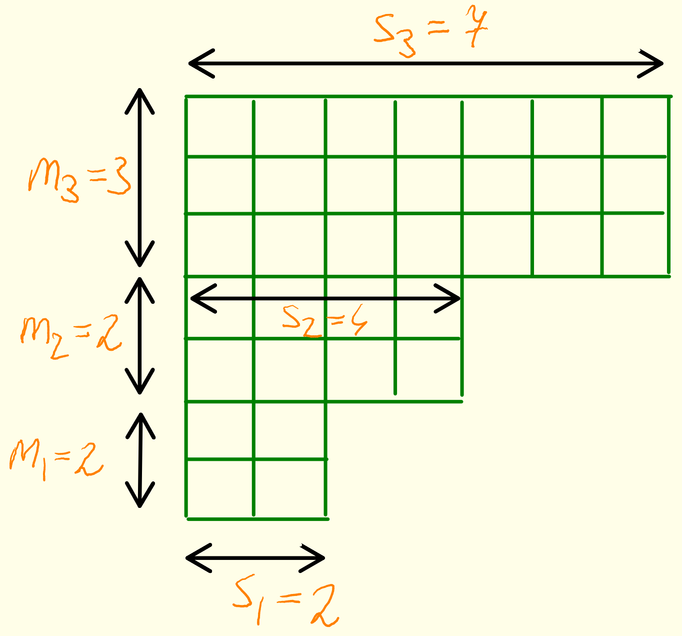


MacMahon partition function

Defn (MacMahon 1920) For $a \geq 1$, we define the MacMahon partition function

$$M_a(n) := \sum_{\substack{n = m_1 s_1 + \dots + m_a s_a \\ 0 < s_1 < \dots < s_a}} m_1 m_2 \cdots m_a$$

Rk $M_a(n)$ equals the sum of multiplicity products of partitions of n with a part sizes.



$$\lambda = (7^3, 4^2, 2^2) = (7,7,7,4,4,2,2)$$

Consider $\psi(n) := (n^2 - 3n + 2) M_1(n) - 8 M_2(n)$

Ex $n=3$, $a=1$: $\lambda = (3)$ or $\lambda = (1^3)$ $\longrightarrow M_1(3) = 1 + 3 = 4$

$a=2$: $\lambda = (2,1)$ $\longrightarrow M_2(3) = 1 \cdot 1 = 1$

$$\psi(3) = 2M_1(3) - 8M_2(3) = 0$$

MacMahon partition function

Defn

$$M_a(n) := \sum_{\substack{n = m_1 s_1 + \dots + m_a s_a \\ 0 < s_1 < \dots < s_a}} m_1 m_2 \cdots m_a$$

$$\psi(n) := (n^2 - 3n + 2) M_1(n) - 8 M_2(n), \qquad \psi(3) = 0$$

Ex $n=4$, $a=1$: $\lambda = (4)$, $\lambda = (2^2)$ or $\lambda = (1^4)$ $\longrightarrow$ $M_1(4) = 1+2+4 = 7$

$a = 2$: $\lambda = (3,1)$ or $\lambda = (2,1^2)$ $\longrightarrow$ $M_2(4) = 1 \cdot 1 + 1 \cdot 2 = 3$

$$\psi(4) = 6 M_1(4) - 8 M_2(4) = 18$$

n	2	3	4	5	6	7	8	9	10	11
$\psi(n)$	0	0	18	0	120	0	270	192	504	0

Two prime-detecting expressions

Thm (Craig-vI-Ono) For positive integers n, we have

(1) $(n^2-3n+2)M_1(n) - 8M_2(n) \geq 0$

(2) $(3n^3-13n^2+18n-8)M_1(n) + (12n^2-120n+212)M_2(n) - 960M_3(n) \geq 0$

and for $n \geq 2$ these expressions vanish if and only if n is prime.

Rk We call such an expression prime-detecting.
Note that the sum of two prime-detecting expressions is prime-detecting.
Also, multiplying with $f(n)$ for a polynomial yields a prime-detecting expression if $f(n) > 0$ for all n.

Prime Detecting Expression
$(n^2-3n+2)M_1(n)-8M_2(n)$
$(3n^3-13n^2+18n-8)M_1(n)+(12n^2-120n+212)M_2(n)-960M_3(n)$
$(25n^4-171n^3+423n^2-447n+170)M_1(n)+(300n^3-3554n^2+12900n-14990)M_2(n)+$ $(2400n^2-60480n+214080)M_3(n)-725760M_4(n)$
$(126n^5-1303n^4+5073n^3-9323n^2+8097n-2670)M_1(n)+$ $(3024n^4-48900n^3+288014n^2-737100n+695490)M_2(n)+$ $(60480n^3-1510080n^2+10644480n-23496480)M_3(n)+$ $(725760n^2-36288000n+218453760)M_4(n)-580608000M_5(n)$
$(300n^8-1542n^7-33049n^6+377959n^5-1651959n^4+3726801n^3-4575760n^2+2903750n-746500)M_1(n)+$ $(12000n^7-91008n^6-2799900n^5+50637162n^4-351366300n^3+1239098170n^2-2210467000n+1585493500)M_2(n)+$ $(432000n^6-3548160n^5-236343840n^4+5133219840n^3-42370071840n^2+161101416000n-236150560800)M_3(n)+$ $(12096000n^5-72817920n^4-17599680000n^3+396192142080n^2-3123876672000n+8555162112000)M_4(n)+$ $(193536000n^4-1056513024000n^2+21310248960000n-112944125952000)M_5(n)+$ $(-46495088640000n+604436152320000)M_6(n)-1115882127360000M_7(n)$

Conjecture These are all such expressions (up to addition and multiplication as before)

Prime-detecting expressions with constants coefficients

Defn For $\underline{\ell} \in \mathbb{Z}_{\geq 0}^{a}$, we define the generalized MacMahon partition function

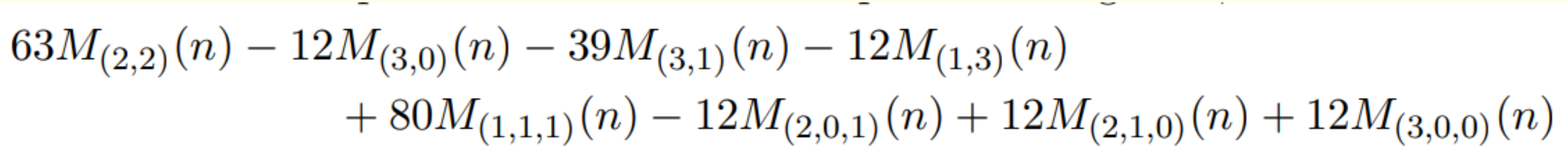
$$M_{\underline{\ell}}(n) := \sum_{\substack{n = m_1 s_1 + \dots + m_a s_a \\ 0 < s_1 < \dots < s_a}} m_1^{\ell_1} m_2^{\ell_2} \cdots m_a^{\ell_a}$$

Thm (Craig-vI-Ono) Let $d \geq 4$.

(1) There exist $c_{\underline{\ell}} \in \mathbb{Z}$ s.t. $\sum_{|\underline{\ell}| \leq d} c_{\underline{\ell}} M_{\underline{\ell}}(n) \geq 0$ $\qquad |\underline{\ell}| = \ell_1 + \ell_2 + \dots + \ell_a$

(2) There are $\gg d^2$ linear independent such expressions

Ex



$$63M_{(2,2)}(n) - 12M_{(3,0)}(n) - 39M_{(3,1)}(n) - 12M_{(1,3)}(n) + 80M_{(1,1,1)}(n) - 12M_{(2,0,1)}(n) + 12M_{(2,1,0)}(n) + 12M_{(3,0,0)}(n) = \frac{11}{3}\psi(n)$$

Prime-detecting quasimodular forms

Let
$$G_k := -\frac{B_k}{2k} + \sum_{n\geq 1} \overset{\sum_{d|n} d^{k-1}}{\sigma_{k-1}(n)}\, q^n, \qquad D := q\frac{\partial}{\partial q} \qquad (q = e^{2\pi i \tau})$$

Consider
$$f_{k,\ell} := (D^\ell+1)\,G_{k+1} - (D^k+1)\,G_{\ell+1}$$
$$= \ast + \sum_{n\geq 1}\sum_{d|n} \left((n^\ell+1)\,d^k - (n^k+1)\,d^\ell\right) q^n$$

Note that for $d=1$ one gets
$d=n$ one gets
$$\begin{array}{rl} n^\ell+1-n^k-1 &= n^\ell-n^k \\ (n^\ell+1)n^k-(n^k+1)n^\ell &= \underline{n^k-n^\ell}\;+ \\ & \quad\; 0 \end{array}$$

Hence
$$f_{k,\ell} = \ast + \sum_{n\geq 1}\sum_{\substack{d|n\\ 0<d<n}} \left((n^\ell+1)\,d^k - (n^k+1)\,d^\ell\right) q^n$$

So, $f_{k,\ell}$ vanishes at prime coefficients. Positivity of coefficients is also easy.

Prime-detecting quasimodular forms

Thm (Craig-vI-Ono) All prime-detecting forms in $\bigoplus_{k \text{ even}} \bigoplus_{n \geq 0} D^n G_k$ are linear combinations of $D^n H_k$ with

$$H_k := \begin{cases} \frac{1}{6}(D^2 - D + 1)\, G_2 - G_4 & k = 6 \\ \frac{1}{24}(-D^2 G_{k-6} + (D^2+1)\, G_{k-4} - G_{k-2} & k \geq 8 \end{cases}$$

Ex $f_{1,3} = (D+1)\, H_4$

Rk The coefficients of H_k are

- non-negative
- coprime integers.

Conj The H_k are all prime-detecting $\underbrace{\text{quasimodular forms}}_{\text{elements of } \mathbb{Q}[G_2, G_4, G_6]}$ (up to linear combinations and derivatives)

q-analogues of multiple zeta values

Let

$$g(\underline{\ell}) := \sum_{n\geq 0} M_{\underline{\ell}}(n)\, q^n \xrightarrow{q\to 1} \frac{(1-q)^{|\underline{\ell}|+a}}{\prod_i \ell_i!} \underbrace{\sum_{s_1<\dots<s_a} \frac{1}{s_1^{\ell_1+1} s_2^{\ell_2+1} \cdots s_a^{\ell_a+1}}}_{\text{multiple zeta value}} \qquad (\ell_a \geq 1, \ell_i \geq 0)$$

$\mathcal{Z}_q := \langle g(\underline{\ell}) \rangle_{\mathbb{Q}}$ was introduced by Bachmann-Kühn.

Facts:

- $\mathcal{Z}_q$ is a differential algebra
- $\widetilde{M} := \mathbb{Q}[G_2, G_4, G_6] \subseteq \mathcal{Z}_q$
- [Hoffman-Ihara] $$\sum_{j\geq 0} g(\underbrace{1,1,\dots,1}_{j})\, X^j = \exp\left(\sum_{n\geq 1} \frac{(-1)^{n+1}}{n} X^n \underbrace{\sum_m \frac{q^m}{(1-q^m)^{2n}}}_{\sum_{m,s\geq 1} \binom{s+n-1}{s-n} q^{ms} \,\in\, \widetilde{M}} \right)$$

$\rightsquigarrow$ $M_a(n)$ are coefficients of a quasimodular form

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(1) $(n^2-3n+2)M_1(n) - 8M_2(n) \geq 0$ (H_4)

(2) $(3n^3-13n^2+18n-8)M_1(n) + (12n^2-120n+212)M_2(n) - 960M_3(n) \geq 0$ (H_6)

and for $n \geq 2$ these expressions vanish if and only if n is prime.

Thank you!