A Kaneko-Zagier equation for Jacobi forms

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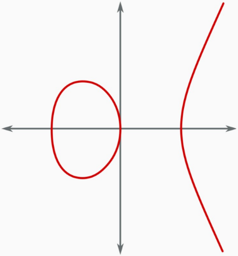
Universiteit Utrecht
Max Planck Institute for Mathematics

The classical picture

Elliptic curves



$y^2 = 4x^3 - g_2 x - g_3$

Parametrization

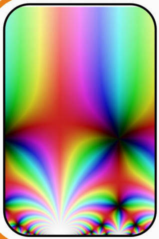
Jacobi forms

$\wp'^2 = 4\wp^3 - g_2\wp - g_3$

Taylor coefficients

Isomorphism class

Modular functions



$\Delta = g_2^3 - 27g_3^2$

$j = 1728\frac{g_2^3}{\Delta}$

Supersingular elliptic curves and modular forms f_p

Supersingular elliptic curves

$$E[p](\overline{\mathbb{F}_p}) = 0$$

$$f_p := \mathrm{Res}_{x=0}\wp'(x)^{p/3}$$

Kaneko–Zagier equation

$$D_\tau^2\left(\frac{f_p}{\Delta^{p/12}}\right) = p^2\frac{g_2}{192\pi^4}\left(\frac{f_p}{\Delta^{p/12}}\right)$$

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Example ($p = 37$)

$$f_{37} = \Delta^3(2945j^3 - 7879680j^2 + 5446434816j - 660451885056).$$

$$2945j^3 - 7879680j^2 + 5446434816j - 660451885056 \equiv 22(j-8)(j^2-6j-6) \mod 37.$$

The roots over $\mathbb{F}_{37^2}$ are the supersingular j-invariants.

Supersingular elliptic curves and modular forms f_p

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Theorem (Kaneko–Zagier)

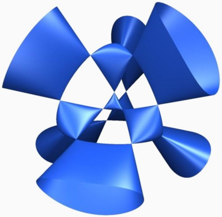
The functions f_p

(i) *satisfy the Kaneko–Zagier equation for all $p \in \mathbb{Z}_{\geq 0}$*

(ii) *determine supersingular j-invariants for all primes $p \geq 5$ (as in the example).*

A Kaneko–Zagier equation for Jacobi forms

Gromov–Witten invariants of K3 surfaces



Jacobi forms $\quad (p = e^{2\pi i z}, q = e^{2\pi i \tau})$

$$\Theta(z) = (p^{1/2} - p^{-1/2}) \prod_{n\geq 1} \frac{(1-pq^n)(1-p^{-1}q^n)}{(1-q^n)^2}$$

$$\varphi_m = \mathrm{Res}_{x=0} \left(\frac{\Theta(x+z)}{\Theta(x)} \right)^m$$

Differential equation

$$D_\tau^2 \varphi_m = m^2 \frac{D_\tau^2(\Theta)}{\Theta} \varphi_m$$

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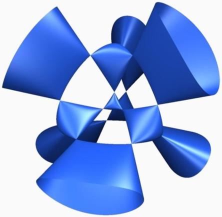
Gromov–Witten invariants of K3 surfaces



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Example

$$\varphi_4 = \left(16\, D_z(\Theta)^3 - 12\, \wp\, D_z(\Theta)\, \Theta^2 - \wp'\, \Theta^3\right) \Theta$$

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Theorem (Oberdieck–Pixton–vI)

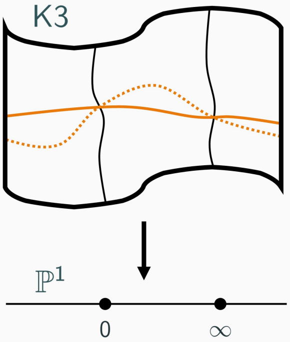
The functions φ_m

(i) *satisfy the the above differential equation for all* $m \in \mathbb{Z}_{\geq 0}$*;*

(ii) *admit quasimodular Taylor coefficients depending polynomially on* m*.*

Conjectural relation to Gromov–Witten theory of K3 surfaces

Conjecture (Oberdieck–Pandharipande)
All double ramification cycle integrals in the Gromov–Witten theory of K3 surfaces are given by an explicit formula in terms of the quasi-Jacobi forms φ_m *and their derivatives.*



Example
For the double ramification cycle integral of type $(n, -n)$ (conjecturally) the computation boils down to computing

$$\lim_{m \to n} \left(\varphi_n \varphi_m + \frac{n}{m-n} D_\tau(\varphi_m) \varphi_n - \frac{m}{m-n} \varphi_m D_\tau(\varphi_n) \right)$$

as an element of $\mathbb{C}[\wp, \wp', D_z(\Theta), \Theta, E_2, g_2]$.

Thank you!