A Kaneko-Zagier equation for Jacobi forms

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The Kaneko–Zagier equation

The study of supersingular elliptic curves leads to the Kaneko–Zagier equation

$$\vartheta_{k+2}\vartheta_k f_k = \frac{k(k+2)}{144} E_4 f_k \qquad (k \in \mathbb{Z}).$$

Theorem This equation has a unique (normalized) *modular solution f_k* of weight k if $k \equiv 0$ or $k \equiv 4 \mod 6$:

$$f_k = \begin{cases} E_4^{k/4}\, F\left(-\frac{k}{12}, -\frac{k-4}{12}, -\frac{k-5}{6}; \frac{E_4^3 - E_6^2}{E_4^3}\right) & k \equiv 0, 4 \mod 12, \\ E_4^{\frac{k-6}{4}} E_6\, F\left(-\frac{k-6}{12}, -\frac{k-10}{12}, -\frac{k-5}{6}; \frac{E_4^3 - E_6^2}{E_4^3}\right) & k \equiv 6, 10 \mod 12. \end{cases}$$

Here:

- $D_\tau = \frac{1}{2\pi i}\frac{\partial}{\partial \tau}$ and the Serre derivative $\vartheta_k = D_\tau - \frac{k}{12}E_2(\tau)$,
- Eisenstein series: $E_k(\tau) = \frac{1}{2}\sum'_{\substack{m,n\in\mathbb{Z}\\(m,n)=1}} \frac{1}{(m\tau+n)^k}$, where $\mathrm{Im}(\tau) > 0$
- Hypergeometric function:

$$F(a,b,c;x) = \sum_{n=0}^{\infty} \frac{a(a+1)\cdots(a+n-1)\, b(b+1)\cdots(b+n-1)}{c(c+1)\cdots(c+n-1)} \frac{x^n}{n!}.$$

The solutions are given by the Weierstrass $\wp$ function

Weierstrass $\wp$ function

$$\wp(\tau, z) = \frac{1}{z^2} + \sum_{m,n\in\mathbb{Z}}^{*} \left(\frac{1}{(z+m\tau+n)^2} - \frac{1}{(m\tau+n)^2} \right).$$

For all $k \in \mathbb{Z}$ one has

$$f_k = \mathrm{Res}_{z=0} \left(\frac{\wp'(\tau, z)}{-2} \right)^{-(k+1)/3}.$$

or, by Lagrange inversion, in terms of generating series

$$x = \sum_{k\geq 0} f_k \frac{y^{k+1}}{k+1} \quad \Longleftrightarrow \quad y = \left(\frac{\wp'(x)}{-2} \right)^{-1/3}.$$

A differential equation for quasi-Jacobi forms

The study of Gromov–Witten invariants of K3 surfaces leads for all $m \in \mathbb{Z}$ to the differential equation

$$D_\tau^2 \varphi_m = m^2 B \varphi_m,$$

where

$$B = \frac{1}{\Theta} D_\tau^2 \Theta, \qquad \Theta = \left(p^{1/2} - p^{-1/2}\right) \prod_{n \geq 1} \frac{(1 - pq^n)(1 - p^{-1}q^n)}{(1 - q^n)^2},$$

with $p = e^{2\pi i z}, q = e^{2\pi i \tau}$.

Compare to the Kaneko–Zagier equation where

$$\frac{E_4}{144} = \frac{1}{\eta} D_\tau^2 \eta, \qquad \eta = q^{1/24} \prod_{n \geq 1} (1 - q^n).$$

The solutions are given by a ratio of Jacobi theta functions

Analogue for Jacobi forms:

$$D_\tau^2 \varphi_m = m^2 B \varphi_m.$$

Theorem (Oberdieck–Pixton–vI) For all $m \geq 0$ we have

$$\varphi_m = \mathrm{Res}_{x=0} \left(\frac{\Theta(x+z)}{\Theta(x)} \right)^m .$$

In terms of generating series

$$x = \sum_{m=1}^{\infty} \varphi_m \frac{y^m}{m} \quad \Longleftrightarrow \quad y = \frac{\Theta(x)}{\Theta(x+z)}.$$

In order to understand the functions φ_m we have to understand the meromorphic Jacobi form $\frac{\Theta(x+z)}{\Theta(x)}$ (to be continued).

A recurring theme: residues of Jacobi forms

Situation: f meromorphic of weight -1, and a differential equation

$$D_\tau^2 \varphi_m = m^2 \frac{D_\tau^2 f}{f} \varphi_m.$$

with solutions for $m \geq 0$ given by

$$\varphi_m = \mathrm{Res}_{x=0} F^m.$$

Examples

	f	F
Kaneko–Zagier	η^2	$\left(-\frac{1}{2}\wp'\right)^{-1/3} = \left(-\frac{1}{2} D_z^3 \log \Theta\right)^{-1/3}$
Oberdieck–Pixton–vI	Θ	$\frac{\Theta(x)}{\Theta(x+z)}$
Tomoaki Nakaya	$\eta(\tau)\eta(2\tau)$	$\left(-D_z \frac{\Theta(2\tau,2z)}{2\Theta(\tau,z)^2}\right)^{-1/2}$

Question Is it possible to formulate this observation as a theorem?

Proof sketch of such a result

How does one proof that

$$D_\tau^2 \varphi_m = m^2 \frac{D_\tau^2 f}{f} \varphi_m \quad \text{is solved by} \quad \varphi_m = \mathrm{Res}_{x=0} F^m?$$

Proof sketch. Observe

$$D_\tau^2 G = \frac{D_\tau^2 f}{f} D_y^2 G, \quad \text{where} \quad G = \sum_{m=1}^{\infty} \varphi_m \frac{y^m}{m} \quad \text{and} \quad D_y = y \frac{\partial}{\partial y}$$

Applying

$$y = \frac{1}{H(x)} \iff x = G(y)$$

yields

$$D_x(H)^2 \, D_\tau^2(H) - 2D_x(H) \, D_x D_\tau(H) \, D_\tau(H) + D_x^2(H) \, D_\tau(H)^2 = \frac{D_\tau^2 f}{f} \, D_x^2 \log(H) \, H^3.$$

'Simply' checking that $F = H$ solves this differential equation suffices!
$\Longrightarrow$ We have to understand the space of derivatives of Jacobi forms.

Intermezzo: transformation of some special functions

Let $A = D_z \log \Theta$. Write $e(x) = e^{2\pi i x}$.

Elliptic transformation For all $m, n \in \mathbb{Z}$

$$\begin{aligned} \wp(\tau, z + m\tau + n) &= \wp(\tau, z) \\ e\left(\tfrac{1}{2}(m^2\tau + 2mz + m^2 + n^2)\right)\Theta(\tau, z + m\tau + n) &= \Theta(\tau, z) \\ A(\tau, z + m\tau + n) &= A(\tau, z) - m. \end{aligned}$$

Modular transformation For all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$

$$\begin{aligned} (c\tau + d)^{-2}\,\wp\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) &= \wp(\tau, z) \\ (c\tau + d)\,e\left(\frac{1}{2}\frac{c^2}{c\tau + d}\right)\Theta\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) &= \Theta(\tau, z) \\ (c\tau + d)^{-1}\,A\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) &= A(\tau, z) + \frac{cz}{c\tau + d} \\ (c\tau + d)^{-2}\,E_2\left(\frac{a\tau + b}{c\tau + d}\right) &= E_2(\tau) + \frac{12}{2\pi i}\frac{c}{c\tau + d}. \end{aligned}$$

Quasi-Jacobi forms

Definition

A quasi-Jacobi form is a polynomial in A and E_2 with Jacobi forms as coefficients.

Proposition

The space of meromorphic quasi-Jacobi forms is

- *finite dimensional after fixing a weight, index and discrete set of allowed poles $z = a\tau + b$;*
- *closed under the differential operators D_τ and D_z.*

Corollary

It is a finite computation to check whether $F = H$ solves a differential equation as

$$D_x(H)^2\, D_\tau^2(H) - 2D_x(H)\, D_x D_\tau(H)\, D_\tau(H) + D_x^2(H)\, D_\tau(H)^2 = \frac{D_\tau^2 f}{f}\, D_x^2 \log(H)\, H^3.$$

The solutions φ_m as quasi-Jacobi forms

Theorem (Oberdieck–Pixtion–vI) *For $m \geq 0$, the functions*

$$\varphi_m = \mathrm{Res}_{x=0} \left(\frac{\Theta(x+z)}{\Theta(x)} \right)^m$$

are quasi-Jacobi forms of weight -1 and index $m/2$, satisfying

$$\frac{\partial}{\partial E_2} \varphi_m = 0$$
$$\frac{\partial}{\partial A} \varphi_m = \frac{1}{2} \sum_{\substack{i+j=m \\ i,j \geq 1}} \frac{m^2}{ij} \varphi_i \varphi_j .$$

Moreover, the Taylor coefficients of φ_m are quasimodular forms, depending polynomially on m:

$$\varphi_m = mz - G_2 m^3 z^3 + \left(\left(\frac{1}{3} G_2^2 - \frac{1}{72} G_4 \right) m^5 + \left(\frac{1}{6} G_2^2 - \frac{5}{72} G_4 \right) m^3 \right) z^5 +$$
$$+ \left(\left(-\frac{1}{18} G_2^3 + \frac{1}{180} G_2 G_4 - \frac{1}{43200} G_6 \right) m^7 + \left(-\frac{1}{9} G_2^3 + \frac{1}{18} G_2 G_4 - \frac{7}{8640} G_6 \right) m^5 + \right.$$
$$\left. + \left(\frac{1}{45} G_2 G_4 - \frac{7}{3600} G_6 \right) m^3 \right) z^7 + \dots$$

Thank you!