Automorphic Forms and L-functions over the Rationals

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Declaration

'This piece of work is a result of my own work except where it forms an assessment based on group project work. In the case of a group project, the work has been prepared in collaboration with other members of the group. Material from the work of others not involved in the project has been acknowledged and quotations and paraphrases suitably indicated.'

Abstract

We begin by reviewing basic concepts from number theory before detailing classical modular and automorphic forms. A discussion of integral quadratic forms, and in particular binary quadratic forms, follows. Using this knowledge we build theta functions associated to positive definite quadratic forms and show that they are automorphic forms. We then discuss various L-functions of number field extensions K over $\mathbb{Q}$. Given these results we look to specialise to the quadratic field case, and see that we must consider imaginary quadratic fields and real quadratic fields separately. Our discussion of the real quadratic case leads us into Maass forms of weight 0 on $SL(2,\mathbb{Z})$. Ending, we state basic results from Galois theory and representation theory which allow us to introduce Artin L-functions over $\mathbb{Q}$.

Contents

1. Introduction

Modular and automorphic forms and L-functions underpin a huge amount of modern mathematics and physics. They appear in areas ranging from algebraic number theory to the physical description of black holes, and from topology to gauge theory. There is a vast amount of literature on the subject, though much is of a level most suitable for PhD and post-doctoral reading, and we could not hope to cover every aspect in this report.

Consider the equation

$$x^2 - Dy^2 = p$$

for a prime p and integers x, D, y. To solve this equation in general is not straightforward - we need to work in field extensions of $\mathbb{Q}$. Factorising the left hand side gives

$$(x - \sqrt{D}y)(x + \sqrt{D}y) = p$$

and so we need knowledge of primes in the quadratic field $\mathbb{Q}[\sqrt{D}]$. Integer primes p, when considered over extensions of $\mathbb{Q}$, exhibit interesting properties. For example, they can remain prime or split into (possibly non-equivalent) elements of norm p.
However, in general, quadratic fields are not unique factorisation domains and so this behaviour is not well-defined. We turn to the theory of ideals to remedy this situation, since they restore a notion of unique decomposition into prime ideals $\mathfrak{P}$. Our questions are then:
When does the ideal (p) split?
How many ideals are there of a given norm n?
For quadratic fields these can be easily answered by elementary methods - but in field extensions of higher degree it is not so easy. Information on the prime ideals of a field extension $K/\mathbb{Q}$ is encoded in Dedekind zeta function $\zeta_K(s)$ and Hecke L-functions $L_K(s)$ of field extensions, motivating the further study of these objects. For example, information about the number of prime ideals of prime norm is seen directly from the coefficients in the expansion of the Dedekind zeta function.
These L-functions exhibit many fascinating properties, perhaps the most important and useful of which is that they admit continuations to all of $\mathbb{C}$ and satisfy functional equations. These functions are closely related to certain theta functions and modular forms, and many proofs pertaining to both the Dedekind zeta function and Hecke L-functions rely heavily on knowledge of both modular forms and theta functions.
Hence number theory is intricately linked to the theory of modular forms. By considering the theory of L-functions and automorphic forms we can answer many questions in number theory. The aim of this report is to give an overview of introductory results of classical automorphic forms (with specific emphasis on modular forms), L-functions, and quadratic fields that is accessible to final year undergraduate students or masters students. At points there is direction given to the interested reader on where to find further material that may be of interest.
Throughout, I have tried to give some brief examples which I hope prove helpful to the reader. While many of the theorems stated in this report are exceedingly beautiful their

proofs are in some cases highly involved and are therefore not presented. I hope that the references provided in lieu of such proofs will suffice for the motivated reader.
The notation used here is fairly standard where possible, with some modifications where it may help the reader. For succinctness we do not underline vectors and the like, however it should be clear where this is. Many authors use their own notation, and readers looking to seek further material should be wary of this. In particular, we use the 'twos-in' notation for our matrices A associated to quadratic forms (i.e. they have even diagonal). The terms L-function and L-series are used interchangeably, though strictly speaking we should use the term L-series before proving the continuation property.

I would like to thank in particular Dr. Jens Funke for his consistent encouragement and guidance, alongside the Grey College mathematicians for their support throughout the year and correcting the inevitable grammatical errors in this report.

2. A brief review of number theory

We begin by introducing some of the results we will need from number theory. In particular, we need definitions and basic facts about number fields, number field extensions, and characters. We assume that the reader is familiar with working with these objects and so we do not present an in-depth discussion here. Results are stated without proof.

2.1 Number fields

Definition We define L to be a **field extension** of a field K if K is a subfield of L. We write this as L/K.

In this report we deal with number field extensions over the field $\mathbb{Q}$, sometimes writing 'field extension' or simply 'extension'. Let $K/\mathbb{Q}$ be a field extension of degree n, written as $[K : \mathbb{Q}] = n$. Then K can be realised by adjoining a root of some polynomial with coefficients in $\mathbb{Q}$ of degree n. In particular we are interested in number field extensions of degree 2 and detail some more concrete results of these in the next section.

Definition Every number field K contains a **ring of integers**, denoted by $\mathcal{O}_K$. This is the ring of all elements $\alpha \in K$ such that α satisfies a monic polynomial over $\mathbb{Q}$. Such elements are known as integral elements.

Definition Each number field K also contains a set of units (elements of norm ± 1), which we denote by $\mathcal{O}_K^*$. We denote units of positive norm by $\mathcal{O}_K^{*,+}$.

One of the most important aspects of field extensions L/K is that primes of the base field K may no longer remain prime in the extension L. For example:

Example 2.1.1
Let $L = \mathbb{Q}[i]$ and $K = \mathbb{Q}$. Then by the above proposition we have $\mathcal{O}_L = \mathbb{Z}[i]$, the Gaussian integers. Consider the prime $p = 2$ in $\mathbb{Z}$. In the extension L we have $2 = (1+i)(1-i)$. Now, $(1+i) = i(1-i)$, so we see that $(1+i) \sim (1-i)$, since i is a unit in $\mathbb{Z}[i]$. Then $2 \sim (1-i)^2$, and we note that $(1-i)$ is prime in $\mathbb{Z}[i]$. In this case we say that $p = 2$ is ramified in L.

Definition An **ideal** I in $\mathcal{O}_K$ is a subgroup of $\mathcal{O}_K$ under addition, and $xa = ax \in I$ for all $x \in I, a \in \mathcal{O}_K$.

$\mathcal{O}_K$ also contain a group of fractional ideals. These are submodules I of $\mathcal{O}_K$ where there exists $\delta \neq 0 \in \mathcal{O}_K$ such that $\delta I \subset \mathcal{O}_K$. (Equivalently, $J = \delta I$ is an ideal in $\mathcal{O}_K$ and $I = \delta^{-1} J$). We denote the group of fractional ideals by $\mathcal{J}(\mathcal{O}_K)$.

Definition The **ideal norm** of an ideal $I \in \mathcal{O}_K$ is given by

$$N(I) = |\mathcal{O}_K/I|,$$

where $\mathcal{O}_K/I$ is the quotient group of $\mathcal{O}$ over the ideal I.

Definition Let K be a number field over $\mathbb{Q}$. Let the group of principal ideals in $\mathcal{O}_K$ be denoted by $\mathcal{P}(\mathcal{O}_K)$. We define the **ideal class group** of K to be the quotient group of fractional ideals over the principal ideals in K, and denote it by $Cl(K)$, i.e.

$$Cl(K) = \mathcal{J}(\mathcal{O}_K)/\mathcal{P}(\mathcal{O}_K).$$

We define the **class number** of a field K as $h = |Cl(K)|$. This class number is always finite [1, page 106].

Definition Let K be a number field over $\mathbb{Q}$. We define the **narrow class group** as

$$Cl^+(K) = \mathcal{J}(\mathcal{O}_K)/\mathcal{P}^+(\mathcal{O}_K),$$

where $\mathcal{P}^+(\mathcal{O}_K)$ is the group of totally positive principal fractional ideals. These are ideals of the form $a\mathcal{O}_K$ where $\sigma(a) > 0$ for every real embedding σ of $K \hookrightarrow \mathbb{R}$.
We define the **narrow class number** of a field K as $h_0 = |Cl^+(K)|$ which again is finite.

Clearly, we have that $Cl(K) \subseteq Cl^+(K)$ with equality if the field K contains a unit of norm -1. Otherwise, we have that the narrow class group is strictly bigger than the class group, since we factor out a smaller group for the narrow class group. In this case we have $h_0 = 2h$.

Definition Two ideals I, J are **equivalent in the narrow sense** if $I = (\alpha)J$ with $\alpha \in \mathcal{O}_K$ and $N(\alpha) > 0$. We write $I \sim J$.

2.1.1 Quadratic number fields

In particular we will be interested in specific results involving quadratic fields. A field extension $K/\mathbb{Q}$ of degree 2 is said to be either real quadratic or imaginary quadratic depending on whether one needs to adjoin a real or imaginary root. If K is quadratic over $\mathbb{Q}$ we write $K = \mathbb{Q}[\sqrt{D}]$. Here again we state definitions and basic results without proof.
The ring of integers of quadratic extensions of $\mathbb{Q}$ is [2, page 670, Theorem 10.31]:

Theorem 2.1.2
Let $K = \mathbb{Q}[\sqrt{D}]$ be a field extension of $\mathbb{Q}$ of degree 2, with D a squarefree integer $\neq 0, 1$. Then

$$\mathcal{O}_K = \begin{cases} \mathbb{Z}[\sqrt{D}] & \text{if } D \equiv 2, 3 \mod 4 \\ \mathbb{Z}[\frac{1+\sqrt{D}}{2}] & \text{if } D \equiv 1 \mod 4. \end{cases}$$

Definition The discriminant of a quadratic field $K = \mathbb{Q}[\sqrt{D}]$ is

$$\begin{cases} D & \text{if } D \equiv 1 \mod 4 \\ 4D & \text{otherwise .} \end{cases}$$

Since the theory differs slightly for real and imaginary quadratic fields we will state particular theorems separately in the following sections. For now we continue with the general theory for quadratic fields and consider how primes can split in such extensions. Integer primes in quadratic fields behave in precisely one of three ways in K, and when K is a unique factorisation domain (UFD) we can categorise this precisely [2, page 683, Theorem 10.41].

When we do not know that K is a UFD then we must use ideals. Working with ideals restores a notion of unique decomposition of any given ideal I into prime ideals $\mathfrak{P}_i$. This is true in any given number field extension of finite degree over $\mathbb{Q}$. We assume familiarity with the concept, and only present the theorems that are most important for us here.

Definition The **norm map** of a quadratic field K is given by

$$\begin{aligned} N : K &\to \mathbb{Z} \\ a + b\sqrt{D} &\mapsto a^2 - Db^2 \end{aligned}$$

We will make use of the norm maps extensively. To extend the above results of splitting of integer primes in quadratic extensions we look at how prime ideals behave in such extensions [1, page 108].

Theorem 2.1.3
Let $K = \mathbb{Q}[\sqrt{D}]$ with D squarefree. Let p be an odd prime in $\mathbb{Z}$.
Then the ideal (p) in $\mathcal{O}_K$ behaves in the exactly one of the following ways:

$$\begin{cases} (p) = \mathfrak{p}_p\overline{\mathfrak{p}_p} \textit{ with } \mathfrak{p}_p = (p, m - \sqrt{D}) \textit{ where } m^2 \equiv D \mod p & \textit{if } \chi_D(p) = 1 \\ (p) = \mathfrak{p}_p^2 \textit{ with } \mathfrak{p}_p = (p, \sqrt{D}) & \textit{if } \chi_D(p) = 0 \\ (p) \textit{ is prime itself in } \mathcal{O}_K & \textit{if } \chi_D(p) = -1, \end{cases}$$

where $\mathfrak{p}_p$ are prime ideals of norm p in $\mathcal{O}_K$ and $\chi_D(p) = \left(\frac{D}{p}\right)$ is the Legendre symbol of D on p.

We say that p splits, ramifies or is inert in K depending on which case it falls into, respectively.

There is also a similar theorem for the prime $p = 2$ [1, page 108], though there are now slightly different conditions. Since $p = 2$ is the only even prime, for succinctness, we leave corrections for the case $p = 2$ to the reader. If we let a_m denote the number of ideals of norm m in $\mathcal{O}_K$ for quadratic fields K then we see directly, by theorem 2.1.3 that for odd prime p we have:

$$a_p = \begin{cases} 2 & \text{if } \chi_D(p) = 1 \\ 1 & \text{if } \chi_D(p) = 0 \\ 0 & \text{if } \chi_D(p) = -1, \end{cases}$$

and we note that this formula is completely multiplicative due to the unique factorisation of ideals. So, we can see exactly how many ideals there are of a given norm m in $\mathcal{O}_K$ by finding the coefficient a_m. This idea will resurface when we look at the Dedekind zeta function of a field extension, $\zeta_K(s)$. We now only need knowledge of the units of K, and consider first the case where K is an imaginary quadratic field.

Units in imaginary quadratic fields

We need knowledge of the group of units of the field K. In the imaginary quadratic case the group is very simple, in contrast to the real quadratic case which we will later see is an infinite group. When K is an imaginary quadratic field, the group of units has finite size (in fact, it is bounded above in size by 6). More precisely we have [2, page 672, Theorem 10.33]:

Theorem 2.1.4
Let $K = \mathbb{Q}[\sqrt{D}]$ with $D \leq -1$. Then

$$\mathcal{O}_K^* = \begin{cases} \{\pm 1, \pm i\} & \text{if } D = -1 \\ \{\omega^j | 0 \leq j \leq 5\} & \text{if } D = -3 \\ \{\pm 1\} & \text{Otherwise} \end{cases},$$

where $\omega = \frac{1+\sqrt{-3}}{2}$.

It is clear that the norm of any unit in K is 1, and so we have that $\mathcal{O}_K^{*,+} = \mathcal{O}_K^*$.

Example 2.1.5
An example we will work with throughout this report is the field extension $K = \mathbb{Q}[\sqrt{-23}]$. This is clearly an imaginary quadratic field over $\mathbb{Q}$. That is, K can be realised by adjoining a square-root of a negative integer to $\mathbb{Q}$. It is of discriminant -23 since we have $-23 \equiv 1 \mod 4$.
By theorem 2.1.4 we have that $\mathcal{O}_K^* = \{\pm 1\}$. We can also see that the norm map gives that $N(1) = N(-1) = 1$ and so $\mathcal{O}_K^* = \mathcal{O}_K^{*,+}$.

Units in real quadratic fields

In the real quadratic case the group of units is more complicated. In fact it is infinite in size, since it is the set of solutions to the associated Pell equation of the field:

$$\begin{cases} a^2 - Db^2 = \pm 1 & \text{for } D \equiv 2, 3 \mod 4 \\ a^2 - Db^2 = \pm 4 & \text{for } D \equiv 1 \mod 4. \end{cases}$$

Moreover, we have the following theorem which tells us precisely what the units of a real quadratic field are.

Theorem 2.1.6
Let K be a real quadratic field. Then

$$\mathcal{O}_K^* = \langle \epsilon \rangle \,,$$

where $\epsilon = a + b\sqrt{D}$ is the fundamental unit of K.

Proof This follows directly from Dirichlet's Unit Theorem [3, page 42, Theorem 7.4]). ■

The fundamental unit ϵ is the smallest positive solution to the Pell equation (note that $a, b \in \mathbb{Z}$). More precisely [4, page 18, Theorem 1.17]:

Theorem 2.1.7
Let $K = \mathbb{Q}[\sqrt{D}]$ with $D > 0$, D squarefree. Then we have:

1. If $\mathcal{O}_K = \mathbb{Z}[\sqrt{D}]$ then $\epsilon = a + b\sqrt{D}$ is given by the solution to $a^2 - Db^2 = \pm 1$ such that $a > 0$, $b > 0$, and b is minimal.

2. If $\mathcal{O}_K = \mathbb{Z}[\frac{1+\sqrt{D}}{2}]$ (i.e. $D \equiv 1 \mod 4$) then we have two cases:

a) If $D = 5$ then $\epsilon = \frac{1+\sqrt{5}}{2}$.

b) If $D \neq 5$ then $\epsilon = a + b\sqrt{D}$ where $a, b > 0$ are the solution to $a^2 - Db^2 = \pm 4$ with b minimal.

Since the group of units is generated by the fundamental unit ϵ and the norm map is completely multiplicative we have that $\mathcal{O}_K^* = \mathcal{O}_K^{*,+}$ if and only if $N(\epsilon) = -1$.

Example 2.1.8
Let $K = \mathbb{Q}[\sqrt{7}]$. It is a real quadratic field, with fundamental unit $\epsilon = 8 + 3\sqrt{7}$. Since $N(\epsilon) = 1 > 0$ we have that the narrow class group of K is strictly bigger than the class group of K. In fact, we have that $h = 1$ and $h_0 = 2$ here.

We next briefly introduce characters of integers and ideals for use later in this report.

2.2 Characters

We concentrate on two different characters, namely the Dirichlet character and Hecke character on the ideal class group. The general Hecke character is more complicated, but allows for much wider reaching generalisations - including ray class groups which we will mention briefly again later.

Definition A **Dirichlet character** χ is a function $\chi : \mathbb{Z} \to \mathbb{C}$ with the following properties:

1. $\exists$ a positive integer q such that $\chi(n+q) = \chi(n)$ for all $n \in \mathbb{Z}$. q is known as the modulus of the character.
2. If $\gcd(n,q) > 1$ then $\chi(n) = 0$; if $\gcd(n,q) = 1$ then $\chi(n) \neq 0$.
3. $\chi(mn) = \chi(m)\chi(n)$ for all $m,n \in \mathbb{Z}$.

We call χ principal if $\chi(n) = 1$ for all $(n,q) = 1$, and denote the trivial character of modulus 1 (taking value 1 on all integers) by χ_0. This is the only time when $\chi(0) \neq 0$.
We say that χ is imprimitive if for some n with $(n,q) = 1$, $\chi(n)$ has a period less than q. Otherwise, χ is primitive.
In particular, if q is prime then χ is primitive, and every imprimitive character can be induced by a primitive one (so we need only look at the theory of primitive characters).

It is easy to see some further properties of Dirichlet characters, arising directly from the definition above. For example, $\chi(1) = \chi(1 \times 1) = \chi(1)\chi(1)$ using property 3. Then property 2. gives that $\chi(1) \neq 0$ so we immediately obtain that $\chi(1) = 1$. We also see a very important fact - χ is completely multiplicative.
The inverse of a Dirichlet character χ is given by $\chi^{-1} = \bar{\chi}$.

Definition The **conductor of a Dirichlet character** of modulus q is the minimal positive q^* such that

1. $q^*|q$
2. $\chi(n+q^*) = \chi(n)$ for all n where $(n,q) = (n,q^*) = 1$.

We will use Dirichlet characters to form L-functions associated to the field $\mathbb{Q}$. However, we will also want to consider L-functions of general number field extensions of $\mathbb{Q}$ and we will need to define characters of ideal classes.

Definition An **ideal class character** χ of a field K is a character which acts on the ideals I_i of K such that:
1. $\chi(I_1 I_2) = \chi(I_1)\chi(I_2)$.
2. $\chi(J) = 1$ for all principal ideals J with $N(J) > 0$.

The most important aspect to note is that the ideal class character is multiplicative, allowing us to decompose the character of any ideal to the character acting on prime ideals instead. Also note that the characters on the class group are really dependent on the norm of a given ideal.
It is clear to see that ideal class characters are a generalisation of Dirichlet characters, with the additional of condition 2. We will use this to generalise results of L-functions of $\mathbb{Q}$ to L-functions of higher degree field extensions $K/\mathbb{Q}$ later in this report.

3. Modular forms

3.1 Modular forms and cusp forms

Here we briefly cover the basic definitions of modular, cusp, and automorphic forms. This section is aimed to be a general recap of material and is in no way an exhaustive list of definitions and properties. We only detail results that are closely linked to the rest of this report. Since modular forms can be defined through the study of elliptic functions over lattices [5, section 1.2, page 6] it makes sense to study the relation of two lattices. Two pairs of complex numbers (ω_1, ω_2) and (ω_1', ω_2') determine the same lattice if and only if

$$\omega_1' = a\omega_1 + b\omega_2$$
$$\omega_2' = c\omega_1 + d\omega_2,$$

for $a, b, c, d \in \mathbb{Z}$ such that $ad - bc = \pm 1$. So the action of unimodular transformations of lattices can be seen as the action of the modular group

$$\Gamma := SL(2, \mathbb{Z}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} | a, b, c, d \in \mathbb{Z}, ad - bc = 1\}$$

on the upper half plane $\mathbb{H}$. This action is given by

$$\gamma\tau = \frac{a\tau + b}{c\tau + d},$$

for all

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,$$

and $\tau \in \mathbb{H}$.

Definition A **meromorphic modular form** of weight k on Γ is a function f defined on the upper half-plane $\mathbb{H}$ satisfying:

1. $f(\gamma\tau) = (c\tau + d)^k f(\tau)$ for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$.

2. f is meromorphic on $\mathbb{H} \cup \{\infty\}$.

The statement that f must be meromorphic at the point at ∞ gives rise to the fact that f has a q-expansion [5, page 13]. That is, we can write

$$f = \sum_{n=m}^{\infty} a_n q^n,$$

where m tells us about the behaviour of f at ∞, and $q := e^{2\pi i\tau}$.
If $m < 0$ then f has a pole of order m at infinity, and we do not deal with this case.
If $m \geq 0$ then f is holomorphic at ∞ and if, in addition, f is holomorphic on the upper half-plane then we say that f is a holomorphic modular form.

3.1.1 Holomorphic modular forms on Γ

We will generalise these definitions shortly to allow for a wider class of functions to be included in our considerations. Firstly, we discuss briefly the theory of holomorphic modular forms on Γ.
We wish to be able to construct modular forms - but checking that functions satisfy the modularity condition for every matrix in Γ is clearly not feasible. Instead, we have that it is enough to prove modularity for the generators of Γ. We also have that the matrices

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$
$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

generate Γ [6, page 6, example 1.1.9]. This in turn leads us to the well-known fact that the fundamental domain of Γ is given by [7, page 1]:

$$\mathcal{F} = \{-\frac{1}{2} \leq \Re(\tau) < \frac{1}{2} \text{ with } |\tau| > 1\} \cup \{-\frac{1}{2} \leq \Re(\tau) \leq 0 \text{ with } |\tau| = 1\},$$

for $\tau \in \mathbb{H}$. Below, the shaded region shows the fundamental domain.

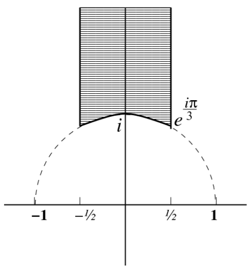

Figure 3.1: The fundamental domain $\mathcal{F}$ of Γ - credit [8]

In particular, we say that the point at ∞ of the fundamental domain is a cusp. Then any holomorphic modular form f that vanishes at the cusp is called a cusp form (this is equivalent to saying that there is no constant term in the q-expansion of f).

Returning to calculating the modularity condition for a function $f(\tau)$ we see that T & S induce two equations that $f(\tau)$ must satisfy:

1. $f(\tau + 1) = f(\tau)$.

2. $f(-\frac{1}{\tau}) = \tau^k f(\tau)$.

If these are satisfied and $f(\tau)$ is holomorphic on $\hat{\mathbb{H}} := \mathbb{H} \cup \{\infty\}$, then $f(\tau)$ is a holomorphic modular form of weight k on Γ. We denote the space of holomorphic modular forms of weight k on Γ by $\mathcal{M}_k(\Gamma)$, and the cusp forms of weight k on Γ by $\mathcal{S}_k(\Gamma)$. Note that we have $\mathcal{M}_k(\Gamma) \supset \mathcal{S}_k(\Gamma)$.

Remark Condition 2. stipulates that there are no modular forms of odd weight. To see this take $\gamma = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \Gamma$ and observe that this imposes that $f(\tau) = (-1)^k f(\tau)$. If k were odd then f would be identically 0.

Example 3.1.1
We define the Eisenstein series of weight $k > 3$ to be

$$E_k(\tau) = \frac{1}{2} \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ \gcd(m,n)=1}} \frac{1}{(m\tau + n)^k}. \tag{3.1.2}$$

It can be shown that the Eisenstein series is a holomorphic modular form of weight k on Γ as follows. We start by checking the modularity conditions for $E_k(\tau)$:
For T

$$\begin{aligned} E_k(\tau+1) &= \frac{1}{2} \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ \gcd(m,n)=1}} \frac{1}{(m(\tau+1)+n)^k} \\ &= \frac{1}{2} \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ \gcd(m,n)=1}} \frac{1}{(m\tau+(m+n))^k} \\ &= E_k(\tau), \end{aligned}$$

where we have used that $E_k(\tau)$ is absolutely convergent [9, page 14] to re-label the sum when moving from the second to the third line.
For S

$$\begin{aligned} E_k(-\frac{1}{\tau}) &= \frac{1}{2} \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ \gcd(m,n)=1}} \frac{1}{(-\frac{m}{\tau}+n)^k} \\ &= \frac{1}{2} \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ \gcd(m,n)=1}} \frac{1}{\tau^{-k}(-m+\tau n)^k} \\ &= \tau^k \frac{1}{2} \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ \gcd(m,n)=1}} \frac{1}{(-m+\tau n)^k} \\ &= \tau^k E_k(\tau). \end{aligned}$$

where we have again made use of absolute convergence to re-arrange the sum when moving from the third to fourth line.

Such convergence also guarantees that $E_k(\tau)$ is holomorphic on $\mathbb{H}$ so it remains to show that it is holomrphic at ∞. This can be easily verified by identifying that

$$\lim_{\tau \to i\infty} E_k(\tau) = 1.$$

Therefore, the Eisenstein series of weight $k > 3$ is a holomorphic modular form of weight k on Γ, and it is not a cusp form since it does not vanish at ∞.
It can be shown that the Eisenstein series of weight $k > 3$ has the q-expansion

$$E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

where B_k are the Benoulli numbers and $\sigma_r(n) = \sum_{d|n} d^r$ is the divisor sum of n [10, page 60,61].

Remark There are generalisations of these Eisenstein series which we will encounter later on. Also note that the series for $k = 2$ is not modular under the action of S. This can be remedied at the expense of the condition that it be holomorphic.

The holomorphic modular forms of Γ are well-understood and it is known that every holomorphic modular form on Γ is a polynomial of Eisenstein series of weight 4 and 6 [9, page 10]. That is, $\mathcal{M} = \mathbb{C}[E_4, E_6]$ as a polynomial ring.
However, modular forms on subgroups of Γ require a lot more work to obtain, as do non-holomorphic modular forms (for example, Maass forms, which we give an introduction to in section 8.3.1).

3.1.2 Modular forms on congruent subgroups

Definition In general we work with the following **congruent subgroups of** Γ:
1. $\Gamma_0(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma |\ c \equiv 0 \mod N\}$.

2. $\Gamma_1(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma |\ a \equiv d \equiv 1 \mod N, \text{ and } c \equiv 0 \mod N\}$.

3. $\Gamma(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma |\ a \equiv d \equiv 1 \mod N, \text{ and } b \equiv c \equiv 0 \mod N\}$.

Notice that we have $\Gamma \supset \Gamma_0(N) \supset \Gamma_1(N) \supset \Gamma(N)$.

Remark Often we will use $\Gamma_0(q)$ instead of $\Gamma_0(N)$ since we will see that the level N can be related to the modulus of a Dirichlet character χ, which typically we denote by q.
For ease of exposition we call the set $G := \{\Gamma, \Gamma(N), \Gamma_0(N), \Gamma_1(N)\}$ so that we may consider the cases simultaneously in our definitions below.

These subgroups have far more complicated structures than that of Γ. For example, each subgroup will come associated to a set of cusps. In particular we have the following definition [7, page 2, Definition 1.5].

Definition A **cusp** of $H \in G$ is an equivalence class in $\mathbb{Q} \cup \{\infty\}$ under the action of H.

Example 3.1.3
Take $H = \Gamma$. Then there is only one equivalence class for $\mathbb{Q} \cup \infty$ and we can choose the point at ∞ to be its representative.

We will not delve into the deep theory of cusps here, but re-direct readers to their discussion at [10, page 10]. A modular form on H which vanishes at all cusps of H is known as a cusp form - note that there are a finite number of cusps [11, page 355].
Then we can define (non-)holomorphic modular forms on $H \in G$.

Definition We say that f is a **(non-)holomorphic modular form** on $H \in G$ if f satisfies

1. $f(\gamma\tau) = (c\tau + d)^k f(\tau)$ for all $\gamma \in H$
2. $f(\tau)$ is (non-)holomorphic on $\hat{\mathbb{H}}$ and at all other cusps

We then denote the space of holomorphic modular forms of weight k on $H \in G$ by $\mathcal{M}_k(H)$. Similarly, we denote the space of holomorphic cusp forms of weight k on $H \in G$ by $\mathcal{S}_k(H)$. Often, we let $\mathcal{M}_k(\Gamma_0(N)) = \mathcal{M}_k(N) = \mathcal{M}_k(q)$ and similar for the cusp form case, $\mathcal{S}_k(\Gamma_0(N)) = \mathcal{S}_k(N) = \mathcal{S}_k(q)$. N is known as the level of the space.

Remark Often when talking about holomorphic modular forms, we will simply use the term modular form - we explicitly mention when something is non-holomorphic.

Similarly to the case for Γ, it is enough to check that a given function is modular for the generators of the subgroup.

3.1.3 Multiplier systems

Now we look to further generalise our definition of a modular form to allow for twists. To do so, we must introduce multiplier systems.

Definition The **automorphy factor** is

$$j_g(\tau) \coloneqq c\tau + d,$$

for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{R})$.

Then, for $g, h \in SL(2, \mathbb{R})$ we have that

$$j_{gh}(\tau) = j_g(h\tau) j_h(\tau).$$

For $\tau \neq 0$ we pick an argument $\in (-\pi, \pi]$ and denote the principal branch of the logarithm of τ by $\log \tau$. We have $\log \tau = \log |\tau| + i \arg \tau$ and the power $\tau^s = e^{s \log \tau}$.
If we let

$$2\pi\omega(g,h) = -\arg j_{gh}(\tau) + \arg j_g(h\tau) + \arg j_h(\tau),$$

then it is clear that this has no dependence on τ. It can be verified that $\omega(g,h) \in \{-1, 0.1\}$. To see this simply take the modulus on each side, then use that we have chosen an argument in $(-\pi, \pi]$. Then $\omega(g,h)$ must be an integer - since g, h are integer valued - with the upper bound on its modulus given by $\frac{3}{2}$.

Definition The **factor system of weight** $k \in \mathbb{R}$ is given by

$$w(g,h) := e^{2\pi i k \omega(g,h)}.$$

In particular, if $k \in \mathbb{Z}$ we have that $w(g,h) = e^{2\pi i k \omega(g,h)} = 1$.
Notice that we have

$$w(g,h) j_{gh}(\tau)^k = j_g(h\tau)^k j_h(\tau)^k.$$

Then we are ready to define our multiplier system [5, page 41, equations 2.52 and 2.53].

Definition Let G be a discrete subgroup of $SL(2, \mathbb{R})$ (these include Γ and its congruent subgroups). Then a **multiplier system** of weight k for G is a function $\vartheta : G \to \mathbb{C}$ such that:

1. $|\vartheta(g)| = 1$ for all $g \in G$
2. $\vartheta(g_1 g_2) = w(g_1, g_2)\vartheta(g_1)\vartheta(g_2)$ for all $g_1, g_2 \in G$.

Remark We require $\vartheta(-1) = e^{-\pi i k}$ if $-1 \in H$ so that we do not generate only the zero modular form.

The multiplier systems defined above allow us to generalise a lot of results from integer weight modular forms to half-integer weight modular forms. These include various theta functions and the Dedekind Eta function. Notice that a multiplier system of weight k is also a multiplier system of any weight $k' \equiv k \mod 2$.

Example 3.1.4
A multiplier system of integer weight k can be viewed as a Dirichlet character χ. More specifically, for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(q)$ we let χ be a Dirichlet character of modulus q. Then, setting $\vartheta(\gamma) = \chi(d)$ we see that this is a multiplier system for $\Gamma_0(q)$. Furthermore, $\chi(d)$ is a multiplier system for any $k \in \mathbb{Z}$ with $k \equiv \frac{1}{2}(1 - \chi(-1)) \mod 2$. We will consider such situations more than their half-integer weight counterparts.

Now we can define a wider class of modular forms.

Definition We say that $f(\tau)$ is an **automorphic form** on $H \in G$ of weight k and multiplier system ϑ (k is either integer or half integer) if it satisfies:

1. $f(\gamma\tau) = \vartheta(\tau)(c\tau+d)^k f(\tau)$ for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in H$.
2. $f(\tau)$ is holomorphic on $\hat{\mathbb{H}}$ and at all other cusps.

We denote the space of automorphic forms on H of half-integer or integer weight k with multiplier system ϑ by $\mathcal{M}_k(H, \vartheta)$, and similarly the space of automorphic cusp forms $\mathcal{S}_k(H, \vartheta)$.

Example 3.1.5
We have that the Dedekind Eta function is an automorphic form of weight $k = \frac{1}{2}$. It is given by the formula

$$\eta(\tau) = e^{\frac{2\pi i\tau}{24}} \prod_{n=1}^{\infty} (1 - e^{2\pi i n\tau}).$$

The multiplier system for the Eta function was found by Dedekind to be [5, page 45, equation 2.71]

$$\vartheta(\gamma) = e^{2\pi i(\frac{a+d-3c}{24c} - \frac{1}{2}s(d,c))},$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $c > 0$. If $c = 0$, $\vartheta(\gamma) = e^{\frac{2\pi i b}{24}}$.

Here we have that $s(d, c)$ is the Dedekind sum

$$s(d,c) = \sum_{n=0}^{c-1} \frac{n}{c}\left(\frac{dn}{c} - \left\lfloor \frac{dn}{c} \right\rfloor - \frac{1}{2}\right),$$

where $\lfloor x \rfloor$ is the floor function of x.

Since we have seen that for integer k the multiplier system can be viewed as a Dirichlet character, it makes sense to define this case separately.

Definition We say that f is a **holomorphic modular form on $H \in G$ of integer weight k and character χ** if it satisfies:
1. $f(\gamma\tau) = \chi(d)(c\tau+d)^k f(\tau)$ for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in H$.
2. $f(\tau)$ is holomorphic on $\hat{\mathbb{H}}$ and at all other cusps.

We denote the space of such modular forms by $\mathcal{M}_k(H, \chi)$ and the space of such cusp forms $\mathcal{S}_k(H, \chi)$.

Remark It is clear that a linear sum of modular (resp. automorphic) forms of the same level, weight and character is again a modular (resp. automorphic) form of the same level, weight and character.

Example 3.1.6
Consider the case where we have integer k, and let $H = \Gamma \in G$. Then we can consider the space of such modular forms with trivial Dirichlet character χ_0. Explicitly we are considering the space of all forms $f(\tau)$ such that

1. *$f(\gamma\tau) = \chi_0(d)(c\tau + d)^k f(\tau)$ for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$*

2. *$f(\tau)$ is holomorphic on $\hat{\mathbb{H}}$ and at all other cusps*

Since χ_0 takes the value 1 on all integers, and $d \in \mathbb{Z}$ by definition of Γ we have that $\mathcal{M}_k(\Gamma, \chi_0) = \mathcal{M}_k(\Gamma)$. This shows that our new definition allows us to consider a wider class of modular forms.

3.1.4 Twists of modular forms

In general, we are interested in relations between modular forms. Taking twists of modular forms allows us to move from spaces of one character to another. In general, twists of modular forms are used extensively to prove converse theorems, and in various other settings including Rankin-Selberg convolution [11, page 133]. More specifically we have the following definition and theorem.

Definition Let $f(\tau) = \sum_{n=0}^{\infty} a_n q^n \in \mathcal{M}_k(q, \chi)$, with χ a Dirichlet character. Let ψ also be a Dirichlet character. Then we define the **twist of** $f(\tau)$ **by** ψ as

$$f_\psi(\tau) = \sum_{n=0}^{\infty} \psi(n) a_n q^n.$$

Theorem 3.1.7
Let χ be a Dirichlet character of conductor $q^*|q$ and $f(\tau) = \sum_{n=0}^{\infty} a_n q^n \in \mathcal{M}_k(q, \chi)$. Let ψ be a primitive Dirichlet character of conductor r.
Then $f_\psi(\tau) \in \mathcal{M}_k(N, \chi\psi^2)$ where $N = lcm(q, q^* r, r^2)$. Furthermore, if $f(\tau)$ is a cusp form then so is $f_\psi(\tau)$.

Proof See proof at [5, page 124, Theorem 7.4]. It is based upon expressing $f_\psi(\tau)$ as a twisted sum of f and using a specific identity of the Gauss sum to prove the modularity property for f_ψ. For the fact that f_ψ is a cusp form whenever f is, one needs only check that f_ψ satisfies the criterion of being a cusp form at [5, page 70], which is clear from the given relation between f and f_ψ. ■

Example 3.1.8
Consider the modular form $E_k(\tau) \in \mathcal{M}_k(\Gamma)$. We know that it has a q-expansion, (see example 3.1.2), and we can write this, as ever, as $\sum_{n=0}^{\infty} a_n q^n$. Then we have that $\chi = \chi_0$ here, and so $q^* = q = 1$. Now consider a twist of the Eisenstein series by a primitive Dirichlet character ψ of conductor r. Then $E_{\psi,k}(\tau) \in \mathcal{M}_k(r^2, \psi^2)$. If ψ is a Dirichlet character of order 2 then clearly $E_{\psi,k}(\tau) \in \mathcal{M}_k(r^2)$.

Remark To see that this theorem of twists leaves the space invariant under the trivial action, simply notice that ψ_0 (the trivial character) has conductor 1, and so $f_{\psi_0}(\tau) \in \mathcal{M}_k(q, \chi)$ for any $f(\tau) \in \mathcal{M}_k(q, \chi)$. (Indeed, by the definition, the twisted modular form under ψ_0 is the original untwisted form).

3.1.5 Hecke operators and Hecke eigenforms on $\Gamma_0(N)$

Here we describe the Hecke operator T_n, and state without proof the main results. An in depth discussion of Hecke operators is given in [12, chapter 4].
While it is often very useful to think of Hecke operators as acting on the lattice that we sum over, here we need only the result that follows and so will pass over this.

Definition Let $k \geq 1$, $q \geq 1$ and χ a Dirichlet character of modulus q such that $\chi(-1) = (-1)^k$. Then we define the **Hecke operator** T_n acting on a function $f(\tau)$ by

$$T_n f(\tau) = \frac{1}{n} \sum_{ad=n} \chi(a) a^k \sum_{0 \leq b < d} f(\frac{a\tau + b}{d}).$$

Hecke operators act on modular forms and preserve the space of such forms. Furthermore, they also preserve the space of cusp forms. More precisely, we have [11, page 370, Proposition 14.8]):

Proposition 3.1.9
For $n \geq 1$ we have that

$$T_n : \mathcal{M}_k(q, \chi) \to \mathcal{M}_k(q, \chi)$$
$$T_n : \mathcal{S}_k(q, \chi) \to \mathcal{S}_k(q, \chi).$$

Hecke eigenfunctions provide a basis for cusp forms of any given level. That is, given a space of cusp forms $\mathcal{S}_k(q, \chi)$ there exists an orthonormal basis consisting of eigenfunctions of all Hecke operators T_n with $\gcd(n, q) = 1$ [11, page 372, Proposition 14.11]. We aim to obtain an Euler product representation for the L-function of a Hecke eigenform f. Hecke eigenforms are non-zero modular forms in $\mathcal{M}_k(q, \chi)$ such that

$$T_n f = \lambda_n f,$$

for some $\lambda_n \in \mathbb{C}$ for every Hecke operator T_n. In other words, they are simultaneous eigenfunctions of every Hecke operator.
There is some subtlety when finding such a product formula - the essence of which is that naively we consider too many modular forms at any given level q. Instead, we should split the modular forms into the so-called newforms (primitive forms) and oldforms. The oldforms are the modular forms which do not originate at a given level q, but rather are carried over from lower levels. The newforms are, naively, the modular forms that truly belong at the level q.
More precisely [11, page 373] we let $\mathcal{S}_k^d(q, \chi)$ be the subspace of $\mathcal{S}_k(q, \chi)$ that is spanned by cusp forms of the form

$$f(d\tau) = \sum_n a_n e^{2\pi i n d \tau},$$

where $f \in \mathcal{S}_k(q', \chi')$ with $dq'|q$ and $q' < q$. These are the 'oldforms', since they stem from lower levels.
It can be shown that letting $\mathcal{S}_k^*(q, \chi)$ be the orthogonal complement of $\mathcal{S}_k^d(q, \chi)$ with

respect to a given inner product of modular forms (the Petersson inner product) gives us the decomposition of all cusp forms into oldforms and newforms as:

$$\mathcal{S}_k(q,\chi) = \mathcal{S}_k^d(q,\chi) \oplus \mathcal{S}_k^*(q,\chi).$$

Example 3.1.10
Let χ be a primitive Dirichlet character. Then there are no $q'|q$ with $q' < q$, so we here have that $\mathcal{S}_k^*(q,\chi) = \mathcal{S}_k(q,\chi)$

The Hecke operators preserve these subspaces of $\mathcal{S}_k(q,\chi)$ [5, page 108, Proposition 6.22]. We also have the following proposition [11, page 373, Proposition 14.13].

Proposition 3.1.11
Let f be a primitive cusp newform with the expansion $f = \sum_{n=1}^{\infty} a_n q^n$. Then f is a Hecke eigenform, i.e.

$$T_n f = \lambda_n f,$$

and we have that $a_n = \lambda_n a_1$ for all $n \geq 1$.

The above proposition shows immediately that $a_1 \neq 0$ so that we can normalise any Hecke eigenform to have $a_1 = 1$. Such a form is called a normalised primitive Hecke eigenform.

Definition For a convergent expansion of the form $f = \sum_{n=0}^{\infty} a_n k^n$, $a_n \in \mathbb{C}$ for some k we define the **L-series of** f as

$$L(f,s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

for $\Re(s) > c$, some $c > 0$ such that the series converges.

Definition We say that an L-series is an L-function if $L(f,s)$ admits an analytic continuation to all $s \in \mathbb{C}$.

We are particularly interested in the L-series of cusp forms and have the following fundamental theorem.

Theorem 3.1.12
Let f be a normalised primitive Hecke eigenform. Then the Hecke L-function of f has the Euler product formula

$$L(f,s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_p \frac{1}{1 - \lambda_p p^{-s} + \chi(p) p^{k-1-2s}}, \tag{3.1.13}$$

where the product is taken over all primes $p \in \mathbb{Z}$.

Remark Hecke also managed to prove that such an L-function has an analytic continuation to $\mathbb{C}$ and that it satisfies a functional equation. We will return to these later in this report.

Proof The proof is rather involved. See [11, page 375, Theorem 14.7]. ■

The converse of this is also true. That is, if $L(f, s)$ has such an Euler product representative then it is a newform [5, page 118, section 6.8].
While this theorem is astonishing and perhaps one of the most beautiful formulae in modern mathematics, in practice is not the most useful description for us. This theorem only tells us that when we have a Hecke eigenform that it arises with such a product formula and vice-versa. Since we are interested in constructing general modular forms, we need a theorem that says when we have a function which satisfies certain properties, then it is a modular form. Thankfully, Hecke was able to prove this in his celebrated converse theorem for $q = 1$ (the case of the full modular group Γ). Weil was then able to generalise this to the $q > 1$ case.

Example 3.1.14
Consider the Eisenstein series of weight $k > 3$ we have introduced before, $E_k(\tau)$. We renormalise this to

$$G_k(\tau) = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \in \mathcal{M}_k(\Gamma, \chi_0).$$

Then we have that $G_k(\tau)$ has the associated L-function

$$L(G_k, s) = \sum_{n=1}^{\infty} \frac{\sigma_{k-1}(n)}{n^s} = \prod_p \frac{1}{1 - \sigma_{k-1}(p)p^{-s} + p^{k-1-2s}}.$$

Now, we know precisely what the divisor sum of a prime is (since the only numbers that divide a prime p are 1 and p we have that $\sigma_{k-1}(p) = 1 + p^{k-1}$), and so the L-function splits as

$$\begin{aligned} L(G_k, s) &= \prod_p \frac{1}{(1 - p^{k-1-s})(1 - p^{-s})} \\ &= \zeta(s - k + 1)\zeta(s), \end{aligned}$$

with $\zeta(s)$ the Riemann zeta function.

Now we state without proof two converse theorems. Firstly, we consider the case where $q = 1$ [5, page 126, Theorem 7.7].

Theorem 3.1.15
Let $q = 1$, and $k > 0 \in \mathbb{Z}$ be even. Let $f(\tau) = \sum_{n=0}^{\infty} a_n e^{2\pi i n \tau}$, with $a_n \ll n^{\alpha}$ for some positive $\alpha \in \mathbb{R}$ for all $n \geq 1$.
Then $f(\tau) \in \mathcal{M}_k(\Gamma)$ if and only if

$$\Lambda(s) = \frac{1}{(2\pi)^s} \Gamma(s) L(f, s)$$

has an analytic continuation to all $\mathbb{C}$, $\Lambda(s) + \frac{a_0}{s} + \frac{i^k a_0}{k-s}$ is entire and bounded in vertical strips, and the functional equation

$$\Lambda(s) = i^k \Lambda(k-s)$$

holds.

The generalisation to $q > 1$ is rather more technical, as now we involve non-trivial characters [5, page 127, Theorem 7.8].

Theorem 3.1.16
Let $k > 0$ be even, and χ a character of modulus q. Let

$$f(\tau) = \sum_{n=0}^{\infty} a_n q^n$$
$$g(\tau) = \sum_{n=0}^{\infty} b_n q^n,$$

with a bound on the coefficients given by $a_n, b_n \ll n^\alpha$ for some positive constant α. Suppose

$$\Lambda(f,s) = (\frac{\sqrt{q}}{2\pi})^s \Gamma(s) L(f,s)$$
$$\Lambda(g,s) = (\frac{\sqrt{q}}{2\pi})^s \Gamma(s) L(g,s)$$

satisfy the following properties:

1. *$\Lambda(f,s) + \frac{a_0}{s} + \frac{b_0 i^k}{k-s}$ is entire and bounded on vertical strips.*
2. *$\Lambda(g,s) + \frac{b_0}{s} + \frac{a_0 i^k}{k-s}$ is entire and bounded on vertical strips.*
3. *$\Lambda(f,s) = i^k \Lambda(g, k-s)$.*

Let $\mathcal{R}$ be a set of primes in $\mathbb{Z}$ coprime to q which meets every primitive residue class. That is, for all $c > 0$ and $(a,c) = 1$ there exists $r \in \mathcal{R}$ such that $r \equiv a \mod c$. Suppose, for any primitive character ψ of conductor $r \in \mathcal{R}$, the functions given by

$$\Lambda(f,s,\psi) = (\frac{\sqrt{N}}{2\pi})^s \Gamma(s) L(f,s,\psi)$$
$$\Lambda(g,s,\psi) = (\frac{\sqrt{N}}{2\pi})^s \Gamma(s) L(g,s,\psi)$$

with $N = qr^2$ are entire and bounded in vertical strips. Furthermore, suppose that they satisfy the functional equation

$$\Lambda(f,s,\psi) = i^k \omega(\psi) \Lambda(g, k-s, \psi^{-1}),$$

with $\omega(\psi) = \chi(r)\psi(q)\tau(\psi)^2 r^{-1}$. Then $f \in \mathcal{M}_k(q,\chi)$, $g \in \mathcal{M}_k(q,\chi^{-1})$ and $g(\tau) = q^{\frac{k}{2}}(q\tau)^{-k} f(\frac{-1}{q\tau})$. Moreover, f and g are cusp forms if $L(f,s)$ or $L(g,s)$ converges absolutely on some line $\Re(s) = \sigma$ for $0 < \sigma < k$.

This gives us a way to construct modular forms from functions that we can verify have the above properties.

Example 3.1.17
We have seen that $\zeta(s)\zeta(s-k+1)$ arises as the L-function of a modular form. We could show this the other way around using the converse theorems. That is, firstly define $L(f,s) = \zeta(s)\zeta(s-k+1)$ for $k \in 2\mathbb{N}$, and show that the RHS satisfies the necessary conditions of the theorem (for this, see the section on the Riemann zeta function). Then we can conclude that f is a modular form of weight k for Γ (in fact, we know it is the Eisenstein series).

In more generality, it can be shown that for any two primitive Dirichlet characters χ_1 and χ_2 of modulus q_1 and q_2 respectively that

$$L(f,s) = L(\chi_1, s)L(\chi_2, s-k+1)$$

arises from a modular form $f \in \mathcal{M}_k(\Gamma_1(q_1q_2))$ [13, page 177, Theorem 4.7.1].

We end this chapter by remarking that whilst these converse theorems are incredibly powerful, they involve rather a lot of work to show that given functions are modular, since one must check all of the necessary conditions. A different way to construct modular and automorphic forms is through theta functions. We build to these by firstly considering integral quadratic forms.

4. Integral quadratic forms

4.1 General integral quadratic forms

Definition An **integral quadratic form** in m variables is a quadratic form given by

$$q(x) = \sum_{i=1}^{m} a_i x_i^2,$$

with $a_i \in \mathbb{Z}$ and $x = (x_1, ..., x_m) \in \mathbb{Z}^m$.

Integral binary quadratic forms (in two dimensions) will be central in our discussion of theta functions associated to quadratic fields later in the report. However, for our discussion on theta functions we will work with the more general integral quadratic forms $q(x)$. These forms can easily split into two categories; positive definite quadratic forms and indefinite forms. We will deal with both cases below, but first detail some of the general basic properties. One of the most important features to note is that to each $q(x)$ we can associate an $m \times m$ matrix A such that $A[x] := x^T Ax = 2q(x)$. Though it is not obvious from first sight, we can use properties of matrices to our advantage, especially in the indefinite case where we are interested in the diagonal form of A.

Example 4.1.1
Clearly for $m = 2$ we have

$$A = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix},$$

which ensures that A has even diagonal entries (for mainly historical reasons). Then we can consider our binary forms to be functions of a vector $x = (x, y)^T$; explicitly we let $Q(x) = q(x, y) = \frac{1}{2}x^T Ax := \frac{1}{2}A[x]$.

Definition Let q and $\tilde{q}$ be two integral quadratic forms. Then we say that q is equivalent to $\tilde{q}$ (and write $q \sim \tilde{q}$) if there exists a square integral substitution matrix S such that

$$q^S := q(Sx) = \tilde{q}(x),$$

and $\det S = \pm 1$.
Notice that the equivalence $\tilde{q} \sim q$ is given by $\tilde{q}^{S^{-1}} = q$, so that this defines a proper equivalence relation.
If $\det S = 1$ we say that q is properly equivalent to $\tilde{q}$.

Example 4.1.2
Consider $q_0(x, y) = x^2 + y^2$. This is the most basic example of a positive definite binary quadratic form. In other words, $q_0(x, y) > 0$ for all non-zero pairs $(x, y) \in \mathbb{Z}^2$.

We can see clearly that $A = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)$. *Take* $S = \left(\begin{smallmatrix} 2 & 1 \\ 3 & 2 \end{smallmatrix}\right)$ *as a substitution matrix (noting that* $\det S = 1$*). Then*

$$\begin{aligned} q_0^S(x,y) &= q_0(2x+y, 3x+2y) \\ &= (2x+y)^2 + (3x+2y)^2 \\ &= 13x^2 + 16xy + 5y^2 \end{aligned}$$

gives that $q_0 \sim q$ *and in fact* q_0 *is properly equivalent to* q.

Since quadratic forms give information on the representation numbers of a given integer n we are led to the following:

Definition We define $r_q(n) = |\{x \in \mathbb{Z}^m | q(x) = n\}|$, the **representation number of** n, given an integral quadratic form q.

Remark For positive definite quadratic forms, these representation numbers are finite. However, in general, they may be infinite and this will cause us some problems when we try to define a theta function associated to indefinite quadratic forms.

So $r_q(n)$ tells us how many integer solutions there are to the equation $q(x) = n$, defined by a quadratic form. Clearly, from a number theoretic perspective, this is a very important and interesting object. For example, if $q(x) = x^2$, then $r_q(n)$ simply gives the number of ways to represent n as a square. We will make use of the following generalisation of [14, page 3, Proposition]:

Proposition 4.1.3
Let $q \sim \tilde{q}$. *Then* $r_q(n) = r_{\tilde{q}}(n)$.

Proof Let $\tilde{q} = q^S$ and S the substitution matrix. Then each solution vector x to $\tilde{q}x = n$ corresponds to a solution $q(Sx) = n$. Every solution to $q(\tilde{x}) = n$ can be found by setting

$$x = S^{-1}\tilde{x}.$$

Moreover, two solutions x and x' give the same vector $\tilde{x}$ if and only if

$$Sx = \tilde{x} = Sx',$$

which clearly gives that $x = x'$, by pre-multiplication of S^{-1}. This gives us that $r_{\tilde{q}}(n) = r_q(n)$ as we desired. ■

Definition We denote the determinant of an integral quadratic form by $d(q) := \det(A)$.

Definition We define the discriminant of a quadratic form $q(x)$ to be

$$D(q) = \begin{cases} (-1)^{\frac{m}{2}} 2^{m-1} d(q) & \text{if } m \text{ is even} \\ (-1)^{\frac{m-1}{2}} 2^{m-2} d(q) & \text{if } m \text{ is odd.} \end{cases}$$

Example 4.1.4
Let $q(x,y) = ax^2+bxy+cy^2$ be an integral binary quadratic form. Then it has associated matrix $A = \left(\begin{smallmatrix} 2a & b \\ b & 2c \end{smallmatrix}\right)$. We clearly have that $d(q) = 4ac - b^2$. Since $m = 2$, according to our definition, the discriminant of $q(x,y)$ is $D(q) = b^2 - 4ac$.

The discriminant of a form encodes information about whether the form is positive definite or not, but note that this must be consistent between equivalent forms. So, we need that $d(\tilde{q}) = d(q)$ for all $\tilde{q} \sim q$. Indeed, this is the case [14, page 5, generalisation of remark 3]):

Lemma 4.1.5
Let $\tilde{q} \sim q$ be two quadratic forms that are equivalent, with a substitution matrix S. Then $d(\tilde{q}) = d(q)$.

Proof We have that

$$\begin{aligned} d(\tilde{q}) &= \det(SAS) \\ &= \det(S)^2 \det(A) \\ &= \det(A) \\ &= d(q). \end{aligned}$$

■

The above results show us that for a given discriminant D we need only look at quadratic forms that are non-equivalent to obtain information on the representation numbers $r(n)$. However, it is a rather subtle problem to obtain the number of non-equivalent quadratic forms there are for a given discriminant. We do know, at least, that the number is finite [15, page 128, Theorem 1.1].

Theorem 4.1.6
Let $d \neq 0$ be an integer, and $Q_i(x)$ integral quadratic forms in n variables. Then there are only finitely many equivalence classes of integral quadratic forms $Q_i(x)$ with $d(Q_i) = d$.

When we consider the construction of integral binary quadratic forms from ideals in an extension $K/\mathbb{Q}$ it turns out that the non-equivalent forms arise from representatives from non-equivalent classes in the ideal class group $Cl(K)$.
Whilst we have seen most of these results for quadratic forms in m variables, we are particularly interested in the binary integral quadratic form case. We begin by outlining the positive definite case for integral binary quadratic forms below, before briefly considering the indefinite case.

4.2 The positive definite case

Definition Let $q(x)$ be an integral quadratic form. Then we say that q is **positive-definite** if $q(x) > 0$ for all non-zero vectors $x \in \mathbb{Z}^m$.

Example 4.2.1
As discussed before, $q_0(x, y) = x^2 + y^2$ *is positive-definite in* $\mathbb{Z}^2$*, since it is positive for all pairs* $(x, y) \neq (0, 0)$*.*
Another example is $q(x, y) = x^2 + 2xy + 2y^2$*. One can verify this by completing the square to give* $q(x, y) = (x + y)^2 + y^2 > 0$ *for non-zero pairs* $(x, y) \in \mathbb{Z}^2$*.*

It is not immediately clear whether a given arbitrary form is positive definite or not, and so we make use of [2, page 339, Theorem 6.23]:

Proposition 4.2.2
Let $q(x, y) = ax^2 + bxy + cy^2$ *be an integral binary quadratic form. Then* q *is positive definite if and only if* $D(q) < 0$ *and* $a, c > 0$*.*

Proof If $D(q) < 0$ then we must have that $0 \leq b^2 < 4ac$, and so if $a > 0$ then $c > 0$ and vice-versa.
If q is positive-definite then $q(1, 0) = a > 0$ and similarly $q(0, 1) = c > 0$.
Notice that

$$\begin{aligned} q(x, y) &= ax^2 + bxy + cy^2 \\ &= \frac{1}{4a}((2ax + by)^2 + (4ac - b^2)y^2 \\ &= \frac{1}{4a}((2ax + by)^2 - d(q)y^2). \end{aligned}$$

So, $q(-b, 2a) = -d(q)a > 0$ and we conclude that $D(q) < 0$. For the converse, simply observe that if $D(q) < 0$ and $a > 0$ then q is positive from the above. ■

It turns out that the representation numbers $r_q(n)$ is finite for positive definite binary integral quadratic forms [2, page 340, Theorem 6.24], and this will be very useful when we form theta functions from such forms in the next chapter. We still need to know which quadratic forms we should be considering as representatives of an equivalence class. Hence we come to the theory of reduced forms.

Definition We say that an integral binary quadratic form is **reduced** if $0 \leq b \leq a \leq c$.

It is much easier to work with reduced forms, since we have the following proposition [14, page, Proposition 8].

Proposition 4.2.3
Two reduced integral binary quadratic forms are equivalent if and only if they are equal.

This proposition shows us that, given two reduced forms we can immediately tell if they are equivalent or not (and hence, in the field theory picture, whether or not they stem from ideals in the same class in the $Cl(K)$. We also have that the number of properly reduced forms of a given discriminant is finite, and that every positive definite form is equivalent to a reduced one (see proposition 10 of [14]). Combining this knowledge with theorem 4.1.6 shows us that we need only look for one form for each equivalence class, and that there are finitely many such forms.

Definition A positive definite integral binary quadratic form is said to be **properly reduced** if $-a < b \leq a < c$ or $0 \leq b \leq a = c$.

Definition A **primitive** positive definite integral binary quadratic form is a properly reduced form such that $\gcd(a, b, c) = 1$.

So we look for forms of negative discriminant $D(q)$. Note that this discriminant will resurface when we look at the theta function arising from the field theory picture. In fact the number of primitive properly reduced quadratic forms for a quadratic field is equal to the class number of the given field. This is because non-equivalent ideals in the field give rise to non-equivalent quadratic forms.

Example 4.2.4
Consider the case when $D = -23 \equiv 1 \mod 4$. Here there are 3 different equivalence classes (note that the class number of $K = \mathbb{Q}[\sqrt{-23}]$ is 3). The representative forms are:

$$\begin{aligned} q_1(x,y) &= x^2 + xy + 6y^2 \\ q_2(x,y) &= 2x^2 + xy + 3y^2 \\ q_3(x,y) &= 2x^2 - xy + 3y^2. \end{aligned}$$

Furthermore, it is clear that q_2 and q_3 represent the same integers (we will make use of this fact when forming a theta function later in this report).

Example 4.2.5
Now consider the case when $D = -20$. These are associated to the imaginary quadratic field $K = \mathbb{Q}[\sqrt{-5}]$ of discriminant -20. The class number of K is 2. Here, there are two non-equivalent primitive properly reduced integral binary quadratic forms. They are given by

$$\begin{aligned} q_1(x,y) &= x^2 + 5y^2 \\ q_2(x,y) &= 2x^2 + 2xy + 3y^2 \end{aligned}$$

These clearly represent different integers.

We will work a lot with positive definite forms, but later on in the report we will touch upon the indefinite case for binary quadratic forms, and so we give a very brief introduction below.

4.3 Indefinite case

We let $q(x)$ be an indefinite binary quadratic form. This means that its associated matrix A has signature $[1, 1]$ (i.e. 1 positive and 1 negative eigenvalue). These are still easy to define, but a lot harder to work with when constructing theta functions, since they now admit infinite representation numbers.

Example 4.3.1
The prototypical example is given by $q(x,y) = x^2 - y^2$. *Clearly this is indefinite as it admits both positive and negative values.*

We still have the general results about equivalences of the indefinite forms, and the finite number of equivalence classes. Again, this re-surfaces when one looks at such forms arising from real quadratic fields - $K = \mathbb{Q}[\sqrt{D}]$ with $D > 0$.

Example 4.3.2
Consider integral binary quadratic forms of discriminant 28. *In the field theory picture these arise from the field* $K = \mathbb{Q}[\sqrt{7}]$, *since* $7 \equiv 3 \mod 4$. *Notice that the* K *has class number* 1, *but narrow class number* 2 *and so we may expect there to be two inequivalent forms. They are given by*

$$\begin{aligned} q_1(x,y) &= x^2 + 6xy + 2y^2 \\ q_2(x,y) &= x^2 + 8y^2 + 9y^2, \end{aligned}$$

each with discriminant 28. *Since the discriminant is positive, we have by proposition 4.2.2 that these forms are not positive definite. To see that they are indefinite consider the diagonal form of each associated matrix. The diagonal matrices are given by*

$$D_1 = \begin{pmatrix} 8\sqrt{2}+10 & 0 \\ 0 & -8\sqrt{2}+10 \end{pmatrix} \ , \ D_2 = \begin{pmatrix} \sqrt{37}+3 & 0 \\ 0 & -\sqrt{37}+3 \end{pmatrix}.$$

Since each has one positive and one negative eigenvalue we see that these quadratic forms are indefinite.

Later in this report we will see that indefinite binary quadratic forms lead us to a definition of an indefinite theta function, and this in turn takes us to Maass forms of weight 0 on Γ. Next, we apply our knowledge of positive definite quadratic forms to build theta functions.

5. Theta functions of positive definite integral quadratic forms

5.1 Theta functions and congruent theta functions

Here we detail some properties of theta functions associated to positive definite binary quadratic forms and show that they are intricately linked to automorphic forms. We will not move into the general indefinite theta function case as it is far more difficult - and indeed there are many open problems. However, we deal with the indefinite binary quadratic case later in this report. A brief overview of the general positive definite case is given in chapter 1 of [12], while the general indefinite case is considered from a field theory perspective in [16].

Definition We say that $P(x)$, of degree m, is a **spherical (or harmonic) polynomial** if it is homogeneous and satisfies the harmonic condition

$$\Delta_A P(x) = 0,$$

where A is a positive definite matrix (here associated to a quadratic form), and

$$\Delta_A = \sum_{i,j} a_{ij}^* \frac{\partial^2}{\partial x_i \partial x_j} \tag{5.1.1}$$

with $A^{-1} = (a_{ij}^*)$. Δ_A is the Laplace operator with respect to A.

Definition Let $Q(x) = \frac{1}{2}x^T Ax$ be an integral positive definite quadratic form in m variables and L be a lattice in $\mathbb{R}^m$ (often we use $L = \mathbb{Z}^m$). Let $P(x)$ be a spherical polynomial with respect to A, of degree ν. Then we define the theta function associated to Q as:

$$\begin{aligned}\theta_Q(\tau, L, P) = \sum_{x\in L} P(x) e^{2\pi i Q(x)\tau} &= \sum_{x\in L} P(x) e^{\pi i x^T Ax\tau} \\ &= \sum_{x\in L} P(x) e^{\pi i A[x]\tau},\end{aligned} \tag{5.1.2}$$

for $\tau = u + iv \in \mathbb{H}$.

This sum definition is convergent due to our quadratic form being positive definite, giving exponential suppression at the cusp at ∞, while $P(x)$ grows as a polynomial. When dealing with the indefinite case we have to deal with the non-convergence problem. We also often suppress the dependence on the lattice and work with the matrix A, i.e. we let $\theta_Q(\tau, L, P) = \theta(\tau, A, P)$. As mentioned before, we are particularly interested in the number theoretics arising from such theta functions and we can let $P(x) = 1$, then re-arrange equation 5.1.2 by the length of the vectors x to get:

$$\theta_Q(\tau, L) = \sum_{n=0}^{\infty} r_Q(n) q^n, \tag{5.1.3}$$

where $q = e^{2\pi i\tau}$ and $r_Q(n)$ are the representation numbers of n given Q. Notice that equivalent quadratic forms will have the same theta function. Hence theta functions carry important information pertaining to number theory.

Returning to the general theta function we see directly from the definition that there is a trivial invariance of theta under the transformation $\tau \mapsto \tau + 1$:

$$\begin{aligned}
\theta(\tau + 1, A, P) &= \sum_{x \in L} P(x) e^{2\pi i Q(x)(\tau+1)} \\
&= \sum_{x \in L} P(x) e^{2\pi i Q(x)\tau} e^{2\pi i Q(x)} \\
&= \sum_{x \in L} P(x) e^{2\pi i Q(x)\tau} = \theta(\tau, A, P),
\end{aligned}$$

where we have used that $Q(x)$ is integer-valued.

Definition We work with the congruent theta functions

$$\theta(\tau; h) = \sum_{\substack{x \in \mathbb{Z}^m \\ x \equiv h \mod N}} P(x) e^{\pi i \frac{A[x]}{N^2}\tau},$$

where $h \in \mathbb{Z}^m$, and $N \in \mathbb{N}$ such that NA^{-1} is integral.

We can relate the congruent theta function back to our definition 5.1.2 by observing

$$\begin{aligned}
\theta(\tau; 0) &= \sum_{\substack{x \in \mathbb{Z}^m \\ N|x}} P(x) e^{\pi i \frac{A[x]}{N^2}\tau} \\
&= \sum_{Ny \in \mathbb{Z}^m} P(Ny) e^{\pi i \frac{A[Ny]}{N^2}\tau} \\
&= \sum_{Ny \in \mathbb{Z}^m} N^\nu P(y) e^{\pi i \frac{A[Ny]}{N^2}\tau} \\
&= \sum_{Ny \in \mathbb{Z}^m} N^\nu P(y) e^{\pi i \frac{(Ny)^T A(Ny)}{N^2}\tau} \\
&= N^\nu \theta(\tau, A, P),
\end{aligned}$$

if we take our theta function's lattice L to be $\mathbb{Z}^m$.

Example 5.1.4
Take $P = 1$, then $\nu = 0$ and we can relate our congruent theta function directly to the theta function by taking $h = 0$. Taking our typical example of $q_0 = x^2 + y^2$ we can see that $N = 4$ and so

$$\begin{aligned}\theta(\tau; h) &= \sum_{\substack{(x,y)\in\mathbb{Z}^2 \\ (x,y)\equiv h \mod 4}} e^{2\pi i \frac{x^2+y^2}{4^2}\tau} \\ &= \sum_{\substack{(x,y)\in\mathbb{Z}^2 \\ (x,y)\equiv h \mod 4}} e^{\pi i \frac{x^2+y^2}{8}\tau}.\end{aligned}$$

Taking $h = 0$ gives us that $4|x$ and $4|y$ so that $16|x^2, y^2$. Then we see that

$$\begin{aligned}\theta(\tau; 0) &= \sum_{\substack{(x,y)\in\mathbb{Z}^2 \\ (x,y)\equiv 0 \mod 4}} e^{2\pi i(x^2+y^2)\tau} \\ &= \sum_{(x,y)\in\mathbb{Z}^2} e^{2\pi i(x^2+y^2)\tau} \\ &= \vartheta^2,\end{aligned}$$

where $\vartheta = \sum\limits_{n\in\mathbb{Z}} e^{2\pi i n^2\tau}$ is Jacobi's theta function.

By using the congruent theta functions one is able to show that both $\theta(\tau; h)$ and $\theta(\tau, A, P)$ are automorphic forms. This requires a fair amount of work - firstly proving the inversion formula for congruent theta functions via Poisson summation, and then working hard to get the automorphy factor of both the congruent theta and theta functions.
We now state the general results that we were aiming to obtain - viewing theta functions as automorphic forms. We will give basic outlines of the proofs, since the details are rather technical and use ideas that we have not discussed. We begin with the theta function transformation formula [5, page 167, Proposition 10.1] and[5, page 170, Proposition 10.4].

Proposition 5.1.5
Let A be a symmetric, positive definite matrix and P a spherical function of A with degree ν. Then for $\tau \in \mathbb{H}$ we have

$$\theta(\tau, A, P) = i^{-\nu}|A|^{-\frac{1}{2}}(\frac{i}{\tau})^{\nu+\frac{m}{2}}\theta(-\frac{1}{\tau}, A^{-1}, P^*),$$

where $P^*(x) := P(A^{-1}x)$ is a spherical function of A^{-1}.

Proof See proof at [5, page 167, Proposition 10.1]. It is based upon a higher dimensional Poisson summation and Fourier transform formula for a generalised result, taking derivatives with respect to a suitable operator defined in terms of the matrix A and certain vectors, and then specialising the resulting equation to obtain the above proposition. ∎

Next we look for the transformation formula of the congruent theta function. To do this we introduce the bilinear form of A.

Definition We define $B(x, y) = x^T Ay$ to be the **bilinear form** associated to A and let

$$\psi(x, y) = e^{2\pi i N^{-2} B(x,y)}.$$

Every character on $H := \{h \mod N | Ah \equiv 0 \mod N\}$ is given by $\psi(h, l)$ for some $l \in H$. Now we can state the transformation formula for the congruent theta function.

Proposition 5.1.6
For $h \in H$ we have

$$\theta(-\frac{1}{\tau}; h) = i^{-\nu} |A|^{-\frac{1}{2}} (-i\tau)^k \sum_{l \in H} \psi(h, l)\theta(\tau; l)$$

Proof See proof at [5, page 170, Proposition 10.4]. It is again based on a higher dimensional Poisson summation formula and the proof of proposition 5.1.5. ∎

Since we have the transformation formula for both the congruent theta function and theta function we are ready to state the main result of this chapter - these functions are automorphic forms. We start with the congruent theta function since it is the slightly more general case [5, page 174, Corollary 10.7].

Theorem 5.1.7
Let $h \in H$. Then the congruent theta function $\theta(\tau; h)$ is an automorphic form for the principal congruent subgroup $\Gamma(4N)$ of Γ. It has weight $k = \nu + \frac{m}{2}$ and multiplier $(\frac{2c}{d})^m$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. If $\nu > 0$ then it is a cusp form.

Proof One begins by showing that

$$\theta(\gamma\tau; h) = e^{2\pi i (abA[h])/2N^2} \vartheta(\gamma)(c\tau + d)^k \theta(\tau; ah),$$

for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ with $d \equiv 1 \mod 2$, and some multiplier system $\vartheta(\gamma)$. Specialising this result to the case of $\gamma \in \Gamma(4N)$ yields the result immediately, where $\vartheta(\gamma)$ becomes $\left(\frac{2c}{d}\right)^r$.
To show that $\theta(\tau; h)$ is holomorphic and vanishes at cusps for $\nu > 0$ one considers the cusps of $\Gamma_0(2N)$. In particular note that $\Gamma(4N) \subset \Gamma_0(2N)$. Cusps of $\Gamma_0(2N)$ are equivalent to the cusp at ∞ of Γ. Then one argues that the matrices $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} 0 & -1 \\ 0 & 1 \end{smallmatrix}\right)$ generate Γ and that the space of congruent theta functions is preserved under the action of T and S. Hence one needs only show that $\vartheta(\tau; h)$ is holomorphic at the cusp at ∞. This is clear by its definition, and since $\nu > 0$ guarantees exponential suppression at ∞ we have a cusp form if $\nu > 0$. ∎

Since the theta function is a special case of the congruent theta function the above theorem specialises immediately to gives us the following - now on $\Gamma_0(2N)$ since we have the relation $\theta(\tau;0) = N^\nu \theta(\tau, A, P)$.

Theorem 5.1.8
Let A be an integral matrix with even diagonal, and N an integer such that NA^{-1} is also integral. Then $\theta(\tau, A, P)$ is an automorphic form on $\Gamma_0(2N)$ of weight $k = \nu + \frac{m}{2}$ associated with a certain multiplier system. Moreover, if $\nu > 0$ then $\theta(\tau, A, P)$ is a cusp form.

Furthermore, we can show that certain theta functions are indeed modular forms, not just automorphic forms. When $m = \operatorname{rank}(A)$ is even the multiplier system reduces to $\left(\frac{D}{d}\right)$ with $D = (-1)^{m/2} \det A$. In this case, we also have that k is an integer, since $m \equiv 0 \mod 2$. This is one of the key results we were looking to obtain [5, page 175, Theorem 10.9].

Theorem 5.1.9
Let A be a symmetric, integral and positive definite matrix of even rank m. Let $N \in \mathbb{N}$ such that NA^{-1} is integral. Suppose A and NA^{-1} have even diagonal entries. Let P be a spherical function with respect to A of degree ν. Then $\theta(\tau, A, P)$ is a modular form for $\Gamma_0(N)$ of weight $k = \nu + \frac{m}{2}$ and multiplier $\vartheta(\tau) = \chi_D(\tau) = (\frac{D}{d})$ with $D = (-1)^{\frac{m}{2}} \det A$. If $\nu > 0$ then $\theta(\tau, A, P)$ is a cusp form.

So we have that theta functions arising from positive definite matrices are modular forms for a subgroup of Γ, while those arising from slightly more general matrices are automorphic forms instead.

Example 5.1.10
Again, focussing on our most basic example, using $q_0(x, y) = x^2 + y^2$, theta function $L = \mathbb{Z}^2$, and $P = 1$ we obtain:

$$\theta(\tau, A, P) = \sum_{x,y \in \mathbb{Z}} e^{2\pi i (x^2+y^2)\tau} = \vartheta^2,$$

where ϑ is Jacobi's theta function $\sum_{n \in \mathbb{Z}} e^{2\pi i n^2 \tau}$.
Since $P = 1$ we can see theta as having the expansion 5.1.3 and so can find the representation numbers to find the q-expansion. Looking for the first few terms in the expansion we must compute the number of integer solutions to $x^2 + y^2 = n$:
So we have the expansion formula as:

$$\theta(\tau, A, P) = 1 + 4q + 4q^2 + 0q^3 + 4q^4 + 4q^5 + 0q^6 + ...$$

Looking for the inversion formula we obtain (using proposition 5.1.5 and noting that $A^{-1} = \frac{1}{4}A$)

$$\theta(\tau, A) = \frac{i}{2\tau}\theta(-\frac{1}{4\tau}, A)$$

$\mathbf{q_0(x,y)=n}$	$\mathbf{r_{q_0}(n)}$
$x^2+y^2=0$	1
$x^2+y^2=1$	4
$x^2+y^2=2$	4
$x^2+y^2=3$	0
$x^2+y^2=4$	4
$x^2+y^2=5$	4
$x^2+y^2=6$	0

Now, we have the matrix $A = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)$ and we look for N an integer such that NA^{-1} also has integer entries. It is easily verified that this is achieved by $N = 4t$ for $t \in \mathbb{N}$. When looking at modular forms we usually have the concept of 'newforms' and 'oldforms' - in essence the 'newforms' are those that truly belong at that level and 'oldforms' are those that are carry-overs from lower levels. So, when we are looking at N in this section, one would normally take the minimal N, here we have $N = 4$.
Now note that A is positive definite with even diagonal entries and is symmetric so we can appeal to theorem 5.1.9. To find the weight we look at $k = \nu + \frac{m}{2}$. Here $P = 1$, so $\nu = 0$, and we have $m = 2$. We thus have that $k = 1$. Then, by theorem 5.1.9 we have that ϑ^2 is a modular form of weight 1 on $\Gamma_0(4)$. It is clearly not a cusp form as it does not vanish at the cusp at ∞. This is one of the most classical results regarding the Jacobi theta function and modular forms.

Example 5.1.11
We can also look at the quadratic forms of discriminant -23. There are 3 different binary quadratic forms here (as seen in our example in the quadratic forms section), with two representing the same integers. The two forms representing distinct integers have matrices

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 12 \end{pmatrix} \ , \ B = \begin{pmatrix} 4 & 1 \\ 1 & 6 \end{pmatrix}$$

with inverses

$$A^{-1} = \frac{1}{23}\begin{pmatrix} 12 & -1 \\ -1 & 2 \end{pmatrix} \ , \ B^{-1} = \frac{1}{23}\begin{pmatrix} 6 & -1 \\ -1 & 4 \end{pmatrix}$$

It is clear that to make NA^{-1} and $\tilde{N}B^{-1}$ integral we can take $N = \tilde{N} = 23$.
Looking at two different cases of P we have:

$\mathbf{P = 1}$

For the expansion we look for the first few representation numbers in a similar fashion to the previous example. Firstly for A:
so that

$$\theta_A = 1 + 2q + 0q^2 + 0q^3 + 2q^4 + 0q^5 + 4q^6 + ...$$

$\mathbf{q_A(x,y) = n}$	$\mathbf{r_{q_A}(n)}$
$x^2+xy+6y^2=0$	1
$x^2+xy+6y^2=1$	2
$x^2+xy+6y^2=2$	0
$x^2+xy+6y^2=3$	0
$x^2+xy+6y^2=4$	2
$x^2+xy+6y^2=5$	0
$x^2+xy+6y^2=6$	4

and for B:

$\mathbf{q_B(x,y) = n}$	$\mathbf{r_{q_B}(n)}$
$2x^2+xy+3y^2=0$	1
$2x^2+xy+3y^2=1$	0
$2x^2+xy+3y^2=2$	2
$2x^2+xy+3y^2=3$	2
$2x^2+xy+3y^2=4$	2
$2x^2+xy+3y^2=5$	0
$2x^2+xy+3y^2=6$	2

so that

$$\theta_B = 1 + 0q + 2q^2 + 2q^3 + 2q^4 + 0q^5 + 2q^6 + ...$$

We now have that $\nu = 0$ and $m = 2$ as before, for both A and B. So we can see that θ_A and θ_B are both of weight 1.
So, invoking our theorem again we find that both θ_A and θ_B are modular forms (and not cusp forms as they do not vanish at the cusp at ∞) of weight 1 on the congruent subgroup $\Gamma_0(23)$ with character χ_{-23}.

$\mathbf{P = x + y}$

The next simplest case is to find a spherical polynomial for each of A and B and then consider the theta functions involving this. From the definition of the Laplace operator 5.1.1 we find that

$$\Delta_A = \frac{1}{23}(12\frac{\partial^2}{\partial x^2} - 2\frac{\partial^2}{\partial x \partial y} + 2\frac{\partial^2}{\partial y^2})$$

and

$$\Delta_B = \frac{1}{23}(6\frac{\partial^2}{\partial x^2} - 2\frac{\partial^2}{\partial x \partial y} + 4\frac{\partial^2}{\partial y^2}).$$

It is clear that the polynomial $P(x,y) = x + y$ will satisfy $\Delta_A P(x,y) = 0 = \Delta_B P(x,y)$, and that $P(x,y)$ is homogeneous of degree 1. Now we have that

$$\theta(\tau, A, P) = \sum_{(x,y)\in\mathbb{Z}^2} (x+y)e^{2\pi i(x^2+xy+6y^2)\tau}$$

$$\theta(\tau, B, P) = \sum_{(x,y)\in\mathbb{Z}^2} (x+y)e^{2\pi i(2x^2+xy+3y^2)\tau}$$

are both modular forms of weight $k = \nu + \frac{m}{2} = 1 + \frac{2}{2} = 2$ *on the congruent subgroup* $\Gamma_0(23)$ *with character* χ_{-23}*, after we again appeal to theorem 5.1.9.*
In this case, since $\deg P = 1 > 0$ *the theorem tells us even more - that in fact, both theta functions here are indeed cusp forms. This is clearly highly non-trivial information, but when the theorem is established it is easy to check.*

Example 5.1.12
Now consider the case where $P = 1$ *and we use our quadratic forms of discriminant* -20*. We have seen that these are given by the associated matrices*

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 10 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 4 & 2 \\ 2 & 6 \end{pmatrix} .$$

Then we build the theta function associated to each form. Again, computing the first few values for A

$\mathbf{q_A(x,y) = n}$	$\mathbf{r_{q_A}(n)}$
$x^2 + 5y^2 = 0$	1
$x^2 + 5y^2 = 1$	2
$x^2 + 5y^2 = 2$	0
$x^2 + 5y^2 = 3$	0
$x^2 + 5y^2 = 4$	2
$x^2 + 5y^2 = 5$	2
$x^2 + 5y^2 = 6$	4

and so

$$\theta_A(\tau) = 1 + 2q + 0q^2 + 0q^3 + 2q^4 + 2q^5 + 4q^6 + ...$$

Similarly for our second quadratic form B

$\mathbf{q_B(x,y) = n}$	$\mathbf{r_{q_B}(n)}$
$2x^2 + 2xy + 3y^2 = 0$	1
$2x^2 + 2xy + 3y^2 = 1$	0
$2x^2 + 2xy + 3y^2 = 2$	2
$2x^2 + 2xy + 3y^2 = 3$	2
$2x^2 + 2xy + 3y^2 = 4$	2
$2x^2 + 2xy + 3y^2 = 5$	0
$2x^2 + 2xy + 3y^2 == 6$	2

and so

$$\theta_B(\tau) = 1 + 0q + 2q^2 + 2q^3 + 2q^4 + 0q^5 + 2q^6 + ...$$

We have that A and B are positive definite, integral matrices with even diagonal entries, and $\nu = 0, m = 2$ so $k = 1$. Again, our theorem tells us that $\theta_A, \theta_B \in \mathcal{M}_1(20, \chi_{-5})$, since it is clear that taking $N = 20$ will ensure that NA^{-1} and NB^{-1} are integral matrices with even diagonal.

This relation between theta functions and automorphic forms gives a way to construct a vast amount of automorphic forms. There is a large amount of literature on the subject for the interested reader, including generalisation of the theta function to include a second variable z - see chapter 10 of [5], chapter 1 of [12], or chapter 4 of [17].
Our knowledge of these theorems regarding theta functions will prove useful for relating L-functions of field extensions to certain automorphic forms. Next we introduce some of the most basic zeta functions and L-series.

6. Zeta functions and L-functions of $\mathbb{Q}$

6.1 Riemann zeta

We begin by detailing the famous Riemann zeta function and some of its properties. In a paper in 1859 [18] Riemann introduced his zeta function, defining

$$\zeta(s) \coloneqq \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t - 1} dt, \tag{6.1.1}$$

where $s \in \mathbb{C}, \Re(s) > 1$.
Notice that we require that $\Re(s) > 1$ so that the series converges. To see the equality of the sum and integral representations in equation 6.1.1 consider first the integral defining the gamma function:

$$\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t} dt, \tag{6.1.2}$$

with $\Re(s) > 0$. In particular, notice that $\Gamma(s)$ has no zeros. If we now let $t = na, n \in \mathbb{N}$ then:

$$\Gamma(s) = n^s \int_0^{\infty} a^{s-1} e^{-na} da.$$

So for any $n \in \mathbb{N}$ and $\Re(s) > 0$

$$\frac{\Gamma(s)}{n^s} = \int_0^{\infty} t^{s-1} e^{-nt} dt.$$

We see that, for $\Re(s) > 1$:

$$\begin{aligned}
\Gamma(s)\zeta(s) &= \sum_{n=1}^{\infty} \int_0^{\infty} t^{s-1} e^{-nt} dt \\
&= \int_0^{\infty} t^{s-1} (\sum_{n=1}^{\infty} e^{-nt}) dt \\
&= \int_0^{\infty} t^{s-1} \frac{e^{-t}}{1 - e^{-t}} dt \\
&= \int_0^{\infty} \frac{t^{s-1}}{e^t - 1} dt.
\end{aligned}$$

This view of our function as an integral over $\mathbb{R}$ will be a useful tool when we extend our argument to obtain the integral representations of more general L-functions; and when we search for their functional equations.
Perhaps the most important known identity of the Riemann zeta function is its Euler

product identity [19, page 2, Theorem 1]. This relates zeta to an infinite product over primes and has extremely deep implications. It is the motivation for much of this report (and indeed, a lot of current research). A discussion of these and many more aspects can be found in [19].

Theorem 6.1.3
We have that

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}} \tag{6.1.4}$$

in its region of convergence, where the product is taken over all prime numbers $p \in \mathbb{N}$.

Proof By definition, in the region of convergence $\Re(s) > 1$, we have

$$\begin{aligned}
\zeta(s) &= \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + ... \\
&= (1 + \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{8^s} + ...)(1 + \frac{1}{3^s} + \frac{1}{9^s} + ...)(1 + \frac{1}{5^s} + \frac{1}{25^s} + ...)... \\
&= \prod_p \sum_{k=0}^{\infty} p^{-ks} \\
&= \prod_p \frac{1}{1-p^{-s}},
\end{aligned}$$

where we have used that integers decompose uniquely into primes. ■

We will use the Euler product identity mainly to show that L-functions associated to various characters and modular forms also arise with Euler products. It has many important implications, but to show the power of the statement we observe the simplest corollary arising from it:

Corollary 6.1.5
The set of all prime numbers $p \in \mathbb{N}$ is infinite in size.

Proof By theorem 6.1.4 we know that $\zeta(s) = \prod\limits_p \frac{1}{1-p^{-s}}$. Letting $s = 1$ we see that $\sum\limits_{n=1}^{\infty} \frac{1}{n} = \prod\limits_p \frac{1}{1-p}$. Note that the left hand side diverges by elementary analysis. Assuming that the set of all primes is finite, the right hand side converges since 1 is not prime. This would be a contradiction of the equality, hence the set of primes has infinite size. ■

It will be necessary to introduce two very important concepts before we can state the main results regarding the continuation and functional equation of the Riemann zeta function. We now introduce the Mellin transform of a function f and the Mellin principle.

Definition We define the **Mellin transform** of $f : \mathbb{R}^+ \to \mathbb{C}$ to be

$$L(f,s) = \int_0^\infty (f(y) - f(\infty))y^s \frac{dy}{y},$$

whenever the limit $f(\infty) = \lim_{y\to\infty} f(y)$ exists and the integral converges.

We call the following theorem the Mellin principle. It is extremely important, and has a wide variety of applications throughout number theory [3, page 423, Theorem 1.4] .

Theorem 6.1.6
Let $f, g : \mathbb{R}_+^* \to \mathbb{C}$ be continuous functions such that we have

$$f(y) = a_0 + O(e^{-cy^\alpha}) \ , \ g(y) = b_0 + O(e^{-cy^\alpha})$$

for $y \to \infty$ and $c, \alpha > 0$ constants. If f and g satisfy

$$f(\frac{1}{y}) = Cy^k g(y)$$

for a real number $k > 0$ and $C \in \mathbb{C}$, then we have

1. *$L(f,s)$ and $L(g,s)$ converge absolutely and uniformly on compact subsets of $\{s \in \mathbb{C} | \Re(s) > k\}$. Therefore,they are holomorphic functions there, and they admit holomorphic continuations to $\mathbb{C}\backslash 0$.*
2. *f and g have simple poles at $s = 0$ and $s = k$ with residues given by*

$$Res_{s=0}L(f,s) = -a_0 \ , Res_{s=o}L(f,s) = Cb_0$$
$$Res_{s=0}L(g,s) = -b_0 \ , Res_{s=o}L(g,s) = C^{-1}a_0.$$

3. *They satisfy the functional equation*

$$L(f,s) = CL(g,k-s).$$

Proof See proof at [3, page 423, Theorem 1.4]. ∎

We will use the above to aid us in the computation of continuations. In particular, we will see it is exceedingly useful in the case where we look to find the continuation of L-functions of modular forms in this chapter. We will show, for each case, how to calculate the functional equation explicitly.

Definition We define the **completed Riemann zeta function** as

$$\Lambda(s) := \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s).$$

We can then prove the following fundamental theorem [13, page 87, Theorem 3.2.2]:

Theorem 6.1.7
Let $\Lambda(s)$ be the completed Riemann zeta function. Then $\Lambda(s)$ has a meromorphic continuation to all of $\mathbb{C}$, which is analytic except at $s = 0$ and $s = 1$. It also satisfies the functional equation $\Lambda(s) = \Lambda(1 - s)$.

Proof Taken and modified from proof of [19, page 8, Theorem 1].
We use the Jacobi theta function $\vartheta(iy = t) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t}$ and its transformation law $\vartheta(\frac{1}{t}) = \sqrt{t}\vartheta(t)$; see example 5.1.10. The completed zeta function can be seen to be the Mellin transform of the Jacobi theta function (simply integrate termwise). Then we have that

$$\begin{aligned}\Lambda(s) &= \int_0^\infty \frac{1}{2}(\vartheta(t) - 1)t^{s/2}\frac{dt}{t}\\ &= \int_0^1 \frac{1}{2}(\vartheta(t) - 1)t^{s/2}\frac{dt}{t} + \int_1^\infty \frac{1}{2}(\vartheta(t) - 1)t^{s/2}\frac{dt}{t}\\ &= -\frac{1}{s} + \int_0^1 \frac{1}{2}\vartheta(t)t^{s/2}\frac{dt}{t} + \int_1^\infty \frac{1}{2}(\vartheta(t) - 1)t^{s/2}\frac{dt}{t}.\end{aligned}$$

Now, in the first integral let $t \mapsto \frac{1}{t}$ and use the theta transformation formula to get

$$\begin{aligned}\frac{1}{2}\int_0^1 \vartheta(t)t^{s/2}\frac{dt}{t} &\mapsto \frac{1}{2}\int_1^\infty \vartheta(t)t^{(1-s)/2}\frac{dt}{t}\\ &= -\frac{1}{1-s} + \frac{1}{2}\int_1^\infty (\vartheta(t) - 1)t^{(1-s)/2}\frac{dt}{t}.\end{aligned}$$

So we have that

$$\Lambda(s) + \frac{1}{s} + \frac{1}{1-s} = \frac{1}{2}\int_1^\infty (\vartheta(t) - 1)(t^{s/2} + t^{(1-s)/2})\frac{dt}{t},$$

and clearly the functional equation is seen by the symmetry of the above under $s \mapsto -s$:

$$\Lambda(s) = \Lambda(1 - s).$$

Clearly, the completed Riemann zeta function has simple poles at $s = 0$ and $s = 1$ each of residue 1. ■

We have shown that the Riemann zeta function encodes information on rational primes, and admits a continuation and satisfies a functional equation. This will be the general theme of functions that appear throughout this report.

6.1.1 The Riemann hypothesis

We could not discuss the Riemann zeta function without at least mentioning possibly the most famous and important open problem in mathematics - the Riemann Hypothesis. It is concerned with the non-trivial zeros of the Riemann zeta function (the trivial zeros are those at even negative integers). If true, it has a wide array of applications (indeed, there is a lot of literature that has been written assuming the conjecture).

The Riemann Hypothesis
The non-trivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$.

Such a seeming simple statement has eluded proof since Riemann conjectured it in his famous paper in 1859. It is such an important conjecture that it was named as one of the millennium problems by the Clay Mathematics Institute [20]. It has been verified numerically that the first 1.5 billion zeros of $\zeta(s)$ (when arranged by increasing positive imaginary part) satisfy the hypothesis. If true, the conjecture would have profound implications on the distribution of prime numbers. There are generalisations of the hypothesis to include more general L-functions, which will be mentioned at the end of the chapter.

6.2 Dirichlet L-series of character χ

The first step of generalisation of Riemann's zeta function is to put a Dirichlet character χ in the numerator. Since we know that these characters are multiplicative this will still carry very nice properties (e.g. Euler product formula), and will still satisfy functional equations and admit a continuation.

Definition We define a **Dirichlet L-series of a character** χ to be

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

for $s \in \mathbb{C}$ with $\Re(s) > 1$.

We begin by showing the most useful property of Dirichlet L-series - that of an Euler product [21, page 114, Corollary 6.5].

Proposition 6.2.1
Let $L(\chi, s)$ be a Dirichlet L-series of a character χ. Then it arises with an Euler product identity given by

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \frac{1}{1 - \chi(p)p^{-s}},$$

in the region of convergence $\Re(s) > 1$.

Proof This follows directly from the Riemann zeta Euler product formula 6.1.4, and noting that $\chi(n)$ is multiplicative by definition. $\blacksquare$

If χ is a principal character of modulus q (see definition 2.2) then it follows immediately from the Euler product definitions that

$$L(\chi, s) = \prod_{p \nmid q} \frac{1}{1 - p^{-s}} = \zeta(s) \prod_{p|q} 1 - p^{-s},$$

and so we can recover the Riemann zeta function by taking the Dirichlet L-series of the trivial character of modulus 1, $L(\chi_0, s) = \zeta(s)$. In the same vein as for the Riemann zeta, we expect Dirichlet series to have a continuation to $\mathbb{C}$ and to satisfy a functional equation. As before, we will look to find the completion of $L(\chi, s)$ and introduce and use properties of a theta function associated to χ. Here we will follow [22]. It will be necessary to define the Gauss sum of primitive χ.

Definition We let $\tau(\chi)$ be the **Gauss sum** of a primitive Dirichlet character χ

$$\tau(\chi) = \sum_{a \mod q} \chi(a) e^{\frac{2\pi i a}{q}}. \tag{6.2.2}$$

Following the lead of the Riemann zeta case we look to introduce a theta function, with the most obvious candidate being

$$\theta_\chi(iy) = \sum_{n=-\infty}^{\infty} \chi(n) e^{-\pi n^2 y}.$$

However, notice that for χ an odd character this sum is identically zero. So we must split the two cases; χ even and χ odd. We will deal with the even case first.

6.2.1 The even case

Looking to find the completion of the L-series leads us to consider, like before, the integral representation of the gamma function:

$$\Gamma(\frac{s}{2}) = \int_0^\infty e^{-y} y^{s/2} \frac{dy}{y}.$$

Definition Let χ be an even primitive Dirichlet character. Then we define its theta function to be

$$\theta_\chi(iy) = \sum_{n=-\infty}^{\infty} \chi(n) e^{-\pi n^2 y} = \chi(0) + 2 \sum_{n=1}^{\infty} \chi(n) e^{-\pi n^2 y}.$$

We then have that [23, page 2]

$$\begin{aligned} \int_0^\infty \frac{1}{2} y^{s/2} \theta_\chi(iy) \frac{dy}{y} &= \sum_{n=1}^{\infty} \chi(n) \int_0^\infty y^{s/2} e^{-\pi n^2 y} \frac{dy}{y} \\ &= \sum_{n=1}^{\infty} \frac{\chi(n)}{\pi^{s/2} n^s} \int_0^\infty y^{s/2} e^{-y} \frac{dy}{y} \\ &= \pi^{-s/2} \Gamma(\frac{s}{2}) L(\chi, s). \end{aligned}$$

Again we look for the completion of this L-function.

Definition Let $L(\chi, s)$ be a Dirichlet L-series of an even primitive character χ with modulus q. We define the completed L-function as

$$\Lambda(\chi, s) = (\frac{q}{\pi})^{s/2}\Gamma(\frac{s}{2})L(\chi, s).$$

This is entire for $q \neq 1$, and otherwise has simple poles at $s = 0$ and $s = 1$, each of residue 1 (this is the Riemann zeta case treated above). For χ non-trivial we have $\chi(0) = 0$.

We expect the completed L-function to admit a continuation to all $\mathbb{C}$ and satisfy a functional equation. This is indeed the case [22, page 4].

Proposition 6.2.3
For $\Lambda(\chi, s)$ the completed L-function of a primitive even Dirichlet character χ we have a meromorphic continuation to $\mathbb{C}$ (which is entire for $\chi \neq \chi_0$). We also have that $\Lambda(\chi, s)$ satisfies the functional equation

$$\Lambda(\chi, s) = \frac{\tau(\chi)}{\sqrt{q}}\Lambda(\chi^{-1}, 1 - s).$$

Proof Proof taken and modified from [23].
Since we have essentially dealt with the case $q = 1$ within the section on Riemann zeta, we assume $q \neq 1$. Now consider our theta function

$$\theta_\chi(iy) = \sum_{n=-\infty}^{\infty} \chi(n)e^{-\pi n^2 y} = \sum_{a \mod q} \chi(a)(\sum_{l\in\mathbb{Z}} e^{-\pi(lq+a)^2 y}).$$

To use the Poisson summation formula - see appendix A - we need knowledge of the Fourier transform of $f(x) := e^{-\pi(xq+a)^2 y}$.
We can compute this directly as - see Appendix B -

$$\begin{aligned} \hat{f}(\xi) &= \int_{-\infty}^{\infty} e^{-\pi(xq+a)^2 y} e^{-2\pi\xi x} dx \\ &= \frac{e^{\frac{2\pi i\xi a}{q}}}{q\sqrt{y}} e^{\frac{-\pi\xi^2}{q^2 y}}. \end{aligned} \tag{6.2.4}$$

So, we have now that

$$\begin{aligned} \theta_\chi(iy) &= \sum_{a \mod q} \chi(a)(\sum_{l\in\mathbb{Z}} e^{-\pi(lq+a)^2 y}) \\ &= \sum_{a \mod q} \chi(a)(\sum_{l\in\mathbb{Z}} \frac{e^{\frac{2\pi i la}{q}}}{q\sqrt{y}} e^{\frac{-\pi l^2}{q^2 y}}) \\ &= \frac{1}{q\sqrt{y}} \sum_{l\in\mathbb{Z}} e^{\frac{-\pi l^2}{q^2 y}} \sum_{a \mod q} \chi(a) e^{\frac{2\pi i la}{q}}. \end{aligned}$$

Now, we want to replace $a \mapsto al^{-1}$. We must, however, be somewhat careful since (l,q) is not necessarily 1. We appeal to appendix C to use that

$$\sum_{a \mod q} \chi(a) e^{\frac{2\pi i l a}{q}} = 0, \tag{6.2.5}$$

for $(l,q) > 1$.
From this we see that we sum over l where l is invertible modulo q. Replacing a with al^{-1} yields

$$\begin{aligned}\theta_\chi(iy) &= \frac{1}{q\sqrt{y}} \sum_{l\in\mathbb{Z}} e^{\frac{-\pi l^2}{q^2 y}} \sum_{al^{-1} \mod q} \chi(al^{-1}) e^{\frac{2\pi i a}{q}} \\ &= \frac{1}{q\sqrt{y}} \sum_{l\in\mathbb{Z}} e^{\frac{-\pi l^2}{q^2 y}} \chi(l)^{-1} \sum_{a \mod q} \chi(a) e^{\frac{2\pi i a}{q}}.\end{aligned}$$

Notice that the second sum is now just the Gauss sum $\tau(\chi)$ and so we have that $\theta_\chi(iy)$ satisfies

$$\theta_\chi(iy) = \frac{\tau(\chi)}{q\sqrt{y}} \theta_{\chi^{-1}}(\frac{i}{q^2 y}). \tag{6.2.6}$$

For ease of exposition consider the case where χ is non-trivial, so that $\chi(0) = 0$. We start by splitting the integral at $\frac{1}{q}$:

$$\begin{aligned}(\frac{1}{\pi})^{s/2}\Gamma(\frac{s}{2})L(\chi,s) &= \int_0^\infty \frac{1}{2}\theta_\chi(iy) y^{s/2} \frac{dy}{y} \\ &= \int_0^{\frac{1}{q}} \frac{1}{2}\theta_\chi(iy) y^{s/2} \frac{dy}{y} + \int_{\frac{1}{q}}^\infty \frac{1}{2}\theta_\chi(iy) y^{s/2} \frac{dy}{y}.\end{aligned}$$

Now let $y \mapsto \frac{1}{q^2 y}$ in the first integral above. We have that

$$\begin{aligned}\int_0^{\frac{1}{q}} \frac{1}{2}\theta_\chi(iy) y^{s/2} \frac{dy}{y} &\mapsto \frac{1}{2}\int_{\frac{1}{q}}^\infty (\frac{1}{q^2 y})^{s/2} \theta_\chi(\frac{i}{q^2 y}) \frac{dy}{y} \\ &= \frac{1}{2q^s}\int_{\frac{1}{q}}^\infty y^{-s/2} \theta_\chi(\frac{i}{q^2 y}) \frac{dy}{y}.\end{aligned}$$

Notice that equation 6.2.6 implies that

$$\theta_\chi(\frac{i}{q^2 y}) = \frac{q\sqrt{y}}{\tau(\chi^{-1})} \theta_{\chi^{-1}}(iy)$$

Putting the above together we have that

$$(\frac{1}{\pi})^{s/2}\Gamma(\frac{s}{2})L(\chi,s) = \int_{\frac{1}{q}}^\infty \frac{1}{2}\theta_\chi(iy) y^{s/2} \frac{dy}{y} + \frac{q^{1-s}}{\tau(\chi^{-1})} \int_{\frac{1}{q}}^\infty \frac{1}{2}\theta_{\chi^{-1}}(iy) y^{(1-s)/2} \frac{dy}{y}.$$

Let $\epsilon(\chi) = \frac{\sqrt{q}}{\tau(\chi)}$. It can be shown that $|\epsilon(\chi)| = 1$ [24, page 6]. We have

$$(\frac{1}{\pi})^{s/2}\Gamma(\frac{s}{2})L(\chi,s) = \int_{\frac{1}{q}}^{\infty} \frac{1}{2}(\theta_\chi(iy)y^{s/2} + \epsilon(\chi)q^{(1-2s)/2}\theta_{\chi^{-1}}(iy)y^{(1-s)/2})\frac{dy}{y}.$$

Multiplying through by $q^{s/2}$ and noting we have that $\epsilon(\chi)\epsilon(\chi^{-1}) = 1$ for even χ gives

$$\begin{aligned}\Lambda(\chi,s) = (\frac{q}{\pi})^{s/2}\Gamma(\frac{s}{2})L(\chi,s) &= \int_{\frac{1}{q}}^{\infty} \frac{1}{2}(\theta_\chi(iy)(qy)^{s/2} + \epsilon(\chi)(qy)^{(1-s)/2}\theta_{\chi^{-1}}(iy))\frac{dy}{y} \\ &= \epsilon(\chi)\int_{\frac{1}{q}}^{\infty} \frac{1}{2}(\epsilon(\chi^{-1})\theta_\chi(iy)(qy)^{s/2} + (qy)^{(1-s)/2}\theta_{\chi^{-1}}(iy))\frac{dy}{y},\end{aligned}$$

from which we can immediately see the functional equation

$$(\frac{q}{\pi})^{s/2}\Gamma(\frac{s}{2})L(\chi,s) = \epsilon(\chi)(\frac{q}{\pi})^{(1-s)/2}\Gamma(\frac{1-s}{2})L(\chi^{-1},1-s).$$

This can be expressed in terms of the completed L-function as

$$\Lambda(\chi,s) = \epsilon(\chi)\Lambda(\chi^{-1},1-s).$$

■

6.2.2 The odd case

For the case when χ is odd we see that

$$\theta_\chi(iy) = \sum_{n=-\infty}^{\infty} \chi(n)e^{-\pi i n^2 y}$$

is identically 0, since pairs $(n,-n)$ cancel perfectly. We must therefore look to define a slightly altered theta function. Perhaps the simplest answer would be to use the following.

Definition Let χ be an odd primitve Dirichlet character. Then we define its theta function as

$$\theta_\chi(iy) = \sum_{n=-\infty}^{\infty} \chi(n)ne^{-\pi i n^2 y}.$$

This is clearly not identically zero, as pairs $(-n,n)$ in the sum no longer cancel. We will skip much of the calculation as it is extremely similar to the even case, but one needs to be slightly more careful to keep track of the extra factors. Similar to the even case we have the following definition.

Definition Let χ be an odd primitive Dirichlet character. Then we define the completed L-function to be

$$\Lambda(\chi,s) = (\frac{q}{s})^{s/2}\Gamma(\frac{s+1}{2})L(\chi,s).$$

After similar calculations and setting $\epsilon(\chi) = \frac{\sqrt{q}}{i\tau(\chi)}$ (the factor of i appears since theta now transforms with a factor of i) we find that:

Proposition 6.2.7
Let χ be an odd primitive Dirichlet character of modulus q. Then $L(\chi, s)$ has an analytic continuation to $\mathbb{C}$ and the completed L-function $\Lambda(\chi, s)$ satisfies the functional equation

$$\Lambda(\chi, s) = \epsilon(\chi)\Lambda(\chi^{-1}, 1-s)$$

Proof As discussed, much of the calculation is similar to the even case. We end up with the equation

$$(\frac{q}{\pi})^{s/2}\Gamma(\frac{s+1}{2})L(\chi, s) = \epsilon(\chi)(\frac{q}{\pi})^{(1-s)/2}\Gamma(\frac{2-s}{2})L(\chi^{-1}, 1-s),$$

from which the functional equation is apparent

$$\Lambda(\chi, s) = \epsilon(\chi)\Lambda(\chi^{-1}, 1-s).$$

■

6.3 Dirichlet L-functions associated to modular forms

Now we cover some of the most classical results from Hecke pertaining to the L-functions associated to modular forms on Γ. We saw in our section on modular forms that the L-series of a normalised primitive form has an Euler product representation - see equation 3.1.13.
Recall that for a modular form $f = \sum_{n=0}^{\infty} a_n q^n$ we let the L-series associated to f be $L(f, s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$.

It will be shown that there is an analytic continuation of $L(f, s)$ to an entire function, and that its completion satisfies a functional equation. This is exceedingly similar to the cases we have seen previously.

Remark We will only consider cusp forms, since the theory here is slightly more concise as there are no poles to deal with. As Zagier notes [9, page 40], the space of modular forms of weight k is spanned by the Eisenstein series together with cusp forms and the L-series of the Eisenstein series is completely understood. Therefore, we lose nothing in dealing only with the cusp form case.

Definition We define the completed L-function associated to a modular form to be

$$\Lambda(f, s) = \frac{1}{(2\pi)^s}\Gamma(s)L(f, s).$$

As before, we can view this completed L-function as an integral over the real line of our original L-function. We begin with the following representation of the gamma function

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt.$$

Letting $t \mapsto \lambda t$ gives

$$\Gamma(s) = \lambda^s \int_0^\infty t^{s-1} e^{-\lambda t} dt,$$

and so, for positive λ,

$$\lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-\lambda t} dt.$$

Now, if we let $\lambda = 2\pi n$, multiply this by our coefficients a_n and sum over all n we obtain

$$\frac{1}{(2\pi)^s}\Gamma(s)L(f,s) = \sum_{n=1}^{\infty} a_n \int_0^\infty t^{s-1} e^{-2\pi n t} dt.$$

Since we have absolute convergence for $\Re(s) > \frac{k}{2} + 1$ - a simple corollary of [9, page 23, Proposition 8] - we are justified in switching the summation and integration to obtain

$$\frac{1}{(2\pi)^s}\Gamma(s)L(f,s) = \int_0^\infty t^s f(it) \frac{dt}{t}.$$

Since f is a cusp form we note that we have exponential suppression at $\tau \to i\infty$, and the transformation property $f(-\frac{1}{\tau}) = \tau^k f(\tau)$ guarantees such suppression as $\tau \to 0$. Together, these imply that our integral converges absolutely for all $s \in \mathbb{C}$, and so $\Lambda(f,s)$ has a continuation that is entire on the whole complex plane.

To see the functional equation we now split the integral at 1, and let $t \mapsto \frac{1}{t}$ in our second term to obtain

$$\Lambda(f,s) = \int_1^\infty t^s f(it) \frac{dt}{t} + \int_1^\infty t^{-s} f(\frac{i}{t}) \frac{dt}{t}.$$

Using the transformation property of f gives that

$$\Lambda(f,s) = \int_1^\infty t^s f(it) \frac{dt}{t} + i^k \int_1^\infty f(it) t^{k-s} \frac{dt}{t}$$

$$= i^k \Lambda(f, k-s).$$

So we have shown the following [25, page 8, Theorem 1.2].

Theorem 6.3.1
Let $f(\tau)$ be a cusp form on Γ. Then its completed L-function $\Lambda(f,s)$ has an analytic continuation to all $\mathbb{C}$, which satisfies the functional equation $\Lambda(f,s) = i^k\Lambda(f,k-s)$.

Remark This theorem is actually true for all modular forms, not just cusp forms.

If we want to extend our arguments to the case of modular forms on $\Gamma_0(N)$ the situation is slightly more complicated. For $f(\tau) \in S_k(\Gamma_0(N))$ it can be shown that [25, page 9]

$$g(\tau) := N^{-k/2}\tau^{-k}f(-\frac{1}{N\tau}) \in S_k(\Gamma_0(N)).$$

Then the normal procedure of finding an integral representation of the completion, splitting the integral at a suitable point and using transformation results gives us the functional equation

$$\Lambda(f,s) = i^k N^{\frac{k}{2}-s}\Lambda(g,k-s),$$

and that $\Lambda(f,s)$ has a continuation to all $\mathbb{C}$.

6.4 The Generalised Riemann Hypothesis

Before concluding this chapter, we mention the generalised version of the Riemann Hypothesis.

The Generalised Riemann Hypothesis
Let $L(f,s)$ be an L-function. Then all zeros of $L(f,s)$ in the critical strip $0 < \Re(s) < 1$ lie on the line $\Re(s) = \frac{1}{2}$.

A discussion and statement of the hypothesis is given at [11, page 113]. Clearly the Generalised Riemann Hypothesis (GRH) is perhaps more important than the standard Riemann Hypothesis - it covers a wide array of different L-functions, from those discussed here, to those that arise from elliptic curves and beyond. Current analytic methods are unable to prove results that the GRH provides easy answers to, and so it is unlikely that a proof will be seen in the near future.

Our next chapter covers a generalisation of the L-functions we have currently seen to those over more general number field extensions $K/\mathbb{Q}$.

7. Zeta functions and L-functions of number fields

7.1 Dedekind zeta function

Now we introduce the idea of a zeta function of a field extension K over $\mathbb{Q}$ of finite degree n. This allows us to obtain information on the prime ideals of $\mathcal{O}_K$ by using an Euler product formula, now taken over all prime ideals.

Definition We define the **Dedekind zeta function** of a field extension K of $\mathbb{Q}$ to be

$$\zeta_K(s) = \sum_I \frac{1}{N(I)^s}, \tag{7.1.1}$$

where $N(I)$ denotes the ideal norm and we sum over all ideals I in $\mathcal{O}_K$.

Note that this will also converge for $\Re(s) > 1$, and we can easily recover the Riemann zeta function as $\zeta(s) = \zeta_{\mathbb{Q}}(s)$. One of the most important properties will again be that $\zeta_K(s)$ comes with an Euler product identity:

$$\zeta_K(s) = \sum_I \frac{1}{N(I)^s} = \prod_{\mathfrak{P}} \frac{1}{1 - N(\mathfrak{P})^{-s}},$$

where the product is taken over prime ideals $\mathfrak{P}$ in $\mathcal{O}_K$. We leave the proof of this in appendix D.
It will sometimes also be useful to view $\zeta_K(s)$ as a sum over ideal classes in the class group of K.

Definition We define the **partial zeta function** as a sum over ideals in a given class $[I]$ in the class group $Cl(K)$, i.e.

$$\zeta_{K,I}(s) = \sum_{J \in [I]} \frac{1}{N(J)^s}.$$

Then we have

$$\begin{aligned}\zeta_K(s) &= \sum_{[I] \in Cl(K)} \sum_{J \in [I]} \frac{1}{N(J)^s} \\ &= \sum_{[I] \in Cl(K)} \zeta_{K,I}(s).\end{aligned}$$

If we write

$$\begin{aligned}\zeta_K(s) &= \sum_{k \in \mathbb{Z}} \sum_{\substack{I \\ N(I)=k}} \frac{1}{k^s} \\ &= \sum_{k=0}^{\infty} \frac{r(k)}{k^s},\end{aligned}$$

then the numbers $r(k)$ are precisely the number of ideals of norm k in $\mathcal{O}_K$.
Since we are mainly interested in the properties and numbers of prime ideals in our field (recall that we have unique decomposition into prime ideals) we consider the numbers $r(p)$ for p prime in more detail. The numbers $r(p)$ tell us exactly when a prime ideal (p) splits in $\mathcal{O}_K$.
Therefore, (p) will split completely if and only if $r(p) = n = [K : \mathbb{Q}]$:

$$(p) = \mathfrak{P}_1\mathfrak{P}_2...\mathfrak{P}_n,$$

where no two ideals in the RHS are equal. Similarly, (p) is totally ramified if and only if $r(p) = 1$ (that is, we can write $(p) = \mathfrak{P}^n$ for some prime ideal $\mathfrak{P}$). The intermediate cases are similar. We also have that (p) is inert if and only if $r(p) = 0$.

Example 7.1.2
Let K be a quadratic field, so we have $n = 2$. Then when we form $\zeta_K(s)$ the coefficients $r(k)$ tell us precisely how many ideals $I \in \mathcal{O}_K$ of norm k there are. We only have two possibilities for a prime ideal where it is not inert (either there are two non-equivalent ideals dividing it, or it is ramified). The ideal (p) for rational prime p can be written as $(p) = \mathfrak{P}_1\mathfrak{P}_2$ with $N(\mathfrak{P}_1) = N(\mathfrak{P}_2) = p$, $\mathfrak{P}_1 \neq \mathfrak{P}_2$ if and only if $r(p) = 2$. We also have that (p) is ramified and can be written as $(p) = \mathfrak{P}^2$ if and only if $r(p) = 1$.

Remark A similar concept holds for the partial zeta functions, where we now only count the number of ideals of norm k in a given class of the class group.

Here there are also analytic continuations to $\mathbb{C}$, and $\zeta_K(s)$ also satisfies a functional equation similar to those of the Riemann zeta function and Dirichlet L-series. However the full proof is rather involved and beyond the scope of this report. Neukirch proves this entire case explicitly in [3, Chapter 5]. We will give a brief overview of the general way in which to obtain the analytic continuation, functional equation, and the analytic class number formula. Firstly, we concentrate on finding the analytic properties of the partial zeta function $\zeta_{K,I}(s)$ as the result for the $\zeta_K(s)$ will follow immediately.

To begin, we state and prove the following lemma [3, page 458, Lemma 5.3]

Lemma 7.1.3
Let I be an integral ideal of K. Let $I^* = I \backslash \{0\}$ and $[I^{-1}]$ the class of I^{-1} in $Cl(K)$. Then there is a bijection

$$\begin{aligned} I^*/\mathcal{O}_K^* &\xrightarrow{\sim} \{J \in [I^{-1}] \mid J \textit{ is integral}\} \\ a &\mapsto J = aI^{-1} \end{aligned} \tag{7.1.4}$$

Proof To see that the map is injective we look at $a \in I^*$. Then $aI^{-1} = (a)I^{-1}$ is clearly integral and is in the class of I^{-1}. Now if $aI^{-1} = bI^{-1}$ then we must have that $(a) = (b)$ which means that $ab^{-1} \in \mathcal{O}_K^*$. To see surjectivity just observe that for every element $J \in [I^{-1}]$ we have $J = aI^{-1}$ where $a \in IJ \subseteq I$. ∎

So we can relate partial zeta functions to a sum over particular ideals. Using concepts from Minkowski theory [26, page 11] one can define a certain theta function related to K with a given theta function [3, page 459]. Similarly to our other cases, we then look to associate this to a higher-dimensional gamma function, Γ_K.
With this relation, one can obtain a relation between the partial zeta function and an integral of the theta function over a fundamental domain $\mathcal{F}$ (denoted by $f_{\mathcal{F}}(I,t)$). In fact, the completed partial zeta function $\Lambda_{K,I}$ turns out to be the Mellin transform of $f_{\mathcal{F}}(I,t)$, up to a scalar factor.
Then, it can be shown that the volume of such a fundamental domain is [3, page 43, Proposition 7.5]

$$\mathrm{vol}\,\mathcal{F} = 2^{r-1}R,$$

where r is the number of infinite places of K, and R is the regulator of the field. Following our normal procedure, we make use of a transformation formula for the theta function. In particular, we have

Proposition 7.1.5
We have the following transformation

$$f_{\mathcal{F}}(I,\frac{1}{t}) = t^{\frac{1}{2}} f_{\mathcal{F}^{-1}}((I\mathfrak{d})^{-1},t), \tag{7.1.6}$$

and that

$$f_{\mathcal{F}}(I,t) = \frac{2^{r-1}}{\omega_K}R + O(e^{-ct^{1/n}}),$$

where $\mathfrak{d}$ is the different of $K/\mathbb{Q}$ and ω_K is the number of roots of unity in K.

Using the above considerations one can state and prove the main theorem for this section:

Theorem 7.1.7
Let K have discriminant d_K, regulator R, and number of infinite places r. Let

$$\Lambda_{K,I}(s) = |d_K|^{\frac{s}{2}}\pi^{-\frac{ns}{2}}\Gamma_K(\frac{s}{2})\zeta_{K,I}(s)$$

be the completed partial zeta function of a field extension K of $\mathbb{Q}$ of degree n. Then $\Lambda_{K,I}(s)$ has a continuation to $\mathbb{C}$ that is analytic except at simple poles at $s=0$ and $s=1$ where it has residues of

$$-\frac{2^r}{\omega_K}R \ , \ \frac{2^r}{\omega_K}R$$

respectively.
Furthermore, it satisfies the functional equation

$$\Lambda_{K,I}(s) = \Lambda_{K,J}(1-s),$$

where $[I][J] = [\mathfrak{d}]$.

Proof Imitating the proof given in [3] we let $f(t) = f_{\mathcal{F}}(I, t)$ and $g(t) = f_{\mathcal{F}^{-1}}((I\mathfrak{d})^{-1}, t)$. Then we have (from proposition 7.1.6)

$$f(\frac{1}{t}) = t^{\frac{1}{2}} g(t),$$

with

$$f(t) = a_0 + O(e^{-ct^{1/n}}) \ , \ g(t) = a_0 + O(e^{-ct^{1/n}}),$$

and $a_0 = \frac{2^{r-1}}{\omega_K} R$. Using theorem 6.1.6, we have that the Mellin transforms of f and g have analytic continuations with simple poles of f at $s = 0$ and $s = \frac{1}{2}$ with residues a_0 and $-a_0$ respectively. It also ensures the functional equation

$$L(f, s) = L(g, \frac{1}{2} - s)$$

is satisfied.
So, $\Lambda_{K,I}(s) = L(f, \frac{s}{2})$ admits a continuation to $\mathbb{C}$ which is analytic except at $s = 0, 1$ where it has simple poles of respective residue

$$-\frac{2^r}{\omega_K} R \ , \ \frac{2^r}{\omega_K} R.$$

Furthermore, $\Lambda_{K,I}$ satisfies the functional equation

$$\Lambda_{K,I}(s) = \Lambda_{K,J}(1 - s),$$

where $[I][J] = [\mathfrak{d}]$. ∎

This result for the completed partial zeta functions generalises immediately to full zeta function of the field K.

Theorem 7.1.8
Let

$$\Lambda_K(s) = \sum_{[I] \in Cl(K)} \Lambda_{K,I}(s)$$

be the completed zeta function of the field K. It admits a continuation to $\mathbb{C}$ that is analytic except at $s = 0, 1$ and satisfies the functional equation

$$\Lambda_K(s) = \Lambda_K(1 - s).$$

Furthermore, the residue at its simple poles at $s = 0$ and $s = 1$ are

$$-\frac{2^r hR}{\omega_K} \ , \ \frac{2^r hR}{\omega_K},$$

with h the class number of K.

Proof This follows directly from the theorem regarding the partial zeta functions, and summing the results h times (one for each member of the class group). ∎

We have shown that $\Lambda_K(s)$ admits a continuation to $\mathbb{C}$ and satisfies a functional equation. We end this chapter by observing a simple corollary of the above; the analytic class number formula.
To obtain the class number formula we need to evaluate the residue of $\zeta_K(s)$ at $s = 1$. We first have to view the Euler factor (the prefactor that completes the zeta function) as

$$|d_k|^{s/2}\pi^{-ns/2}\Gamma_K(s/2) = |d_k|^{s/2}(\pi^{-s/2}\Gamma(\frac{s}{2}))^{r_1}(2(2\pi)^{-s}\Gamma(s))^{r_2},$$

where r_1 is the number of real places and r_2 is the number of complex places of the field K. In particular, the definition of Γ_K allows it to be split into the standard gamma functions (up to scalars). Evaluating this expression at $s = 1$ yields

$$\begin{aligned}|d_k|^{1/2}(\pi^{-1/2}\Gamma(\frac{1}{2}))^{r_1}(2(2\pi)^{-1}\Gamma(1))^{r_2} &= |d_k|^{1/2}(\pi^{-1/2}\pi^{\frac{1}{2}})^{r_1}(2(2\pi)^{-1})^{r_2}\\ &= \frac{|d_K|^{\frac{1}{2}}}{\pi^{r_2}},\end{aligned}$$

since $\Gamma(1) = 1$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.
Finally, putting the above together, we arrive at our corollary - the analytic class number formula [5, page 210, Theorem 12.2]

Corollary 7.1.9

$$Res_{s=1}\zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2}}{\omega_K|d_K|^{1/2}}hR$$

7.2 Hecke L-series

Now we introduce the idea of a Hecke L-series - here we will detail the most basic version, using a narrow class group character. There are generalisations to so-called ray-class group characters which the interested reader can find more information on in [13, Section 3.3] or [27, page 89].

Definition Let χ be an ideal class character and K a field extension of $\mathbb{Q}$ of degree n. We define the **Hecke L-series** associated to K and χ to be

$$L_K(\chi, s) = \sum_I \frac{\chi(I)}{N(I)^s},$$

where we sum over all ideals I in $\mathcal{O}_K$.

Firstly note that we also have an Euler product representation as a product over all prime ideals in $\mathcal{O}_K$ given by [11, page 129]

$$L_K(\chi, s) = \prod_{\mathfrak{P}} \frac{1}{1 - \chi(\mathfrak{P})N(\mathfrak{P})^{-s}}.$$

This follows immediately from the Euler product formula for $\zeta_K(s)$, and noting that χ is multiplicative by definition. Notice that this shares many similarities to the generalisation of the Riemann zeta to Dirichlet L-series from before. We can recover the Dedekind zeta function from such L-series by taking χ to be the trivial class group character χ_0 - taking value 1 on all ideals, i.e. $L_K(\chi_0, s) = \zeta_K(s)$.
The Riemann zeta function is the Hecke L-series over the field $K = \mathbb{Q}$. We can see this since $\mathbb{Q}$ has a class number of 1, the only character on the class group is the trivial character. Hence we have $L_{\mathbb{Q}}(\chi, s) = \zeta(s)$.
We would expect the Hecke L-series to again admit a meromorphic continuation to $\mathbb{C}$, and to satisfy some sort of functional equation. This is indeed the case [11, page 129] and follows very easily from the previous section.
Note that since we consider only narrow class group characters, χ is a constant on ideal classes in the narrow class group. To show that genus characters are constant on ideal classes in the ideal class group consider two ideals that are equivalent in the narrow sense. That is $I = (\alpha)J$ for $N(\alpha) > 0$. Then it suffices to show that $\chi(I) = \chi(J)$ for $I \sim J$. This follows immediately from the definition of genus characters, taking the value 1 on all ideals (α) where $N(\alpha) > 0$. We get

$$L_K(\chi, s) = \sum_{[I]\in Cl^+(K)} \sum_{J\in[I]} \frac{\chi(J)}{N(J)^s} = \sum_{[I]\in Cl^+(K)} \chi([I]) \sum_{J\in[I]} \frac{1}{N(J)^s}$$

$$= \sum_{[I]\in Cl^+(K)} \chi([I])\zeta_{K,I}.$$

Here again we can write the Hecke L-series as a sum $\sum_{n=0}^{\infty} \frac{r(n)}{n^s}$, where the coefficients $r(n)$ now arise by counting the number of ideals of a given norm in each class, and taking a weighted sum over the classes (with weights given by the character of the class). We will return to this with an explicit example in the quadratic field section.

Since we can write our L-series as a twisted sum over $\zeta_{K,I}(s)$, all of our results about the partial zeta functions in the previous section carry over to here. However, when we now consider the full L-function of the field, it arises with a character on the narrow class group. We can define the completed L-function in a similar way to before, letting

$$\Lambda_K(s,\chi) = |d_k|^{s/2}\pi^{-ns/2}\Gamma_K(s/2)L(\chi,s).$$

Then clearly this is equivalent to a twisted sum over ideal classes of completed partial zeta functions. That is

$$\Lambda_K(s,\chi) = \sum_{[I]\in Cl^+(K)} \chi([I])\Lambda_{K,I}(s).$$

So, given our previous results we see that this has a meromorphic continuation to $\mathbb{C}$, which is entire except for possibly at $s = 0$ and $s = 1$ (this is only the case when $\chi = \chi_0$, the trivial character). We can also see that the functional equation is, via theorem 7.1.7 [3, page 503, Corollary 8.6]

$$\Lambda_K(s,\chi) = \chi(\mathfrak{d})\Lambda_K(1-s,\chi^{-1}).$$

This is analogous to the Dirichlet L-series case that we considered before, but is now defined on field extensions K of $\mathbb{Q}$.

We have shown that the Dedekind zeta function and Hecke L-function carry non-trivial information on the prime ideals of the field extensions, and are therefore of great importance. We have also seen that, similarly to the rest of the zeta functions and L-functions considered, that they have continuations and satisfy functional equations. In the next chapter we will specialise our results and see more concrete examples for the case where K is a quadratic field over $\mathbb{Q}$.

8. The quadratic field theory case

In this chapter we will concentrate on quadratic fields over $\mathbb{Q}$, and see that we need to discuss the two cases of real fields and imaginary fields separately. We begin by detailing some general results that hold for both cases before discussing each in more detail.

8.1 General results

Theorem 8.1.1
Let $\zeta_K(s)$ be the Dedekind zeta function of a quadratic field $K = \mathbb{Q}[\sqrt{D}]$. Then we have

$$\zeta_K(s) = \zeta(s)L(\chi_D, s).$$

Proof Consider the Euler product formula:

$$\zeta_K(s) = \prod_{\mathfrak{P}} \frac{1}{1 - N(\mathfrak{P})^{-s}},$$

where the product is taken over all prime ideals in $\mathcal{O}_K$. There are three different cases for the splitting of primes into prime ideals in a quadratic field (see theorem 2.1.3). From these we have a decomposition into three distinct products (since a prime is in exactly one of the three cases).

$$\zeta_K(s) = \prod_{\mathfrak{P}} \frac{1}{1 - N(\mathfrak{P})^{-s}} = \prod_{\substack{p \\ (\frac{D}{p})=-1}} \frac{1}{1 - p^{-2s}} \prod_{\substack{p \\ (\frac{D}{p})=1}} (\frac{1}{1 - p^{-s}})^2 \prod_{\substack{p \\ (\frac{D}{p})=0}} \frac{1}{1 - p^{-s}}.$$

Now consider the right-hand-side of equation 8.1.1, and expand the Euler products. This gives us:

$$\begin{aligned}
\zeta(s)L(\chi_D, s) &= \prod_p \frac{1}{1 - p^{-s}} \prod_p \frac{1}{1 - (\frac{D}{p})p^{-s}} \\
&= \prod_p (\frac{1}{1 - p^{-s}})(\frac{1}{1 - (\frac{D}{p})p^{-s}}) \\
&= \prod_p \frac{1}{1 - (\frac{D}{p})p^{-s} + (\frac{D}{p})p^{-2s} - p^{-s}}, \\
&= \zeta_K(s)
\end{aligned}$$

where the last equality can simply be checked on a case-by-case basis for each value of $(\frac{D}{p})$ and comparing with the expanded Euler product of $\zeta_K(s)$ from above. ■

8.1.1 Class number formula for quadratic fields

Before our main exposition of quadratic forms associated to ideals in quadratic fields and their theta functions, we briefly consider the class number formula for quadratic fields.
We have seen that in general the class number for a number field extension of finite degree of $\mathbb{Q}$ can be found by considering the residue at $s = 1$ of $\zeta_K(s)$ (see corollary 7.1.9). For quadratic extensions of $\mathbb{Q}$ this result becomes split for imaginary and real fields. Letting $\omega = |\mathcal{O}_K^*|$ for imaginary quadratic fields we immediately see that

$$\mathrm{Res}_{s=1}\zeta_K(s) = \begin{cases} \dfrac{2\pi}{\omega\sqrt{|D|}}h & \text{if } K \text{ is imaginary} \\ \dfrac{\log\epsilon}{\sqrt{D}}h & \text{if } K \text{ is real,} \end{cases}$$

since the regulator of an imaginary quadratic field is 1, and of a real quadratic field is $\log\epsilon$ (where ϵ is the fundamental unit).
On the other hand, by theorem 8.1.1 we have that $\zeta_K(s) = \zeta(s)L(\chi_D, s)$. Now we take residues of each side at $s = 1$, and use our section on the Riemann zeta function to see that it has residue 1, to obtain

$$L(1, \chi_D) = \begin{cases} \dfrac{2\pi}{\omega\sqrt{|D|}}h & \text{if } K \text{ is imaginary} \\ \dfrac{\log\epsilon}{\sqrt{D}}h & \text{if } K \text{ is real.} \end{cases}$$

So we have a formula for the class number of a quadratic field in terms of the L-function, $L(1, \chi_D)$:

$$h = \begin{cases} \dfrac{\omega\sqrt{|D|}}{2\pi}L(1, \chi_D) & \text{if } K \text{ is imaginary} \\ \dfrac{\sqrt{D}}{\log\epsilon}L(1, \chi_D) & \text{if } K \text{ is real.} \end{cases}$$

Example 8.1.2
We continue with our example of discriminant $D = -23$. It is well known that there are three equivalence classes in the ideal class group of $K = \mathbb{Q}[\sqrt{23}]$, i.e. $h = 3$. Since $D < -3$ we have from theorem 2.1.4 that $\omega = 2$ so that our formula becomes

$$L(1, \chi_{23}) = \frac{3\pi}{\sqrt{23}}.$$

Remark This gives an effective way to compute the value of our L-function at $s = 1$ when the class number is known, and equally the class number when we know the value of our L-function at $s = 1$. Dirichlet managed to show that the value of $L(1, \chi)$ is finite for primitive Dirichlet characters χ [28, page 2, Lemmas 1.5 and 1.6] implying the finiteness of the class number for quadratic fields.

8.1.2 Genus characters

We briefly consider the case where the character of Hecke L-series is of order 2 - a so-called genus character. Here, $D = D_1 D_2$ is the product of two discriminants. The character χ_D is defined as [5, page 219, equation 12.59]

$$\chi_D = \begin{cases} \chi_{D_1}(N(\mathfrak{P})) & \text{if } \mathfrak{P} \nmid D_1 \\ \chi_{D_2}(N(\mathfrak{P})) & \text{if } \mathfrak{P} \nmid D_2. \end{cases}$$

Theorem 8.1.3
Let D be the product of two discriminants, i.e. $D = D_1 D_2$. Then the Hecke L-series decomposes as

$$L_K(s, \chi_D) = L_K(s, \chi_{D_1}) L_K(s, \chi_{D_2}).$$

Proof Consider the Euler product definition of the LHS and proceed case-by-case for the splitting of the prime ideals (using theorem 2.1.3), as in [27, page 60]. For simplicity denote $\chi_{D_1} = \chi_1$ and $\chi_{D_2} = \chi_2$.

$$\begin{aligned} L_K(s, \chi_D) &= \prod_{\mathfrak{P}} \frac{1}{1 - \chi_D(\mathfrak{P}) N(\mathfrak{P})^{-s}} \\ &= \prod_p \prod_{\mathfrak{P}|(p)} \frac{1}{1 - \chi_D(\mathfrak{P}) N(\mathfrak{P})^{-s}}. \end{aligned}$$

Now consider the case where (p) is inert. That is, $\chi_D(\mathfrak{p}) = -1$ and $N(\mathfrak{P}) = p^2$. Then one of χ_1 and χ_2 is 1 while the other must be -1. Then the inner Euler product here becomes

$$\begin{aligned} \prod_{\mathfrak{P}|(p)} \frac{1}{1 - \chi_D(\mathfrak{P}) N(\mathfrak{P})^{-s}} &= \frac{1}{1 - p^{-2s}} \\ &= \frac{1}{1 - p^{-s}} \frac{1}{1 + p^{-s}} \\ &= \frac{1}{1 - \chi_1(p) p^{-s}} \frac{1}{1 - \chi_2(p) p^{-s}}. \end{aligned}$$

If, instead, p splits completely then we have $\chi_D(p) = 1$ so $\chi_1(p) = \chi_2(p)$. Then

$$\prod_{\mathfrak{P}|(p)} \frac{1}{1 - \chi_D(\mathfrak{P}) N(\mathfrak{P})^{-s}} = \frac{1}{1 - \chi_1(p) p^{-s}} \frac{1}{1 - \chi_2(p) p^{-s}}.$$

The final case is where p is ramified in K. Here, $\chi_D(p) = 0$, and so $p|D_1$ or $p|D_2$. Assuming that $p|D_2$ (the argument is precisely the same for $p|D_1$), we have that $\chi_2(p) = 0$ and that its factor in the Euler product is 1. Hence

$$\prod_{\mathfrak{P}|(p)} \frac{1}{1 - \chi_D(\mathfrak{P}) N(\mathfrak{P})^{-s}} = \frac{1}{1 - \chi_1(p) p^{-s}} \frac{1}{1 - \chi_2(p) p^{-s}}.$$

Combining the three cases above we have that

$$\begin{aligned} L_K(s,\chi_D) &= \prod_p \prod_{\mathfrak{P}|(p)} \frac{1}{1-\chi_D(\mathfrak{P})N(\mathfrak{P})^{-s}} \\ &= \prod_p \frac{1}{1-\chi_1(p)p^{-s}} \frac{1}{1-\chi_2(p)p^{-s}} \\ &= L(\chi_1,s)L(\chi_2,s) \end{aligned}$$

■

Example 8.1.4
Take $K = \mathbb{Q}[\sqrt{7}]$ with discriminant 28 since $7 \equiv 3 \mod 4$. One can verify that the narrow class number $h_0 \neq h$ is 2, and so there is a genus character. The trivial character χ_0 acts as the identity and we have already seen that

$$L_K(\chi_0,s) = \zeta_K(s) = \zeta(s)L(\chi_7,s).$$

Note that

$$\chi_{28} = \left(\frac{28}{}\right) = \left(\frac{2}{-}\right)^2 \left(\frac{7}{-}\right) = \chi_7$$

Looking at the genus character, which we denote by χ, we see that it acts on principal ideals (α) as:

$$\chi((\alpha)) = \begin{cases} +1 & \text{if } N(\alpha) > 0 \\ -1 & \text{if } N(\alpha) < 0. \end{cases}$$

We take the two representative ideals of the narrow class group $I = (1)$ and $J = (\sqrt{7})$ with $I \not\sim J$. Note that $N(I) = 1 > 0$ and $N(J) = -7 < 0$. So $\chi(I) = 1$ and $\chi(J) = -1$. Consider the three separate cases of the splitting of (p):

1. When $\left(\frac{7}{p}\right) = -1$ then the ideal (p) is inert and already prime in the extension. Then it is has norm $p^2 > 0$ and the genus character χ acts as the identity.

2. When $\left(\frac{7}{p}\right) = 1$ then (p) splits as $\mathfrak{P}\mathfrak{P}'$.

a) If $\mathfrak{P} \in [I]$ then we are looking for solutions of $x^2 - 7y^2 = p$, where we see that p is a quadratic residue $\mod 7$, i.e. $\left(\frac{p}{7}\right) = 1$.

b) If $\mathfrak{P} \in [J]$ then we are looking for solutions of $x^2 - 7y^2 = -p$, where we see that $-p$ is a quadratic residue $\mod 7$, i.e. $\left(\frac{-p}{7}\right) = 1$.

Both conditions cannot hold at the same time. To see this, we can compute the squares modulo 7.

x mod 7	x^2 mod 7
0	0
1	1
2	4
3	2
4	2
5	4
6	1

We note that there are no pairs of $\pm p \mod 7$*, apart from* $p \equiv 0$*. However choosing* $p \equiv 0 \mod 7$ *forces* $\left(\frac{p}{7}\right) = 1$*. In both cases we know that* $\left(\frac{7}{p}\right) = 1$ *and using Euler's quadratic reciprocity law we can write this as*

$$1 = \left(\frac{p}{7}\right)(-1)^{\frac{p-1}{2}}.$$

So we need $p \equiv 1 \mod 4$ *for a) to hold, implying* $\chi_{-4}(p) = 1$*. For b) to hold we need* $p \equiv 3 \mod 4$*, giving us that* $\chi_{-4}(p) = -1$*.*

3. The ramified primes are those dividing 28*, i.e.* 2 *and* 7*. An element of norm* 2 *is given by* $3 + \sqrt{7}$ *and so* $\chi(3 + \sqrt{7}) = 1$*. Meanwhile, we have that* $(\sqrt{7})|(7)$*.*

Putting these three cases together in the definition of $L_K(\chi, s)$ *we obtain*

$$L_K(\chi, s) = \prod_p \frac{1}{1 - \chi_{-7}(p)p^{-s}} \frac{1}{1 - \chi_{-4}(p)p^{-s}},$$

and we note that this agrees with our theorem above as $28 = -7 \times -4$*.*

A further discussion of genus characters and their implications is given at [5, page 220].

8.1.3 Building quadratic forms from ideals

We begin by defining the construction of a quadratic form associated to an ideal. It is well known that any integral ideal of a quadratic field is generated by at most two elements, $I = (\alpha, \beta) \in \mathcal{O}_K$. The construction of the quadratic form will be the same in both the real and imaginary cases.

Definition Let $I = (\alpha, \beta) = \mathbb{Z}\alpha + \mathbb{Z}\beta$ be an integral ideal in $\mathcal{O}_K$. Then we define the **quadratic form associated to** I as

$$Q_I(x, y) = \frac{N(x\alpha + y\beta)}{N(I)} \tag{8.1.5}$$

Notice that under the action of $\gamma \in SL(2,\mathbb{Z})$ we have that the quadratic form is unchanged:

$$\begin{aligned} Q_{\gamma I}(x,y) &= \frac{N(x(\gamma\alpha)+y(\gamma\beta))}{N(\gamma I)} \\ &= \frac{N(\gamma)N(x\alpha+y\beta)}{N(\gamma)N(I)} \\ &= \frac{N(x\alpha+y\beta)}{N(I)} \\ &= Q_I(x,y) \end{aligned}$$

We note a particularly important lemma that occurs in both cases of quadratic fields:

Lemma 8.1.6
Let K be a quadratic field (either real or imaginary). Then there is a bijection

$$\{Q_I(x)=n\}/\mathcal{O}_K^{*,+} \xrightarrow{\sim} |\{J \in [I] | N(J)=n\}|.$$

Proof Let J be in the class group $[I]^{-1}$ such that IJ is principal. So we have that $IJ=(\xi)$ for some $\xi \in \mathcal{O}_K$, with $N(\xi)>0$. Therefore $J=(\xi)I^{-1}$. Now we can see that J is a rational ideal precisely when $I|(\xi)$. If we again define a map

$$\begin{aligned} \pi : \{\xi \in [I] | N(\xi)>0\} &\to \{J \in [I]^{-1} | J \text{ is integral}\} \\ \xi &\mapsto (\xi)I^{-1} \end{aligned}$$

we see that it is surjective and well-defined, i.e. two elements ξ_1 and ξ_2 have the same image under π if and only if $\xi_1=\epsilon\xi_2$ for $\epsilon \in \mathcal{O}_K^*$ and $\epsilon^{-1} \in \mathcal{O}_K^*$. So we have our bijection between $\{Q_I(x)=n\}/\mathcal{O}_K^{*,+}$ and $\{J \in [I] | N(J)=n\}$, proving the lemma. ■

We call $|\{Q_I(x)=n\}|$ the representation number of n of Q_I - it is the number of solutions to the equation $Q_I(x)=n$. Notice that this is the same as in the discussions of quadratic forms above. For real quadratic fields the representation numbers will be infinite and this will cause us problems.
Now we are ready to discuss the two separate cases, beginning with the easier case - imaginary quadratic fields.

8.2 Imaginary quadratic fields

The theory of theta functions and modular forms is far simpler for imaginary quadratic fields than it is for real quadratic fields, as we will see in these chapters. Our main objective here is to show that the modular forms arising from the imaginary fields are holomorphic. We begin by forming quadratic forms and theta functions from ideals in the field extension of $\mathbb{Q}$ and then relate these to L-functions of modular forms. Throughout this section we let $K=\mathbb{Q}(\sqrt{D})$ be an imaginary quadratic field (so $D<0$). This ensures that we also need only sum over the standard class group for our L-series since

$Cl^+(K) = Cl(K)$.
The norm map in imaginary quadratic fields is given by:

$$N(a + b\sqrt{D}) = (a + b\sqrt{D})(a - b\sqrt{D}) = (a^2 - b^2 D) > 0 \text{ for } (a, b) \neq (0, 0).$$

This forces every quadratic form in our construction to be positive definite here - and therefore we have finite representation numbers.

Remark There is a one-to-one correspondence between primitive ideals and positive definite quadratic forms of discriminant $D \leq 0$.

Recall our definition of a theta function associated to a positive definite quadratic form Q_I:

$$\theta_{Q_I}(\tau, L) = \sum_{x \in L} e^{2\pi i Q_I(x)\tau} = \sum_{n=0}^{\infty} r_{Q_I}(n) q^n. \tag{8.2.1}$$

In particular, this converges since our quadratic form is positive definite hence the representation numbers are finite for fixed n. The theta function is holomorphic on $\hat{\mathbb{H}}$.

8.2.1 Dedekind zeta for imaginary quadratic fields

Using our definition of the Dedekind zeta function we observe that it can be split into the sum over ideal classes:

$$\zeta_K(s) = \sum_{I \in \mathcal{O}_K} \frac{1}{N(I)^s} = \sum_{[I] \in Cl(K)} (\sum_{J \in [I]} \frac{1}{N(J)^s}).$$

Here we see that the partial zeta function is

$$\begin{aligned}
\zeta_{K,I}(s) &= \sum_{J \in [I]} \frac{1}{N(J)^s} \\
&= \sum_{n=1}^{\infty} (\sum_{\substack{J \in [I] \\ N(J) = n}} J) n^{-s} \\
&= \sum_{n=1}^{\infty} |\{J \in [I] | N(J) = n\}| n^{-s} \\
&= \frac{1}{\omega} \sum_{n=1}^{\infty} \frac{r_{Q_I}(n)}{n^s} \qquad \text{by lemma 8.1.6.}
\end{aligned}$$

Then we have

$$\zeta_K(s) = \sum_{[I] \in Cl(K)} \zeta_{K,I}(s) = \frac{1}{\omega} \sum_{[I] \in Cl(K)} (\sum_{n=1}^{\infty} \frac{r_{Q_I}(n)}{n^s}), \tag{8.2.2}$$

where each class $[I]$ in the class group $Cl(K)$ is represented by a quadratic form Q_I with representation numbers $r_{Q_I}(n)$, and $\omega = |\mathcal{O}_K^*|$.

Now consider the L-function associated to a theta function θ_{Q_I}:

$$L(\theta_{Q_I}, s) = \sum_{n=1}^{\infty} \frac{r_{Q_I}(n)}{n^s}.$$

We have that $\zeta_{K,I}$ is a scalar multiple of the L-series associated to θ_{Q_I}. Explicitly

$$\zeta_{K,I} = \frac{1}{\omega} L(\theta_{Q_I}, s).$$

We have given a proof of the following theorem:

Theorem 8.2.3
Let K be an imaginary quadratic field with ring of integers $\mathcal{O}_K$, and $\omega = |\mathcal{O}_K^*|$. Then

$$\zeta_K(s) = \frac{1}{\omega} \sum_{[I] \in Cl(K)} L(\theta_{Q_I}, s).$$

Definition We define

$$f_{\chi_0}(\tau) = \frac{1}{\omega} \sum_{[I] \in Cl(K)} \theta_{Q_I}(\tau).$$

Then we have the following

Proposition 8.2.4
Let $f_{\chi_0}(\tau)$ be defined as above. Then $\zeta_K(s)$ is the L-series associated to $f_{\chi_0}(\tau)$, and $f_{\chi_0}(\tau) \in \mathcal{M}_1(|D|, \chi_D)$.

Proof First, we note that by construction we have that $\zeta_K(s)$ is the L-series associated to $f_{\chi_0}(\tau)$. To see that we have a weight one modular form we refer back to our section on theta functions. Since here we are considering a linear sum of theta functions, we first consider each theta function separately.
For a holomorphic theta function of this type associated to a positive definite integral binary quadratic form we have, by theorem 5.1.9, that it is a modular form of weight 1 on some $\Gamma_0(N)$ and character χ_D.
It remains to show that this N is the same for all theta functions in our sum. This must be the case, since each different theta is associated to a matrix of the same determinant, D. So certainly, NA^{-1} is integral for some N for all the theta functions. In fact, taking the value $N = |D|$ will clearly suffice.
Therefore, we have that $f_{\chi_0}(\tau)$ is indeed a modular form of weight 1 on the congruent subgroup of $\Gamma_0(|D|)$ with character χ_D. ∎

8.2.2 Hecke L-Series

We can further generalise the above results by taking a twist of $\zeta_K(s)$. Recall that for a Hecke character $\chi(I)$ on the class group $Cl(K)$ we have the Hecke L-series

$$L_K(\chi, s) = \sum_I \frac{\chi(I)}{N(I)^s} = \prod_{\mathfrak{P}} \frac{1}{1 - \chi(\mathfrak{P})N(\mathfrak{P})^{-s}}.$$

Since χ is a class function on the class group - and here the narrow class group is the same as the standard class group - we have that

$$L_K(\chi, s) = \sum_{[I] \in Cl(K)} \chi([I])(\sum_{J \in [I]} \frac{1}{N(J)^s}).$$

Much of the computation stays the same as in the Dedekind zeta function case, and we define the following:

$$L_K(\chi, s) = \sum_{[I] \in Cl(K)} \chi([I])\zeta_{K,I}(s),$$

and

$$f_\chi(\tau) = \frac{1}{\omega} \sum_{[I] \in Cl(K)} \chi([I])\theta_{Q_I}(\tau),$$

since the character $\chi([I])$ simply factors through in the calculations.
In much the same vein as before, we have:

Theorem 8.2.5
Let $f_\chi(\tau)$ be as defined above. Then $L_K(\chi, s)$ is the L-function associated to $f_\chi(\tau)$, and $f_\chi(\tau) \in \mathcal{M}_1(|D|, \chi_D)$.

Proof Similarly to before, we know that each individual theta function in the linear sum is a modular form of weight 1 on the congruent subgroup $\Gamma_0(|D|)$. Then since we take a twist of this modular form by a character on the ideal class group we can appeal to theorem 3.1.7. Then we still have a modular form of weight 1 on $\Gamma_0(|D|)$ with character χ_D. ■

Remark For χ not real it can be shown that $f_\chi(\tau)$ is in fact a primitive cusp form [5, page 213, Theorem 12.5].

Example 8.2.6
Take $K = \mathbb{Q}[\sqrt{-23}]$. We have already seen that the class number of K is $h = 3$, and that two of the quadratic forms represent the same integers. From example 5.1.11 we have our two theta functions as

$$\theta_A = 1 + 2q + 0q^2 + 0q^3 + 2q^4 + 0q^5 + 4q^6 + ...$$

and

$$\theta_B = 1 + 0q + 2q^2 + 2q^3 + 2q^4 + 0q^5 + 2q^6 + ...$$

with each theta belonging to $\mathcal{M}_k(23)$*. We know* $\omega = 2$ *and form*

$$\begin{aligned} f_{\chi_0}(\tau) &= \frac{1}{\omega} \sum_{[I] \in Cl(K)} \theta_{Q_I}(\tau, L) \\ &= \frac{1}{2}(\theta_A + 2\theta_B) \\ &= \frac{3}{2} + q + 2q^2 + 2q^3 + 3q^4 + ... \end{aligned}$$

and we have that $f_{\chi_0}(\tau) \in \mathcal{M}_1(23, \chi_{-23})$*. It is clearly not a cusp form since it does not vanish at* ∞*.*
For the non-trivial characters we have an action of $e^{\pm 2\pi i/3}$ *on the two theta functions that represent the same integers. Hence when we form* $f_\chi(\tau)$ *we use that* $1 + e^{2\pi i/3} + e^{-2\pi i/3} = 0$ *to obtain*

$$\begin{aligned} f_{\chi}(\tau) &= \frac{1}{\omega} \sum_{[I] \in Cl(K)} \chi([I])\theta_{Q_I}(\tau, L) \\ &= \frac{1}{2}(\theta_A - \theta_B) \\ &= q - q^2 - q^3 + q^6 + ... \end{aligned}$$

Our theorem now tells us that $f_\chi(\tau) \in \mathcal{M}_1(23, \chi_{-23})$*. In fact, since* χ *is not real, we have that this is a cusp form here, and it arises with an L-series*

$$L(f_\chi, s) = \prod_p \frac{1}{1 - a_p p^{-s} + \chi_{-23}(p)p^{-2s}},$$

with

$$a_p = \begin{cases} 1 & \textit{if } p = 23 \\ 0 & \textit{if } \left(\frac{p}{23}\right) = -1 \\ 2 & \textit{if } \left(\frac{p}{23}\right) = 1 \textit{ and } p \textit{ can be represented as } x^2 + xy + 6y^2 \\ -1 & \textit{if } \left(\frac{p}{23}\right) = 1 \textit{ and } p \textit{ can be represented as } 2x^2 + xy + 3y^2. \end{cases}$$

Example 8.2.7
Take $K = \mathbb{Q}[\sqrt{-5}]$ *of discriminant* -20*. We have seen in example 5.1.12 that we our two theta functions associated to quadratic forms of discriminant* -20 *are given by*

$$\theta_A(\tau) = 1 + 2q + 0q^2 + 0q^3 + 2q^4 + 2q^5 + 4q^6 + ...$$

and

$$\theta_B(\tau) = 1 + 0q + 2q^2 + 2q^3 + 2q^4 + 0q^5 + 2q^6 + ...$$

Since we know that the class number of K *is* 2*, these are the only two non-equivalent thetas. We also know that we have only the trivial character and genus character on the*

class group since $Cl(K) \cong \mathbb{Z}_2$.
Looking at the trivial character first, we have

$$\begin{aligned} f_{\chi_0}(\tau) &= \frac{1}{\omega} \sum_{[I]\in Cl(K)} \theta_{Q_I}(\tau, L) \\ &= \frac{1}{2}(\theta_A + \theta_B) \\ &= 1 + q + q^2 + q^3 + 2q^4 + q^5 + 3q^6 + ... \end{aligned}$$

and that $f_{\chi_0}(\tau) \in \mathcal{M}_1(20, \chi_{-5})$.
Similarly to before we now consider the second character, here a genus character

$$\begin{aligned} f_{\chi}(\tau) &= \frac{1}{\omega} \sum_{[I]\in Cl(K)} \chi([I])\theta_{Q_I}(\tau, L) \\ &= \frac{1}{2}(\theta_A - \theta_B) \\ &= 2q - 2q^2 - 2q^3 + 2q^5 + 2q^6 + ... \end{aligned}$$

Here we have that $f_{\chi}(\tau) \in \mathcal{M}_1(20, \chi_{-5})$ *by our theorem.*

We can go even further than this, and show that the Hecke L-series actually arises as a twisted sum of Eisenstein series. For any primitive ideal $I = a(1, \frac{b+\sqrt{D}}{2a})$ we consider the twisted sum [11, Section 22.3]

$$\sum_{\chi} \chi(I) L_K(\chi, s) = h \sum_{I\sim J} \frac{1}{N(J)^s}.$$

Now, since we have that $J = (a)I$ for some $a \in \mathcal{O}_K$ we can rewrite this as

$$\begin{aligned} h \sum_{I\sim J} \frac{1}{N(J)^{-s}} &= \frac{h}{\omega} a^{-s} \sum_{\substack{\alpha\in I^{-1} \\ \alpha\neq 0}} \frac{1}{|\alpha|^{2s}} \\ &= \frac{h}{\omega} \sum_{(m,n)\neq(0,0)} \frac{1}{|m\tau + n|^{2s}}, \end{aligned}$$

where $\tau = \frac{b+\sqrt{D}}{2a} = \frac{b}{2a} + i\frac{\sqrt{|D|}}{2a}$.
Then we have that

$$\begin{aligned} \sum_{(m,n)\neq(0,0)} \frac{1}{|m\tau + n|^{2s}} &= \zeta(2s) \sum_{\gcd(m,n)=1} \frac{1}{|m\tau + n|^{2s}} \\ &= 2\zeta(2s)(\frac{\sqrt{|D|}}{2a})^{-s} \frac{1}{2} \sum_{\gcd(m,n)=1} \frac{(\frac{\sqrt{|D|}}{2a})^s}{|m\tau + n|^{2s}} \\ &= 2\zeta(2s)(\frac{\sqrt{|D|}}{2a})^{-s} E_s(\tau), \end{aligned}$$

with $E_s(\tau) = \sum_{\gcd(m,n)=1} \frac{v^s}{|m\tau+n|^{2s}}$ the Eisenstein series for the modular group Γ.
Combining these results gives us that

$$\sum_{\chi} \chi(I) L_K(\chi, s) = \frac{2h}{\omega} \zeta(2s) \left(\frac{\sqrt{|D|}}{2a}\right)^{-s} E_s(\tau).$$

This expression can be inverted [11, page 511, equation 22.45] to obtain the following expression of Λ as a function of Eisenstein series

$$\Lambda_K(\chi, s) = \frac{2}{\omega} \sum_{I} \chi(I) \pi^{-s} \Gamma(s) \zeta(2s) E_s(\tau_I),$$

where we sum over a set of representatives inequivalent primitive ideals, and τ_I is clearly dependent on our ideal I. Hence we see that the Eisenstein series inherits properties of the analytics of the L-function (and vice-versa).

8.3 Real quadratic fields

Now we turn our attention to real quadratic fields $K = \mathbb{Q}[\sqrt{D}]$ with $D > 0$ a discriminant. Here we will discover that the theta function we used for the imaginary quadratic case will no longer converge. We will define a new theta function, leaving the theory of L-functions behind here (of course, we have covered some general aspects of these in generality in previous sections) - and we will have to work hard to obtain our first example of a Maass form.
We begin by noting that much of the theory from the imaginary case no longer holds, since the quadratic forms are now indefinite. We do, however, still have that our Hecke L-series is sum of the L-functions of the theta functions (one theta for each representative in the narrow class group). That is,

$$L_K(s) = \sum_{[I] \in Cl^+(K)} \chi([I]) \sum_{J \in [I]} \frac{1}{N(J)^s}$$

is the L-function of

$$f_\chi(\tau) = \sum_{[I] \in Cl^+(K)} \chi([I]) \theta_{Q_I}(\tau),$$

for some theta function that we define below.
We have the same construction of quadratic forms from ideals as before. However, a generic quadratic form here will now be indefinite (that is, it admits both positive and negative values).

Example 8.3.1
The prototypical example of an indefinite form is $Q(x) = x^2 - y^2$. This has an associated matrix $A = \left(\begin{smallmatrix} 2 & 0 \\ 0 & -2 \end{smallmatrix}\right)$ of signature $[1, -1]$. Clearly, $Q(x)$ can take both positive and negative values, e.g. $Q((1,0)) = 1^2 - 0^2 = 1$ whilst $Q((0,1)) = 0^2 - 1^2 = -1$.

We would like to use the same approach as in the imaginary quadratic case. Naively, we begin with the theta function as before

$$\theta_{Q_I}(\tau, L) = \sum_{x \in L} e^{2\pi i Q_I(x)\tau} = \sum_{n=0}^{\infty} r_{Q_I}(n) q^n$$

However, now we see that since our representation numbers are infinite this theta diverges. We must think of a new strategy to define a theta functions for indefinite quadratic forms.
Since we know how to deal with positive definite quadratic forms we will look at how we can relate indefinite forms to positive definite ones.
Begin with an indefinite integral binary quadratic form $Q(x)$. We can look at its associated matrix A of signature $[1,1]$ and diagonalise it to get $A = UDU^{-1}$ with D a diagonal matrix. D will be of the form

$$D = \begin{pmatrix} a & 0 \\ 0 & -b \end{pmatrix}.$$

Now we can relate $Q(x)$ to a positive definite quadratic form by simply changing the sign of the negative eigenvalue of D. Here, we let

$$D^* = \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right).$$

So we end up with a matrix $A^* = UD^*U^{-1}$ which now has signature $[2,0]$, i.e. is positive definite. We denote this positive definite quadratic form by $Q_0(x)$.
However, this diagonalisation process is not unique and we must account for this non-uniqueness. Consider the group orthogonal to $Q(x)$ of determinant 1, $SO(Q)$. We have that

$$SO(Q) = \{B \in SL(2,\mathbb{R}) | B^T D B = D\}.$$

Letting $B = \left(\begin{smallmatrix} c & d \\ e & f \end{smallmatrix}\right)$ we find that we have several conditions on $c, d, e, f \in \mathbb{R}$. For ease of exposition we can take $Q(x) = x^2 - y^2$. Then the conditions on the elements of B will, after some simple algebra, reduce down to

$$\begin{cases} cf - de = 1 \\ cd - ef = 0 \\ c^2 - e^2 = 1 \\ d^2 - f^2 = -1. \end{cases}$$

These conditions can be satisfied by either taking

$$c = f = -1 \ , \ d = e = 0$$

or

$$c = f = \cosh x \ , \ d = e = \sinh x,$$

$x \in \mathbb{R}$, or some combination of the two. Explicitly then, we have

$$SO(Q) = \left\langle \begin{pmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle,$$

for $x \in \mathbb{R}$.
We will denote the connected component of this group by

$$SO_0(Q) = \left\langle \begin{pmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{pmatrix} \right\rangle \cong \mathbb{R}.$$

For $g_x \in SO_0(Q)$ we let $Q_x(v) = Q_0(g_x^{-1}v)$. Now we are ready to define our theta function for indefinite quadratic forms. Brunier and Funke define a general theta function for a general indefinite form of signature $[p, q]$ at [29, page 54], including extra variables z and h. We take this definition and immediately specialise to the case of signature $[1, 1]$, and take $z = h = 0$.

Definition Let $Q(x)$ be an indefinite binary quadratic form, $\tau = u + iv \in \mathbb{H}$ and L a lattice in $\mathbb{R}^2$. Then we define its theta function as

$$\theta(\tau, x, L) = \sqrt{v} \sum_{\lambda \in L} e^{2\pi i Q(\lambda) u} e^{-2\pi Q_x(\lambda) v}.$$

In particular, notice that this will now converge - since the indefinite form Q is only attached to a term of $e^{2\pi i}$, and the form Q_x being attached to a negative exponential ensures convergence of the second part. Also note the square root term before the sum - this ensures modularity of θ on $\Gamma_0(D)$. However, in direct contrast to the imaginary quadratic field case, this theta function is now non-holomorphic.
We know that the automorphism group of a theta function $L \in \mathcal{O}_K$ is given by $\mathrm{Aut}(L) = \mathrm{Aut}(\mathcal{O}_K^*) = \mathcal{O}_K^{*,+}$, since automorphisms must preserve both the size and sign of units. Clearly $\mathrm{Aut}(L) \subset SO_0(Q)$ since they are of determinant 1 by definition of SO and the automorphisms are orthogonal to Q. We can also immediately see an invariance of theta. By applying $\epsilon \in SO_0(Q)$ to x we have

$$\begin{aligned} \theta(\tau, \epsilon x, L) &= \sqrt{v} \sum_{\lambda \in L} e^{2\pi i Q(\lambda) u} e^{-2\pi Q_{\epsilon x}(\lambda) v} \\ &= \sqrt{v} \sum_{\lambda \in L} e^{2\pi i Q(\lambda) u} e^{-2\pi Q_x(\epsilon^{-1}\lambda) v} \\ &= \theta(\tau, x, L), \end{aligned}$$

where we have used that, for $\epsilon \in SO_0(Q)$,

$$\begin{aligned} Q_{\epsilon x}(\lambda) &= Q_0((\epsilon x)^{-1}\lambda) \\ &= Q_0(x^{-1}\epsilon^{-1}\lambda) \\ &= Q_x(\epsilon^{-1}\lambda). \end{aligned}$$

Since we have these conditions, we can look to integrate away the symmetries of theta. We take the integral over $\mathcal{O}_K^{*,+}\backslash SO_0(Q)$ and define

$$I(\tau, L) = \int\limits_{\mathcal{O}_K^{*,+}\backslash SO_0(Q)} \theta(\tau, x, L)dx.$$

To compute this integral we must use a process known as 'unfolding', where we look at the isomorphisms of the group we are integrating over. Here, we have that

$$\mathcal{O}_K^{*,+}\backslash SO_0(Q) \cong \log\epsilon\backslash\mathbb{R} \cong \mathbb{Z}\backslash\mathbb{R}$$

where ϵ is the fundamental unit of K. So we are integrating over a circle. It is clear that theta, and therefore $I(\tau, L)$, has the invariance when replacing $\tau \mapsto \tau + 1$. Hence we expect it to arise with a Fourier series of the form

$$I(\tau, L) = a_0(v)\sqrt{v} + \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} a_n(v)e^{2\pi i n u}\sqrt{v}.$$

Now we need to compute what these $a_n(v)$ are. Begin with the expression

$$\begin{aligned} a_n(v) &= \int\limits_{\mathbb{Z}\backslash\mathbb{R}} \sum_{\substack{\lambda\in L\\ N(\lambda)=n}} e^{-2\pi Q_x(\lambda)v}dx \\ &= \int\limits_{\mathbb{Z}\backslash\mathbb{R}} \sum_{b\in\mathcal{O}_K^*} \sum_{\substack{\lambda\in\mathcal{O}_K\\ N(\lambda)=n\\ \bmod \mathcal{O}_K^*}} e^{-2\pi Q_x(b\lambda)v}dx \\ &= \int\limits_{\mathbb{Z}\backslash\mathbb{R}} \sum_{b\in\mathcal{O}_K^*} \sum_{\substack{\lambda\in\mathcal{O}_K\\ N(\lambda)=n\\ \bmod \mathcal{O}_K^*}} e^{-2\pi Q_{b^{-1}x}(\lambda)v}dx. \end{aligned}$$

Now we use the fact that

$$\int_{\mathbb{Z}\backslash\mathbb{R}} \sum_{n\in\mathbb{Z}} f(x+n)dx = \int_{\mathbb{R}} f(x)dx$$

to find that we have, after taking the finite sum term in front of the integral:

$$a_n(v) = \sum_{\substack{\lambda\in\mathcal{O}_K\\ N(\lambda)=n\\ \bmod \mathcal{O}_K^*}} \int_{\mathbb{R}} e^{-2\pi Q_x(\lambda)v}dx.$$

We concentrate on the integral term in the formula for $a_n(v)$. We can assume that $\lambda = (\pm\sqrt{|n|}, 0)$, of length n. Then we have that, for any x, using the definition $Q_x(\lambda)$:

$$\begin{aligned}
Q_x(\lambda) &= Q_0(\left(\begin{smallmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{smallmatrix}\right)^{-1} \lambda) \\
&= Q_0(\left(\begin{smallmatrix} \cosh x & -\sinh x \\ -\sinh x & \cosh x \end{smallmatrix}\right) (\pm\sqrt{|n|}, 0)^T) \\
&= Q_0(\pm\sqrt{|n|}\cosh x, \mp\sqrt{|n|}\sinh x) \\
&= |n|(\cosh^2 x + \sinh^2 x).
\end{aligned}$$

So we now look at the integral

$$\begin{aligned}
\int_{\mathbb{R}} e^{-2\pi|n|v(\cosh^2 x + \sinh^2 x)} dx &= \int_{\mathbb{R}} e^{-2\pi|n|v(\frac{1}{2}e^{2x} + e^{-2x})} dx \\
&= \int_{\mathbb{R}} e^{-\pi|n|v(e^t + e^{-t})} \frac{dt}{2}.
\end{aligned}$$

where we have used the transformation $x \mapsto \frac{t}{2}$. Converting this into an integral over $[0, \infty)$ gives

$$2\int_0^\infty e^{-\pi|n|v(e^t+e^{-t})} \frac{dt}{2} = \int_0^\infty e^{-\pi|n|v(e^t+e^{-t})} dt.$$

Definition We define the K-Bessel function by

$$K_\nu(x) = \int_0^\infty e^{-x\cosh t} \cosh(\nu t) dt.$$

Using the definition of the K-Bessel function we can see that

$$\int_0^\infty e^{-\pi|n|v(e^t+e^{-t})} dt = K_0(2\pi|n|v).$$

Now consider the action of $\alpha \in \mathcal{O}_K^*$ on $SO_0(Q)$. It has a finite number of orbits since the action simply moves which ideal class we are looking at, and we know that there are a finite number of classes in the narrow class group. Hence the coefficients $a_n(v)$ are finite. Denote the number of orbits $r_Q(n)$ for fixed n.

So we have that, for $n \neq 0$

$$a_n(v) = r_Q(n) K_0(2\pi|n|v).$$

Concentrating now on the $n = 0$ case, we have that

$$a_0(v) = \int_{\log\epsilon \backslash \mathbb{R}} 1 dx = \log\epsilon.$$

and so we have shown the following theorem:

Theorem 8.3.2

$$I(\tau, L) = \sqrt{v}\log\epsilon + \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} r_Q(n)\sqrt{v}K_0(2\pi|n|v)e^{2\pi inu}.$$

This is a weight 0 Maass form, a non-holomorphic modular form. So in the real quadratic field case we have associated quadratic forms to a non-holomorphic theta function and then to a Maass form of weight 0. This contrasts the imaginary quadratic case where we had a relatively simple holomorphic theta function, which was modular.

8.3.1 Maass forms of weight 0

Clearly, the Maass form given in the previous section is of significant interest at it carries information on the indefinite quadratic form. Here, we introduce slightly more general Maass forms (those of weight $k = 0$) and detail some of their properties. We will follow [30] in their discussion of classical automorphic forms.

Definition We define the **hyperbolic Laplace operator** on $\mathbb{H}$ as

$$\Delta = -v^2(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}).$$

Definition Let χ be a Dirichlet character of modulus q, $\lambda \in \mathbb{C}$. Then we call a smooth function f on $\mathbb{H}$ a **Maass form of weight** $k = 0$ **and level** q if it satisfies

1. $\Delta f = \lambda f$.

2. $f(\gamma\tau) = \chi(a)f(\tau)$ for $\gamma \in \Gamma_0(q)$ and $\tau \in \mathbb{H}$.

3. f is of moderate growth at all cusps - that is, there exists $C > 0$ such that $f(\tau) \ll |v|^C$ for $|v| \geq 1$.

Now, we have seen that for our Maass form in the previous section that we have a Fourier expansion so we may expect there to be Fourier expansion representations of all weight 0 Maass forms. This is indeed the case; we have an expansion of the form

$$f(u + iv) = \sum_{n\in\mathbb{Z}} a_n(v)e^{2\pi inu}.$$

Now, since we know by definition that f satisfies $\Delta f = \lambda f$ we can apply Δ to both sides of the above equation. For a fixed n we have that

$$f(u + iv) = a_n(v)e^{2\pi inu}$$

so that

$$\Delta f = \lambda f = \lambda a_n(v)e^{2\pi inu}.$$

We also have that, for the right hand side

$$\begin{aligned}
\Delta(a_n(v)e^{2\pi inu}) &= -v^2(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2})a_n(v)e^{2\pi inu} \\
&= -v^2 a_n(v)(2\pi in)^2 e^{2\pi inu} - v^2\frac{\partial^2 a_n(v)}{\partial v^2}e^{2\pi inu} \\
&= (4\pi^2 n^2 v^2 a_n(v) - v^2\frac{\partial^2 a_n(v)}{\partial v^2})e^{2\pi inu}.
\end{aligned}$$

Let $a_n(v) = \omega$ for convenience and cancel the factors of $e^{2\pi inu}$ to see that ω must satisfy the 2nd order ordinary differential equation

$$v^2\omega'' + (\lambda - 4\pi^2 n^2 v^2)\omega = 0. \tag{8.3.3}$$

This equation can be solved by Bessel functions [31, page 5, equations 3.3 and 3.4]. Specifically, we write $\lambda = s(1-s)$, and have two types of function which will solve the ODE above and provide linearly independent solutions

$$\omega_1(v) = \sqrt{v}K_{\frac{s-1}{2}}(2\pi|n|v)$$

and

$$\omega_2(v) = \sqrt{v}I_{\frac{s-1}{2}}(2\pi|n|v)$$

Here, K_s and I_s denote Bessel functions of the second and first kind respectively. Now consider the asymptotic behaviour of these solutions as we tend toward infinity. We have that $\omega_1(v) \sim \frac{\pi}{2}e^{-2\pi|n|v}$ and $\omega_2(v) \sim \frac{1}{2\pi}e^{2\pi|n|v}$. Since we require (in the definition of a Maass form) that we have moderate growth at the cusps we see that we cannot have any terms for $n \neq 0$ of the type given by ω_2 - else we would have a contradiction at the cusp at ∞.

When $n = 0$ there are two cases to consider. Firstly, when $s \neq \frac{1}{2}$ we have that v^s and v^{1-s} will solve the ODE and provide two linearly independent solutions. Secondly, when $s = \frac{1}{2}$ we must modify our results and use $\sqrt{v}$ and $\sqrt{v}\log v$ instead.

Putting this information together we have that (for $s \neq \frac{1}{2}$)

$$f(u+iv) = a_0(v)v^s + b_0(v)v^{1-s} + \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} a_n(v)\sqrt{v}K_{\frac{s-1}{2}}(2\pi|n|v)e^{2\pi inu}$$

and a similar expression for $s = \frac{1}{2}$, with the constant factors replaced with appropriate powers of v as discussed.

We say that a Maass form is a Maass cusp form if $a_0 = b_0 = 0$. If we now compare this result to our previous section's Maass form we can see that we do indeed find a weight 0 Maass form arising from indefinite theta functions (with $s = \frac{1}{2}$).

Example 8.3.4
In our section focussing on holomorphic modular forms we introduced the $E_k(\tau)$ of weight $k > 3$ (since there were no modular forms of odd weight). Now, we look at the case of a modified Eisenstein series - it turns out that this is a Maass form of weight 0 on Γ.
We let

$$E^*(\tau, s) = \pi^{-s}\Gamma(s)\frac{1}{2}\sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{v^s}{|m\tau + n|^{2s}}$$

on $\Re(s) > 1$. We have that $E^*(\gamma\tau, s) = E^*(\tau, s)$ for all $\gamma \in SL(2,\mathbb{Z})$ - this is easily verified since absolute convergence allows us to rearrange the sum.
We also have

$$\begin{aligned}
\Delta E^*(\tau, s) &= -v^2(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2})E^*(\tau, s) \\
&= (\pi^{-s}\Gamma(s)\frac{1}{2})(-v^2\frac{\partial^2}{\partial v^2}) \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{v^s}{|m\tau + n|^{2s}} \\
&= (\pi^{-s}\Gamma(s)\frac{1}{2})(-s(s-1)) \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{v^s}{|m\tau + n|^{2s}} \\
&= s(1-s)E^*(\tau, s).
\end{aligned}$$

It is of moderate growth at the cusp at ∞ - which can be checked by finding the analytic continuation of the Eisenstein series (we do not present this here), so we have that $E^*(\tau, s)$ is a weight zero Maass form on Γ.

One can define a Maass form of weight k by modifying the differential operator in the definition, and adding a factor to the modularity condition. In general, Maass forms have a wide variety of applications and appear throughout mathematics and physics. For example, the theory of harmonic Maass forms can be applied to black holes, gauge theory, representation theory of Lie superalgebras, knot theory and topology. Clearly, their further study is of importance - interested readers could find extra material in section 2 of [25] for a modular approach, or [31] for a more direct approach.

Next, we include some basic definitions and facts that will allow us to generalise the L-functions we have associated to field extensions to more general L-functions involving Galois representations.

9. A brief review of Galois theory and representations

9.1 Galois theory

We detail some useful results here, without proof, which will be used when discussing our introduction to Artin L-functions.

Firstly, consider a tower of field extensions $K \subseteq L \subseteq M$.

Theorem 9.1.1
The degree of M over K is given by

$$[M : K] = [M : L][L : K]$$

Proof See proof of [32, page 45, Theorem 4.2]. ■

Definition A field extension L of K of finite degree n is called a **splitting field** for a polynomial $f(x) \in K[x]$ if $L = K(\alpha_1, ..., \alpha_n)$ where $f(x) = c(x - \alpha_1)...(x - \alpha_n)$ with $c \neq 0 \in K$.

Example 9.1.2
Consider the polynomial $f(x) = x^2 + 1 \in \mathbb{R}[x]$. Then f has roots $\pm i$, and joining these roots to the base field $\mathbb{R}$ gives that $\mathbb{R}(i, -i) = \mathbb{C}$ is a splitting field for f over $\mathbb{R}$.

While the existence of such a field is clear (we can always take the base field and join all roots of the given polynomial), we also have that it is unique up to isomorphism.
We will also need to define the normal field extensions as follows.

Definition A field extension L of K is called **normal** if every irreducible polynomial with at least one root in L, denoted by $f(x) \in K[x]$, splits in L.

Example 9.1.3
All quadratic field extensions are normal. This can seen by considering $[L : K] = 2$. Let $f(x) \in K[x]$ (of degree 2) have a root x_1 in L. Then we have that

$$f(x) = (x - x_1)\tilde{f}(x) \in L[x].$$

Then comparing degrees of each side we see that $\tilde{f}(x)$ is a linear polynomial and therefore f must split completely in L.

We also have that a field extension is normal if and only if it is a splitting field for some polynomial in the base field.

Definition A field extension L of K is called **separable** if all elements in L have minimal polynomials with no multiple roots.

We will make use of the following theorem, since we are always working with fields K over $\mathbb{Q}$.

Theorem 9.1.4
A finite field extension L of K is separable if K contains $\mathbb{Q}$ as a subfield.

Proof See proof at [32, page 86, Proposition 8.6]. ■

Definition A finite field extension L of K is a **Galois extension** if it is a normal and separable extension.

The Galois extension has degree equal to the size of the group of automorphisms of L that are the identity map on K. We can denote these automorphisms by $\mathrm{Aut}_K L$. In this way, the automorphisms of any number field extension K of $\mathbb{Q}$ are simply the automorphisms of K.

Definition We denote the group of automorphisms of a Galois extension L of K by

$$\mathrm{Gal}(L/K) := \mathrm{Aut}_K L.$$

This is the Galois group of the extension.

The Galois group encodes the symmetries of the field extension and has many properties that we do not discuss here.

Example 9.1.5
Firstly, consider the quadratic field $K = \mathbb{Q}[\sqrt{D}]$. It is of degree 2 and contains $\mathbb{Q}$ as a subfield and so by our previous considerations must be a Galois extension. The Galois group Gal$(K/\mathbb{Q})$ must have cardinality 2. Since the identity map is always a member of any Galois group, we have that

$$Gal(K/\mathbb{Q}) = \{e, \sigma\} \cong \mathbb{Z}_2,$$

for some element σ.
We can identify σ by considering its action on an element in K. Since it is not the identity element, we must have that it changes some part of an arbitrary element $a+b\sqrt{D}$. Now, a, b are rational numbers and since σ is the identity map there (by definition) we must have that

$$\sigma(\sqrt{D}) = -\sqrt{D},$$

where we use that we are in a group, so that $\sigma^2 = Id$.

Note that the Galois group of a field extension is not necessarily abelian - however the theory is a lot easier when this is the case.

9.2 Representations

Here we state the basic definitions and facts that we need, without proof. However, for a comprehensive guide to representation theory we refer readers to [33].

Definition A **representation** of a finite group G on a finite-dimensional vector space V is a homomorphism

$$\rho : G \to \mathrm{GL}(V)$$

taking the group G to the group of automorphisms of V.

Definition A **sub-representation** U of V is a subspace which is invariant under all operators $\rho(g)$ for $g \in G$.

Definition A representation $V \neq 0$ is called **irreducible** if the only sub-representations are 0 and V.

Definition We define the **character** of a representation as $\chi_\rho = \chi(\rho(g)) = \mathrm{Tr}(\rho(g))$. This takes values in $\mathbb{C}$.

Through representation theory one is able to show that characters completely characterise representations, and so we use characters and representations interchangeably. When looking at the Artin L-function we will want to consider some simple examples, and as stated the Galois theory of abelian groups is rather more simple than non-abelian groups, and so we hope to have a similar statement here.

Theorem 9.2.1
Any finite dimensional representation of any finite abelian group decomposes into the direct sum of 1*-dimensional irreducible representations.*

Proof See proof at [34, page 5, Proposition 2.15].

We can also take a representation ρ of a subgroup H of G and form an induced representation on all G, which we denote by $\mathrm{Ind}_H^G \rho$. The induced representation is unique for a given representation on H.
In particular, we are interested in representations of the Galois group of a Galois extension of $\mathbb{Q}$. Such theory is a rich source of mathematics, and is used in some very deep proofs - including Wiles' proof of Fermat's Last Theorem [35]. Next we use the concepts we have discussed to introduce the Artin L-functions.

10. Artin L-series over the rationals

We give the full-blown definition of the Artin L-function [36, page 27], and show how it generalises what we have seen in the quadratic field extension of $\mathbb{Q}$ case. Note that Artin L-functions generalise the L-functions we have seen to higher dimensions by using representation theory and Galois theory. For every prime ideal $\mathfrak{P}$ in L consider a prime ideal $\mathfrak{p}$ in K dividing $\mathfrak{P}$. Let $G = \text{Gal}(L/K)$.

Definition We let

$$D_{\mathfrak{p}} = \{\sigma \in G | \mathfrak{p}^{\sigma} = \mathfrak{p}\}$$

be the **decomposition group** of $\mathfrak{p}$.

Definition We define

$$I_{\mathfrak{p}} = \{\sigma \in G | \sigma(x) \equiv x^{N(\mathfrak{p})} \mod \mathfrak{p} \text{ for all } x \in \mathcal{O}_L\}$$

as the **inertia group** of $\mathfrak{p}$.

Example 10.0.1
Consider the quadratic field extension $K = \mathbb{Q}[i]$ over $\mathbb{Q}$. Its Galois group is $\{Id, \sigma\} \cong \mathbb{Z}_2$ where $\sigma : a + bi \to a - bi$.
Consider the inert prime 7. It has decomposition group $D_7 = G$ and inertia group $I_7 = \{Id\}$.
Now take a ramified prime, $2 = (1 + i)^2$. It also has decomposition group $D_2 = G$, but here the inertia group is given by $I_2 = G$.

We have that $I_{\mathfrak{p}}$ is a normal subgroup of $D_{\mathfrak{p}}$, and that the quotient group of $D_{\mathfrak{p}}$ over $I_{\mathfrak{p}}$ is given by [37, page 3, Theorem 8.1.8]

$$D_{\mathfrak{p}}/I_{\mathfrak{p}} \cong \text{Gal}((\mathcal{O}_L/\mathfrak{p})/(\mathcal{O}_K/\mathfrak{P})),$$

and that such a group is cyclic and is generated by the so-called Frobenius automorphism $\text{Frob}_{\mathfrak{P}} : x \to x^{N(\mathfrak{P})}$.

We call the 'pull-back' of this automorphism to $D_{\mathfrak{p}}$ the Frobenius element, which is clearly defined up to an element of the intertia group. We denote the Frobenius element by $\sigma_{\mathfrak{p}}$. Considering all such $\sigma_{\mathfrak{p}}$ where $\mathfrak{p}|\mathfrak{P}$ gives a conjugacy class $\sigma_{\mathfrak{P}}$, the so-called Artin symbol of $\mathfrak{P}$.

Example 10.0.2
Take $K = \mathbb{Q}$ and $L = \mathbb{Q}[\sqrt{D}]$. Then the Artin symbol becomes the Legendre symbol.

Definition Let L/K be a finite Galois extension with Galois group G. Let ρ be an n-dimensional representation of G with character χ_ρ. Then we define the **Artin L-function** to be

$$L(s, \chi_\rho, L/K) = \prod_{\mathfrak{P}} \frac{1}{\det(\mathrm{Id}_n - N(\mathfrak{P})^{-1}\rho(\sigma_{\mathfrak{P}})|V_\rho^{I_{\mathfrak{P}}})}.$$

In particular note that the inertia subgroup is simply $\{\mathrm{Id}\}$ if and only if (p) does not ramify anywhere in the extension. In this case, we have no restriction on our subspace V.

Example 10.0.3
Since we have been mainly concerned with quadratic extensions over $\mathbb{Q}$ we now let K be such a field. Then we have that Gal$(K/\mathbb{Q}) \cong \mathbb{Z}_2$. From our section on representation theory we know that all irreducible representations of cyclic groups are one-dimensional. So here we are in the one-dimensional case.
Now we have two choices for our representation ρ. Either ρ is the trivial representation or it is a non-trivial genus character. When ρ is the trivial representation we see that the Artin L-function reduces to

$$L(s, \chi_\rho, K) = \prod_{\mathfrak{P}} \frac{1}{1 - N(\mathfrak{P})^{-s}},$$

and is the same as the Dedekind zeta function $\zeta_K(s)$ as defined in 7.1.1.

Example 10.0.4
Let $K = \mathbb{Q}[\xi_m]$ where ξ_m is a primitive m^{th} root of unity. Then we have $G = \mathbb{Z}_m^\times$, and again we have one-dimensional representations only since the Galois group is cyclic. In this case, they are precisely the Dirichlet characters $\chi : \mathbb{Z}_m^\times \to \mathbb{C}^*$, and so the Artin L-function reduces down to the Dirichlet L-series we saw earlier, $\sum\limits_{n=1}^{\infty} \frac{\chi(n)}{n^s}$.

So the Artin L-functions really do generalise what we have seen in this report. In general, Artin L-functions cannot be written as infinite sums, but they do exhibit many of the properties that we would expect. That is [3, page 522, Proposition 10.4]:

Proposition 10.0.5

1. If χ_1 and χ_2 are two characters of G then

$$L(s, \chi_1 + \chi_2, L/K) = L(s, \chi_1, L/K)L(s, \chi_2, L/K).$$

2. Let $K \subseteq L \subseteq L'$ with $L' \supseteq K$ a Galois extension. Then for a character χ of Gal(L'/K) we have

$$L(s, \chi, L'/K) = L(s, \chi, L/K).$$

3. If $K \subseteq L \subseteq M$ and χ a character then

$$L(s, \chi, M/L) = L(s, \chi^*, M/K),$$

where χ^* is the character arising from the induced representation $Ind_G^H \rho$ of the representation ρ on H, where $H = Gal(M/L)$ and $G = Gal(M/K)$.

Proof See proof at [3, page 522, Propsition 10.4]. It is far beyond the scope of this report. ∎

To end, we mention Artin's conjecture and the Langlands Program. Artin believed that all of his L-functions should be entire functions (expect possibly at $s = 1$), and enjoy analytic continuations (like the ones we have seen in this report). More precisely we have

Artin's Conjecture
Let L be a Galois extension of K with Galois group G. Let ρ be a finite dimensional, irreducible representation. Then $L(s, \chi_\rho, L/K)$ is an entire function (except at $s = 1$ for the trivial representation) for all $s \in \mathbb{C}$.

Results on this conjecture are still partial - for example,a proof exists for one-dimensional irreducible representations ρ, but many results are still yet to be proved. This has intricate links to the famous Langlands Program, which looks to relate Galois groups to automorphic forms and representations. For the interested reader, [30] is an excellent starting point for further reading.

11. Summary

In this report we have seen how algebraic number theory is intricately linked to the theory of automorphic forms on congruent subgroups of Γ. We detailed results pertaining to various L-functions of both $\mathbb{Q}$ and number field extensions of finite degree, $K/\mathbb{Q}$. Considering such L-functions allowed us to show how they were related to certain theta functions and hence modular forms.

We began in chapter 2 by detailing some basic results from algebraic number theory. Throughout the next chapter we discussed modular and automorphic forms on congruent subgroups of $SL(2,\mathbb{Z})$. When we looked to expand our considerations to half-integer weights we needed to introduce multiplier systems to allow for twists of functions. We saw that a multiplier system of half-integer weight k was also a multiplier system for any $k' \equiv k \mod 2$, and that we were able to define more general automorphic forms on the congruent subgroups. Since the multiplier system for integer k could be viewed as a Dirichlet character we split this case off separately and defined holomorphic modular forms of weight k and character χ. Next we considered twists of modular forms and Hecke operators and eigenforms. We stated several key theorems over these sections, including how we could see primitive cusp newforms as Euler products (theorem 3.1.13). Ending this chapter we saw two converse theorems showing us how to prove that a given function is a modular form on some congruent subgroup of Γ without having to resort to the definition.

In chapter 4 we detailed some basic results about integral quadratic forms. In particular we had that for positive definite forms we had finite representation numbers. We also saw that there were a finite number of equivalence classes of integral quadratic forms of a given discriminant, which we were able to relate to the equivalence classes of ideals in the class group of a number field $K/\mathbb{Q}$.

Using our knowledge of positive definite integral quadratic forms we started to construct theta functions in chapter 5. We defined congruent theta functions and showed how these are related to theta functions. Then we saw that both the theta functions and congruent theta functions are automorphic forms on a congruent subgroup of Γ. Further to this we saw that in some cases the theta functions were actually modular forms with character on a congruent subgroup.

In chapter 6 we looked at L-functions on $\mathbb{Q}$, and specifically at the Riemann zeta function and Dirichlet L-function. We saw how these were related to certain theta functions, and also considered their completions. In particular, each admitted a continuation to $\mathbb{C}$ and satisfied a functional equation. This is the general theme of the L-functions we considered. Next we briefly considered the Dirichlet L-function of a modular form f and saw its completion and continuation.

We looked to generalise our results to more general extensions $K/\mathbb{Q}$ of finite degree, and gave an overview of this in chapter 7. After introducing the Dedekind zeta function and partial zeta function of K we saw that they encode information on the prime ideals of $\mathcal{O}_K$. Via Neukirch, we saw that we could relate these to some given theta function and that again they admitted a continuation to $\mathbb{C}$, whilst also satisfying a functional equation. As a simple corollary of these facts we obtained the analytic class number formula for K. In section 7.2 we briefly considered the Hecke L-series for a narrow class group character χ. It inherited properties of the partial zeta function and therefore had similar analytics.

Next, we specialised results to the quadratic field theory case. Beginning by detailing some general results we found a basic decomposition of $\zeta_K(s)$ and how we could use this to find information pertaining to the class number h of K. We also saw how genus characters enabled us to decompose the Hecke L-series into two L-functions of lower discriminants. We considered the cases $D < 0$ and $D > 0$ separately, and began with the simpler case where K was an imaginary quadratic field. Here, we saw that our quadratic fields built from ideals were always positive definite, and hence our theta function was holomorphic. We proceeded to show that the Dedekind zeta function and Hecke L-series of an imaginary quadratic field arose as L-functions of weight 1 modular forms with character χ_D. To end the imaginary case we showed that the Hecke L-series was actually the twisted sum of Eisenstein series. For the real quadratic case we had to define a new theta function, since the previous definition was no longer convergent. We saw how this definition was non-holomorphic - contrasting the previous case. By integrating over the symmetries of the theta function we obtained a Maass form of weight 0 on Γ. Since we obtained a Maass form we dedicated the next section to defining Maass forms of weight 0, and detailing their Fourier expansion.

Chapter 9 was used to state results from Galois theory and representation theory that would allow us to introduce Artin L-functions in the following chapter. When we introduced the Artin L-function we saw that it generalised the theory of Hecke L-series over $\mathbb{Q}$ to higher dimensional L-functions via the use of Galois representations.

11.1 Further work

In chapter 10 we introduced the Artin L-function, and saw specifically how it generalised L-functions to include higher dimensional objects. We also stated Artin's Conjecture, and remarked that though there are partial results, there is no known proof or disproof of the full conjecture. It is the focus of a lot of current research, and is just one such conjecture in the far-reaching Langlands Program, which seeks to reconcile the theory of Galois groups in algebraic number theory and the theory of automorphic forms and representation theory of algebraic groups. This is an area of great importance, and I would encourage readers to further investigate the theory surrounding the Langlands Program. It is a rich source of mathematics, and the conjectures it covers would have profound implications if proven.

Appendix A

Definition Let $f \in L^1(\mathbb{R})$. We define the **Fourier transform** of f as

$$\hat{f}(y) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}dx.$$

Proposition A.1
$f(x) = e^{-\pi x^2}$ is its own Fourier transform

Proof Differentiating with respect to y yields

$$\begin{aligned}\frac{d}{dy}\hat{f}(y) &= -2\pi i \int_{-\infty}^{\infty} xf(x)e^{-2\pi ixy}dx \\ &= -2\pi y\hat{f}(y).\end{aligned}$$

Now we solve the differential equation for $\hat{f}(y)$. This gives $\hat{f}(y) = Ce^{-\pi y^2}$, and setting $y = 0$ gives us that $C = 1$. ■

Proposition A.2
(Poisson Summation Formula)
[11, page 69, Theorem 4.4] Let f and $\hat{f}$ be in $L^1(\mathbb{R})$, with bounded variation. Then

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n),$$

where both series converge absolutely.

Proof Let $F(x) = \sum_{m=-\infty}^{\infty} f(x+m)$. It is clearly periodic of period 1. Now consider the Fourier expansion of such a function, given by

$$F(x) = \sum_{n=-\infty}^{\infty} a_F(n)e^{-2\pi inx},$$

where

$$a_F(n) = \int_0^1 F(t)e^{-2\pi int}dt = \int_{-\infty}^{\infty} f(t)e^{-2\pi int}dt = \hat{f}(t).$$

Taking $F(0)$ yields the result. ■

Appendix B

Proposition B.1

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-\pi(xq+a)^2 y} e^{-2\pi\xi x} dx = \frac{e^{\frac{2\pi i\xi a}{q}}}{q\sqrt{y}} e^{\frac{-\pi\xi^2}{q^2 y}}.$$

Proof

$$\begin{aligned}
\hat{f}(\xi) &= \int_{-\infty}^{\infty} e^{-\pi(xq+a)^2 y} e^{-2\pi\xi x} dx \\
&= \frac{1}{q} \int_{-\infty}^{\infty} e^{-\pi(x+a)^2 y} e^{-2\pi\xi \frac{x}{q}} dx \\
&= \frac{e^{2\pi i\xi a/q}}{q} \int_{-\infty}^{\infty} e^{-\pi(x)^2 y} e^{-2\pi\xi x/q} dx \\
&= \frac{e^{2\pi i\xi a/q}}{q\sqrt{y}} \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi\xi x/q\sqrt{y}} dx \\
&= \frac{e^{2\pi i\xi a/q}}{q\sqrt{y}} e^{-\pi\xi^2/q^2 y}. \qquad \blacksquare
\end{aligned}$$

Appendix C

Proposition C.1

$$\sum_{a \mod q} \chi(a) e^{\frac{2\pi i l a}{q}} = 0$$

for $(l, q) > 1$ *and* χ *a primitive character of modulus* q.

Proof Proof taken and modified from [23, Claim 1.3.1].
Let $(l, q) = q/m > 1$ and prove via a contradiction. Firstly, replace $a \mapsto b(1 + xm)$ for any $x \mod q/m$ to get

$$\begin{aligned}\sum_{a \mod q} \chi(a) e^{2\pi i l a/q} \mapsto &\sum_{a \mod q/m} \sum_{\beta \mod m} \chi(a(1 + xm)) e^{2\pi i l \beta(1+xm)/q} \\ &= \chi(1 + xm) \sum_{a \mod q/m} \sum_{\beta \mod m} \chi(a) e^{2\pi i l \beta/q},\end{aligned}$$

for all $x \in \mathbb{Z}$.
Now, since we have that χ is primitive we have that $x \mapsto \chi(1 + xm)$ is non-trivial, and so the sum must be zero. ■

Appendix D

Proposition D.1
We have the following Euler product representation of the Dedekind zeta function, for $\Re(s) > 1$

$$\zeta_K(s) = \prod_{\mathfrak{P}} \frac{1}{1 - N(\mathfrak{P})^{-s}},$$

where the product is taken over all prime ideals of K*.*

Proof Taken from [38, page 5].
We have that $1 - (N(\mathfrak{P})^{-s})^2 < 1$ and $1 + N(\mathfrak{P})^{-s} - 2N(\mathfrak{P})^{-2s} \geq 1$. So then we have the inequality $1 - N(\mathfrak{P})^{-s} < \frac{1}{1-N(\mathfrak{P})^{-s}} \leq 1 + 2N(\mathfrak{P})^{-s}$.
Letting $1 + b_p = (1 - N(\mathfrak{P})^{-s})^{-1}$ for $\mathfrak{P}|(p)$ we have that $b_p \leq 2NN(\mathfrak{P})^{-s} = 2p^{-fs}$ with f the residual degree of $\mathfrak{P}$.
So we then have

$$\sum_p b_p \leq 2[K : \mathbb{Q}] \sum \lim_p p^{-s} \leq 2[K : \mathbb{Q}] \sum_n n^{-s},$$

as the degree of the field extension $[K : \mathbb{Q}]$ is the number of primes $\mathfrak{P}$ above (p). Notice that we have absolute convergence, since $b_n > 0$ for all n, and $\Re(s) > 1$.
Since we have the geometric series expansion

$$\frac{1}{1 - N(\mathfrak{P})^{-s}} = 1 + \sum_{\mathfrak{P},i} N(\mathfrak{P})^{-is}$$

we need only show for a finite set of prime ideals S_m where each ideal has norm $\leq m$, that we have

$$\prod_{\mathfrak{P} \in S_m} (1 + \sum_{\mathfrak{P},i} N(\mathfrak{P})^{-is}) = 1 + \sum_{\substack{I \in \mathcal{O}_K \\ P|I \in S_m}} N(I)^{-s}.$$

This can be shown for the integer prime case [38, page 3], and this result generalises immediately, since we have unique decomposition into prime ideals.
Now, take the limit $m \to \infty$ to obtain our result. ■

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