Permutation principles for the change analysis of stochastic processes under strong invariance

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Abstract

Approximations of the critical values for change-point tests are obtained through permutation methods. Both, abrupt and gradual changes are studied in models of possibly dependent observations satisfying a strong invariance principle, as well as gradual changes in an i.i.d. model. The theoretical results show that the original test statistics and their corresponding permutation counterparts follow the same distributional asymptotics. Some simulation studies illustrate that the permutation tests behave better than the original tests if performance is measured by the α- and β-error, respectively.

1 Introduction

A series of papers has been published on the use of permutation principles for obtaining reasonable approximations to the critical values of change-point tests. This approach was first suggested by Antoch and Hušková [1] and later pursued by other authors (cf. Hušková [7] for a recent survey). But, so far, it has mostly been dealt with abrupt changes and independent observations. In many practical applications, however, smooth (gradual) changes are more realistic, so are dependent observations.

In this paper we shall discuss the use of permutation principles in the following three models:

1) (*Gradual change in the mean of independent, identically distributed (i.i.d.) observations*) Hušková and Steinebach [8] investigated the following model:

$$X_i = \mu + d\left(\frac{i-m}{n}\right)_+^{\gamma} + e_i, \quad i = 1, \ldots, n, \tag{1.1}$$

where $x_+ = \max(0, x)$; μ, $d = d_n$, and $m = m_n \leqslant n$ are unknown parameters, and $e_1, \ldots, e_n$ are i.i.d. random variables with

$$Ee_i = 0, \qquad 0 < \text{var } e_i = \sigma^2 \ (< \infty), \qquad E|e_i|^{2+\delta} < \infty \quad \text{for some} \ \ \delta > 0. \tag{1.2}$$

The parameter γ is supposed to be known.

Note that – in contrast to abrupt changes – the biggest difference in the mean here is not d, but $d\left(\frac{n-m}{n}\right)^{\gamma}$, and thus depends on n, m and γ.

One is interested in testing the hypotheses

$$H_0 : m = n \quad \text{vs.} \quad H_1 : m < n, \ d \neq 0.$$

The following test statistic, which is based on the likelihood ratio approach in case of normal errors $\{e_i\}$, has been used:

$$T_n^{(1)} = \frac{1}{\hat{\sigma}_n} \max_{1 \leqslant k < n} \frac{\left|\sum_{i=1}^{n}(i-k)_+^{\gamma}(X_i - \overline{X}_n)\right|}{\left(\sum_{i=1}^{n-k} i^{2\gamma} - \frac{1}{n}\left(\sum_{i=1}^{n-k} i^{\gamma}\right)^2\right)^{1/2}},$$

where $\hat{\sigma}_n$ denotes a suitable estimator of σ. Asymptotic critical values for the corresponding test can be chosen according to the following null asymptotics (cf. Hušková and Steinebach [8]):

Theorem 1.1. *Let $X_1, X_2, \ldots$ be i.i.d. r.v.'s with* $\operatorname{var} X_1 = \sigma^2 > 0$, *and* $\mathrm{E}\,|X_1|^{2+\delta} < \infty$ *for some* $\delta > 0$. *Then, for all $x \in \mathbb{R}$, as $n \to \infty$,*

$$P\left(\alpha_n T_n^{(1)} - \beta_n \leqslant x\right) \to \exp\left(-2e^{-x}\right),$$

where $\alpha_n = \sqrt{2\log\log n}$ and $\beta_n = \beta_n(\gamma)$ is as follows:

(i) for $\gamma > \frac{1}{2}$:

$$\beta_n = 2\log\log n + \log\left(\frac{1}{4\pi}\left(\frac{2\gamma+1}{2\gamma-1}\right)^{1/2}\right);$$

(ii) for $\gamma = \frac{1}{2}$:

$$\beta_n = 2\log\log n + \frac{1}{2}\log\log\log\log n - \log(4\pi);$$

(iii) for $0 < \gamma < \frac{1}{2}$:

$$\beta_n = 2\log\log n + \frac{1-2\gamma}{2(2\gamma+1)}\log\log\log n + \log\left(\frac{C_\gamma^{1/(2\gamma+1)} H_{2\gamma+1}}{\sqrt{\pi}\,2^{2\gamma/(2\gamma+1)}}\right),$$

with H_γ as in Remark 12.2.10 *of Leadbetter et al.* [10] *(e.g. $H_1 = 1$, $H_2 = 1/\sqrt{\pi}$), and*

$$C_\gamma = -(2\gamma+1)\int_0^\infty x^\gamma\left((x+1)^\gamma - x^\gamma - \gamma x^{\gamma-1}\right)dx.$$

Moreover, $\hat{\sigma}_n$ is assumed to be an estimator of σ satisfying $\hat{\sigma}_n - \sigma = o_P((\log\log n)^{-1})$ as $n \to \infty$.

2) *(Abrupt change in the mean or variance of a stochastic process under strong invariance)* This model has been considered by Horváth and Steinebach in [6]. Suppose one observes a stochastic process $\{Z(t) : 0 \leqslant t < \infty\}$ having the following structure:

$$Z(t) = \begin{cases} at + bY(t) & ,\quad 0 \leqslant t \leqslant T^*, \\ Z(T^*) + a^*(t - T^*) + b^*Y^*(t - T^*) & ,\quad T^* < t \leqslant T, \end{cases} \tag{1.3}$$

where a, b, a^*, b^* are unknown parameters, and $\{Y(t) : 0 \leqslant t < \infty\}$ resp. $\{Y^*(t) : 0 \leqslant t < \infty\}$ are (unobserved) stochastic processes satisfying the following strong invariance principles:

For every $T > 0$, there exist two independent Wiener processes $\{W_T(t) : 0 \leqslant t \leqslant T^*\}$ and $\{W_T^*(t) : 0 \leqslant t \leqslant T - T^*\}$, and some $\delta > 0$, such that, for $T \to \infty$,

$$\sup_{0 \leqslant t \leqslant T^*} |Y(t) - W_T(t)| = O\left(T^{1/(2+\delta)}\right) \quad \text{a.s.} \tag{1.4}$$

and

$$\sup_{0 \leqslant t \leqslant T - T^*} |Y^*(t) - W_T^*(t)| = O\left(T^{1/(2+\delta)}\right) \quad \text{a.s.} \tag{1.5}$$

Moreover, we assume $Y(0) = 0$ and $Y^*(0) = 0$. It should be noted that only weak invariance has been assumed in [6], instead of the strong rates of (1.4) and (1.5), which are required for later use here. Moreover, the processes $\{Z(t)\}, \{Y(t)\}$, and $\{Y^*(t)\}$ could be replaced by a family of processes $\{Z_T(t)\}, \{Y_T(t)\}$, and $\{Y_T^*(t)\}$, $T > 0$, since the asymptotic analysis is merely based on the approximating family of Wiener processes $\{W_T(t)\}$ and $\{W_T^*(t)\}$, respectively.

One is interested in testing the hypothesis of "no change", i.e.

$$H_0 : T^* = T,$$

against the alternative of "a change in the mean at $T^* \in (0, T)$", i.e.

$$H_1^{(1)} : 0 < T^* < T \quad \text{and} \quad a \neq a^*,$$

resp. "a change in the variance at $T^* \in (0, T)$", i.e.

$$H_1^{(2)} : 0 < T^* < T \quad \text{and} \quad b \neq b^*, \ \text{ but } a = a^*.$$

Basic examples satisfying conditions (1.3)-(1.5) are partial sums of i.i.d. random variables and renewal processes based on i.i.d. waiting times, but also sums of dependent observations (for details we refer to Horváth and Steinebach [6]).

It is assumed, that the process $\{Z(t) : t \geqslant 0\}$ has been observed at discrete time points $t_i = t_{i,N} = i\frac{T}{N}$, $1 \leqslant i \leqslant N = N(T)$. Let $\Delta Z_{i,T} = Z(t_i) - Z(t_{i-1})$ and $\widetilde{\Delta Z}_{i,T} = Z(t_i) - Z(t_{i-1}) - \overline{\Delta Z}_T$. The following statistics will be used:

$$M_T = \max_{1\leqslant k\leqslant N} \left\{ \frac{1}{\sqrt{T}} \frac{1}{\widehat{b}_T} \left| \sum_{i=1}^{k} \left(\Delta Z_{i,T} - \overline{\Delta Z}_T\right) \right| \right\}, \tag{1.6}$$

where $\overline{\Delta Z}_T = \frac{1}{N}\sum_{i=1}^{N} \Delta Z_{i,T}$, and

$$\widehat{b}_T^2 = \frac{1}{T}\sum_{i=1}^{N} \left(\Delta Z_{i,T} - \overline{\Delta Z}_T\right)^2,$$

resp.

$$\widetilde{M}_T = \max_{1\leqslant k\leqslant N} \left\{ \frac{1}{\sqrt{T}} \frac{1}{\widehat{c}_T} \left| \sum_{i=1}^{k} \left(\widetilde{\Delta Z}_{i,T}^2 - \overline{\widetilde{\Delta Z}_T^2}\right) \right| \right\}, \tag{1.7}$$

where $\overline{\widetilde{\Delta Z}_T^2} = \frac{1}{N}\sum_{i=1}^{N} \widetilde{\Delta Z}_{i,T}^2$, and

$$\widehat{c}_T^2 := \frac{1}{T}\sum_{i=1}^{N} \left((\Delta Z_{i,T} - \overline{\Delta Z}_T)^2 - \frac{1}{N}\sum_{l=1}^{N} \left(\Delta Z_{l,T} - \overline{\Delta Z}_T\right)^2 \right)^2.$$

Remark 1.1. The statistic $\widetilde{M}_T$ uses a slightly different variance estimator $\widehat{c}_T^2$ than the one given in Horváth and Steinebach [6]. It possesses, however, the same asymptotic behavior, since the ratio of the two normalizations converges in probability to 1 under the null hypothesis, and to some positive constant under the alternative (cf. Theorem 4.5.2 in Kirch [9]). This modification is necessary for applying the permutation method, since, under the alternative, the permutation statistic (corresponding to the statistic used in [6]) does not converge to $\sup_{0\leqslant t\leqslant 1} |B(t)|$, but to $c \sup_{0\leqslant t\leqslant 1} |B(t)|$, $c > 0$, $c \neq 1$ in general, where c is the asymptotic ratio of the two variance estimators. Here $\{B(t) : 0 \leqslant t \leqslant 1\}$ denotes a Brownian bridge.

The following null asymptotics hold under the above conditions (cf. Horváth and Steinebach [6]):

Theorem 1.2. *If $N = N(T) \to \infty$ and $N = o\left(T^{1-2/(2+\delta)}\right)$ as $T \to \infty$, then, under H_0,*

$$M_T \xrightarrow{\mathcal{D}} \sup_{0\leqslant t\leqslant 1} |B(t)|,$$

where $\{B(t) : 0 \leqslant t \leqslant 1\}$ is a Brownian bridge.

Theorem 1.3. *If $N = N(T) \to \infty$ and $N = o\left(T^{1/2-1/(2+\delta)}\right)$ as $T \to \infty$, then, under H_0,*

$$\widetilde{M}_T \xrightarrow{\mathcal{D}} \sup_{0\leqslant t\leqslant 1} |B(t)|,$$

where $\{B(t) : 0 \leqslant t \leqslant 1\}$ is a Brownian bridge.

3) *(Gradual change in the mean of a stochastic process under strong invariance)* This model has been considered by Steinebach in [13]. Suppose one observes a stochastic process $\{S(t) : 0 \leqslant t < \infty\}$ having the following structure:

$$S(t) := \begin{cases} at + bY(t) & , \ 0 \leqslant t \leqslant T^*, \\ S(T^*) + a(t - T^*) + \tilde{d}(t - T^*)^{1+\gamma} + b^*Y^*(t - T^*) & , \ T^* < t \leqslant T, \end{cases} \tag{1.8}$$

where a, b, b^* and $\{Y(t)\}, \{Y^*(t)\}$ are as in model 2) above, $\tilde{d} = \tilde{d}_T$ is unknown, $\gamma > 0$ is known. Again, the biggest difference in the mean here depends on T, T^* and γ, similarly as in the first model. Note that, instead of (1.4), Steinebach [13] assumed the following weak invariance principle for the process

$\{Y(t) : 0 \leqslant t < \infty\}$, namely that, for every $T > 0$, there is a Wiener process $\{W_T(t) : 0 \leqslant t \leqslant T^*\}$ such that

$$\sup_{1\leqslant t\leqslant T^*} |Y(T^*) - Y(T^* - t) - W_T(t)| / t^{1/(2+\delta)} = O_P(1) \qquad (T \to \infty). \tag{1.9}$$

The reason is that small approximation rates were required near the change-point T^*, but only in a weak sense, whereas we need strong approximations for our permutation principles below. Here, too, the processes $\{Z(t)\}, \{Y(t)\}$, and $\{Y^*(t)\}$ could be replaced by a family of processes $\{Z_T(t)\}, \{Y_T(t)\}$, and $\{Y_T^*(t)\},\ T > 0$.

One is now interested in testing the null hypothesis of "no change in the drift", i.e.

$$H_0 : T^* = T$$

against the alternative of "a smooth (gradual) change in the drift", i.e.

$$H_1 : 0 < T^* < T, \quad \tilde{d} \neq 0.$$

Basic examples fulfilling the conditions above are again partial sums of i.i.d. random variables and renewal processes based on i.i.d. waiting times (cf. Steinebach [13] for more details). As in model 2), we assume that we have observed $\{S(t) : t \geqslant 0\}$ at discrete time points $t_i = i\frac{T}{N}$, and set $\Delta S_{i,T} = S(t_i) - S(t_{i-1})$. The following test statistic is used:

$$T_N^{(2)} = \sqrt{\frac{N}{T\widehat{b}_T^2}} \max_{1\leqslant k<N} \frac{\left|\sum_{i=1}^N (i-k)_+^\gamma (\Delta S_{i,T} - \overline{\Delta S}_N)\right|}{\left(\sum_{i=1}^{N-k} i^{2\gamma} - \frac{1}{N}\left(\sum_{i=1}^{N-k} i^\gamma\right)^2\right)^{1/2}}, \tag{1.10}$$

where $\overline{\Delta S}_T = \frac{1}{N}\sum_{i=1}^N \Delta S_{i,T}$, and $\widehat{b}_T^2 = \frac{1}{T}\sum_{i=1}^N \left(\Delta S_{i,T} - \overline{\Delta S}_T\right)^2$.

Steinebach assumed in [13] a slightly different weight, which is asymptotically equivalent to the one used above. However, it turns out, that the above weight gives much better results for the permutation statistic, which is due to the fact, that it is the maximum-likelihood statistic under Gaussian errors. The results obtained in [13] remain valid.

Remark 1.2. The magnitude of $\tilde{d}$ is completely different from that of d in the first model. However, $\overline{d} := \tilde{d}(1+\gamma)\frac{T^{1+\gamma}}{N}$ is comparable to d, which can easily be seen via the mean value theorem.

Similar to Theorem 1.1, the following null asymptotic applies (cf. Steinebach [13]):

Theorem 1.4. *If (1.9) holds, $N = N(T) \to \infty$ and $N = O(T)$ as $T \to \infty$, then, under H_0, for all $x \in \mathbb{R}$:*

$$P\left(\alpha_N T_N^{(2)} - \beta_N \leqslant x\right) \to \exp\left(-2e^{-x}\right),$$

where $\alpha_N = \sqrt{2\log\log N}$ and $\beta_N = \beta_N(\gamma)$ is as in Theorem 1.1 (with N replacing n).

2 Rank and permutation statistics in case of a gradual change under i.i.d. errors

In order to derive distributional asymptotics for the permutation statistics, we shall make use of the following theorem for the corresponding rank statistics. In the case $\gamma = 1$ was proven by Slabý in [12].

Theorem 2.1. *Let $\mathbf{R} = (R_1, \ldots, R_n)$ be a random permutation of $(1, \ldots, n)$, and $a_n(1), \ldots, a_n(n)$ be scores satisfying*

$$\frac{1}{n}\sum_{i=1}^n (a_n(i) - \overline{a}_n)^2 \geqslant D_1, \tag{2.1}$$

and

$$\frac{1}{n}\sum_{i=1}^n (a_n(i) - \overline{a}_n)^{2+\delta} \leqslant D_2, \tag{2.2}$$

where D_1, D_2 *and* δ *are some positive constants, and* $\overline{a}_n = \frac{1}{n}\sum_{i=1}^{n} a_n(i)$. *Then, for fixed* $\gamma > 0$ *and all* $x \in \mathbb{R}$, *as* $n \to \infty$

$$P\left(\alpha_n T_n(\mathbf{a}) - \beta_n \leqslant x\right) \to \exp\left(-2e^{-x}\right),$$

where

$$T_n(\mathbf{a}) = \frac{1}{\sigma_n(\mathbf{a})} \max_{1 \leqslant k < n} \frac{\left|\sum_{i=1}^{n}(i-k)_+^\gamma (a_n(R_i) - \overline{a}_n)\right|}{\left(\sum_{i=1}^{n-k} i^{2\gamma} - \frac{1}{n}\left(\sum_{i=1}^{n-k} i^\gamma\right)^2\right)^{1/2}};$$

Here $\sigma_n^2(\mathbf{a}) = \frac{1}{n}\sum_{i=1}^{n}\left(a_n(i) - \overline{a}_n\right)^2$, *the variance of* $a_n(R_1)$, $\alpha_n = \sqrt{2\log\log n}$ *and* $\beta_n = \beta_n(\gamma)$ *is as in Theorem* 1.1.

In the proof of this theorem we apply the following weak embedding:

Theorem 2.2. *Let* $a_n(1), \ldots, a_n(n)$ *be scores satisfying* (2.1) *and* (2.2). *Then, on a rich enough probability space, there is a sequence of stochastic processes* $\left\{\widetilde{\Pi}_n(k) : 1 \leqslant k \leqslant n\right\}$ $(n = 1, 2, \ldots)$ *with*

$$\left\{\widetilde{\Pi}_n(k) : 1 \leqslant k \leqslant n\right\} \stackrel{\mathcal{D}}{=} \left\{\frac{1}{\sqrt{\sigma_n^2(\mathbf{a})}} \sum_{i=1}^{k} (a_n(\pi_n(i)) - \overline{a}_n) : 1 \leqslant k \leqslant n\right\},$$

where $(\pi_n(1), \ldots, \pi_n(n))$ *is a random permutation of* $(1, 2, \ldots, n)$, $\sigma_n^2(\mathbf{a}) := \frac{1}{n}\sum_{i=1}^{n}(a_n(i) - \overline{a}_n)^2$, $\overline{a}_n = \frac{1}{n}\sum_{i=1}^{n} a_n(i)$, *and there is a fixed Brownian bridge* $\{B(t) : 0 \leqslant t \leqslant 1\}$ *such that, for* $0 \leqslant \nu < \min\left(\frac{\delta}{2(2+\delta)}, \frac{1}{4}\right)$,

$$\max_{1 \leqslant k < n} \left(\frac{k(n-k)}{n}\right)^\nu \frac{n}{\sqrt{k(n-k)}} \left|\frac{1}{\sqrt{n}}\widetilde{\Pi}_n(k) - B(k/n)\right| = O_P(1).$$

The proof goes along the lines of Theorem 1 of Einmahl and Mason [4], by replacing the Hájek-Rényi inequality (cf. [4], p. 110) resp. Lemma 3 there with the following lemmas:

Lemma 2.1. *Let* $M(0) = 0$, $M(1), \ldots, M(m)$, $m \geqslant 1$, *be a mean* 0, *square-integrable martingale, and* $a(1) \geqslant \ldots \geqslant a(m) \geqslant 0$ *be constants. Then, for* $1 < s \leqslant 2$ *and* $\lambda > 0$,

$$P\left(\max_{1 \leqslant i \leqslant m} a_i |M(i)| > \lambda\right) \leqslant 2^{s-1}\frac{1}{\lambda^s}\sum_{i=1}^{m} a_i^s \,\mathrm{E}\left|M(i) - M(i-1)\right|^s.$$

Proof. Confer Lemma 1 in Häusler and Mason [5], or Lemma 5.1.2 in Kirch [9] together with Einmahl [3]. ■

Lemma 2.2. *Let* $a_n(1), \ldots, a_n(n)$ *be scores with* $\sum_{i=1}^{n} a_n(i) = 0$, *and* $(\pi_n(1), \ldots, \pi_n(n))$ *be a random permutation as in Theorem* 2.2. *Then, for* $1 \leqslant i \leqslant n$ *and* $1 \leqslant s \leqslant 2$,

$$\mathrm{E}\left|\sum_{j=1}^{i} a_n(\pi_n(j))\right|^s \leqslant 2\min(i, n-i)\,\frac{1}{n}\sum_{j=1}^{n}|a_n(j)|^s.$$

Proof. Confer Lemma 5.1.3. in Kirch [9] and Mason [11]. ■

Now we have the tools to prove Theorem 2.1:

Proof of Theorem 2.1. First note that

$$n\sum_{i=1}^{n-k} i^{2\gamma} - \left(\sum_{i=1}^{n-k} i^{\gamma}\right)^2 = (n-k)\sum_{i=1}^{n-k}\left(i^{\gamma} - \frac{1}{n-k}\sum_{j=1}^{n-k} j^{\gamma}\right)^2 + k\sum_{i=1}^{n-k} i^{2\gamma} \geqslant k\int_0^{n-k} x^{2\gamma}dx = k\frac{1}{2\gamma+1}(n-k)^{2\gamma+1}. \tag{2.3}$$

Now, from Theorem 2.2 with $\nu = 0$, uniformly in $k \in [1, \frac{n}{2}]$:

$$\frac{1}{\sigma_n(\mathbf{a})}\sum_{i=1}^{k}\left(a_n\left(R_{n-i+1}\right) - \bar{a}_n\right) = \sqrt{n}B\left(\frac{k}{n}\right) + O_P\left(\sqrt{\frac{k(n-k)}{n}}\right) = \sqrt{n}B\left(\frac{k}{n}\right) + O_P\left(\sqrt{k}\right).$$

Since $\left\{\sqrt{n}B\left(\frac{k}{n}\right) : k = 0, \ldots, n\right\} \stackrel{\mathcal{D}}{=} \left\{W(k) - \frac{k}{n}W(n) : k = 0, \ldots, n\right\}$, where $\{W(t) : t \geqslant 0\}$ is a standard Wiener process, we conclude from the law of the iterated logarithm

$$\begin{aligned}
&\frac{1}{\sigma_n(\mathbf{a})}\max_{n-\log n<k<n}\frac{\left|\sum_{i=1}^{n}(i-k)_+^{\gamma}(a_n(R_i) - \bar{a}_n)\right|}{\left(\sum_{i=1}^{n-k} i^{2\gamma} - \frac{1}{n}\left(\sum_{i=1}^{n-k} i^{\gamma}\right)^2\right)^{1/2}} \\
&= \frac{1}{\sigma_n(\mathbf{a})}\max_{1<k<\log n}\frac{\left|\sum_{l=1}^{k}(l^{\gamma} - (l-1)^{\gamma})\sum_{i=1}^{k-l+1}(a_n(R_{n-i+1}) - \bar{a}_n)\right|}{\left(\sum_{i=1}^{k} i^{2\gamma} - \frac{1}{n}\left(\sum_{i=1}^{k} i^{\gamma}\right)^2\right)^{1/2}} \\
&= O_P\left(\max_{1<k<\log n}\frac{\left|\sum_{l=1}^{k}(l^{\gamma} - (l-1)^{\gamma})\left(W(k-l+1) - \frac{k-l+1}{n}W(n)\right)\right|}{\left(\sum_{i=1}^{k} i^{2\gamma} - \frac{1}{n}\left(\sum_{i=1}^{k} i^{\gamma}\right)^2\right)^{1/2}}\right) \\
&\quad + O_P\left(\max_{1<k<\log n}\frac{\left|\sum_{l=1}^{k}(l^{\gamma} - (l-1)^{\gamma})\sqrt{k-l+1}\right|}{\left(\sum_{i=1}^{k} i^{2\gamma} - \frac{1}{n}\left(\sum_{i=1}^{k} i^{\gamma}\right)^2\right)^{1/2}}\right) \\
&= o_P\left(\sqrt{\log\log n}\right).
\end{aligned}$$

Hence it suffices to investigate the maximum over $k \in [1, n - \log n]$. Let

$$\widehat{T}_n := \max_{1\leqslant k\leqslant n-\log n}\frac{\left|\sum_{i=1}^{n}(i-k)_+^{\gamma}(X_i - \frac{1}{n}\sum_{l=1}^{n} X_l)\right|}{\left(\sum_{i=1}^{n-k} i^{2\gamma} - \frac{1}{n}\left(\sum_{i=1}^{n-k} i^{\gamma}\right)^2\right)^{1/2}}$$

resp.

$$\widetilde{T}_n := \max_{1\leqslant k\leqslant n-\log n}\frac{\left|\sum_{i=1}^{n}(i-k)_+^{\gamma}\left(\widetilde{\Pi}_n(i) - \widetilde{\Pi}_n(i-1)\right)\right|}{\left(\sum_{i=1}^{n-k} i^{2\gamma} - \frac{1}{n}\left(\sum_{i=1}^{n-k} i^{\gamma}\right)^2\right)^{1/2}}$$

be the corresponding test statistics based on i.i.d. $N(0,1)$ random variables X_i resp. on the distributionally equivalent versions of $a_n(R_i)$. We choose X_i such that $B\left(\frac{k}{n}\right) = \frac{1}{\sqrt{n}}\left(\sum_{i=1}^{k} X_i - \frac{k}{n}\sum_{i=1}^{n} X_i\right)$, with $\{B(t)\}$ denoting the Brownian bridge of Theorem 2.2.

By the same application of the law of the iterated logarithm as above,

$$\max_{n-\log n\leqslant k\leqslant n}\frac{\left|\sum_{i=1}^{n}(i-k)_+^{\gamma}(X_i - \frac{1}{n}\sum_{i=1}^{n} X_i)\right|}{\left(\sum_{i=1}^{n-k} i^{2\gamma} - \frac{1}{n}\left(\sum_{i=1}^{n-k} i^{\gamma}\right)^2\right)^{1/2}} = o_P\left(\sqrt{\log\log n}\right).$$

Since $\alpha_n\widetilde{T}_n - \beta_n = \left(\alpha_n\widehat{T}_n - \beta_n\right) + \alpha_n\left(\widetilde{T}_n - \widehat{T}_n\right)$, and since Theorem 1.1 implies that $\alpha_n\widehat{T}_n - \beta_n$ has a limiting Gumbel distribution, it suffices to show that $\alpha_n\left(\widetilde{T}_n - \widehat{T}_n\right) = o_P(1)$. We set $Y_{in} := \widetilde{\Pi}_n(i) -$

$\widetilde{\Pi}_n(i-1) - (X_i - \overline{X}_n)$, where $\overline{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$, and $S_n(l) := \sum_{i=1}^l Y_{in}$. Then,

$$\begin{aligned}
\left|\widetilde{T}_n - \widehat{T}_n\right| &\leqslant \max_{1\leqslant k\leqslant n-\log n} \sqrt{\frac{n}{n\sum_{i=1}^{n-k} i^{2\gamma} - \left(\sum_{i=1}^{n-k} i^{\gamma}\right)^2}} \left|\sum_{i=1}^{n}(i-k)_+^{\gamma} Y_{in}\right| \\
&\leqslant \max_{1\leqslant k\leqslant n-\log n} \sqrt{\frac{n}{n\sum_{i=1}^{n-k} i^{2\gamma} - \left(\sum_{i=1}^{n-k} i^{\gamma}\right)^2}} \sum_{l=1}^{n-k} |S_n(l+k-1)|(l^{\gamma} - (l-1)^{\gamma}) \\
&\leqslant \max_{1\leqslant k<n} \left(\frac{k(n-k)}{n}\right)^{\nu} \frac{n}{\sqrt{k(n-k)}} \left|\frac{1}{\sqrt{n}}\widetilde{\Pi}_n(k) - B\left(\frac{k}{n}\right)\right| \\
&\quad \times \max_{1\leqslant k\leqslant n-\log n} n^{\nu} \sum_{l=1}^{n-k} \frac{((l+k-1)(n-l-k+1))^{1/2-\nu}}{\sqrt{n\sum_{i=1}^{n-k} i^{2\gamma} - \left(\sum_{i=1}^{n-k} i^{\gamma}\right)^2}} (l^{\gamma} - (l-1)^{\gamma}),
\end{aligned}$$

where $0 < \nu < \min\left(\frac{\delta}{2(2+\delta)}, \frac{1}{4}\right)$ as in Theorem 2.2. This theorem also implies

$$\max_{1\leqslant k<n} \left(\frac{k(n-k)}{n}\right)^{\nu} \frac{n}{\sqrt{k(n-k)}} \left|\frac{1}{\sqrt{n}}\widetilde{\Pi}_n(k) - B\left(\frac{k}{n}\right)\right| = O_P(1),$$

which means, that it suffices to show

$$\max_{1\leqslant k\leqslant n-\log n} n^{\nu} \sum_{l=1}^{n-k} \frac{((l+k-1)(n-l-k+1))^{1/2-\nu}}{\sqrt{n\sum_{i=1}^{n-k} i^{2\gamma} - \left(\sum_{i=1}^{n-k} i^{\gamma}\right)^2}} (l^{\gamma} - (l-1)^{\gamma}) = o\left((\log\log n)^{-1/2}\right).$$

The latter rate can be obtained through a straightforward calculation, taking (2.3) into account together with the following estimate:

$$n\sum_{i=1}^{n-k} i^{2\gamma} - \left(\sum_{i=1}^{n-k} i^{\gamma}\right)^2 \geqslant c_{\gamma} n(n-k)^{2\gamma+1} \quad \text{for all } n \geqslant n_{\gamma}, \tag{2.4}$$

where $c_{\gamma} > 0$ and n_{γ} depends only on γ. This completes the proof. For details we refer to Kirch [9], Corollary 5.2.3. ■

We are now ready to study the following permutation statistic:

$$T_n^{(1)}(\mathbf{R}) = \frac{1}{\hat{\sigma}_n} \max_{1\leqslant k<n} \frac{\left|\sum_{i=1}^n (i-k)_+^{\gamma}(X_{R_i} - \overline{X}_n)\right|}{\left(\sum_{i=1}^{n-k} i^{2\gamma} - \frac{1}{n}\left(\sum_{i=1}^{n-k} i^{\gamma}\right)^2\right)^{1/2}},$$

where $\mathbf{R} = (R_1, \ldots, R_n)$ is a random permutation of $(1, \ldots, n)$. We consider now the conditional distribution of $T_n^{(1)}(\mathbf{R})$ given the original observations $X_1, \ldots, X_n$, i.e. the randomness is only generated by the random permutation $\mathbf{R} = (R_1, \ldots, R_n)$.

The following theorem proves that this statistic conditionally on the given observations has a.s. the same asymptotic behavior - both under the null hypothesis and under the alternative - as that of $T_n^{(1)}$ under the null hypothesis (cf. Theorem 1.1).

Theorem 2.3. *Let $X_1, \ldots, X_n$ be observations satisfying* (1.1) *and* (1.2). *Moreover, let $|d| = |d_n| \leqslant D$. Then, for all $x \in \mathbb{R}$, as $n \to \infty$,*

$$P\left(\alpha_n T_n^{(1)}(\boldsymbol{R}) - \beta_n \leqslant x \middle| X_1, \ldots, X_n\right) \to \exp(-2e^{-x}) \qquad a.s.,$$

where α_n, $\beta_n = \beta_n(\gamma)$ are as in Theorem 1.1.

Proof. It is sufficient to verify the assumptions of Theorem 2.1 with $a_n(i) = X_i$, $i = 1, \ldots, n$. First we have

$$\overline{X}_n = \mu + \overline{e}_n + d_n n^{-\gamma-1} \sum_{l=1}^{n} (l - m_n)_+^{\gamma}.$$

Hence

$$\frac{1}{n}\sum_{i=1}^{n}(X_i-\overline{X}_n)^2$$
$$\geqslant \frac{1}{n}\sum_{i=1}^{n}(e_i-\overline{e}_n)^2+2d_nn^{-\gamma}\frac{1}{n}\sum_{i=1}^{n}(i-m_n)_+^{\gamma}e_i-2d_nn^{-\gamma-1}\sum_{l=1}^{n-m_n}l^{\gamma}\frac{1}{n}\sum_{i=1}^{n}e_i.$$

It is enough to show that the second term converges to 0 a.s., because then, by the strong law of large numbers,

$$\liminf_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}(X_i-\overline{X}_n)^2\geqslant \operatorname{var}e_1 \qquad \text{a.s.}$$

Now, by partial summation,

$$\sum_{i=1}^{n}(i-m_n)_+^{\gamma}e_i=S_n(n-m_n)_+^{\gamma}-\sum_{i=1}^{n-1}S_i\left((i+1-m_n)_+^{\gamma}-(i-m_n)_+^{\gamma}\right), \tag{2.5}$$

where $S_i:=\sum_{j=1}^{i}e_j$, and, from the law of the iterated logarithm,

$$\frac{1}{n^{\gamma+1}}\sum_{i=1}^{n-1}S_i\left((i+1-m_n)_+^{\gamma}-(i-m_n)_+^{\gamma}\right)=O\bigg(\frac{1}{n^{\gamma+1}}\sum_{i=1}^{n-1}i^{3/4}\left((i+1-m_n)_+^{\gamma}-(i-m_n)_+^{\gamma}\right)\bigg)$$
$$=o(1) \qquad \text{a.s.},$$

where the last estimate follows via the mean value theorem. Using (2.5) together with the strong law of large numbers, we get indeed, as $n\to\infty$,

$$\frac{d_n}{n^{\gamma+1}}\sum_{i=1}^{n}(i-m_n)_+^{\gamma}e_i$$
$$=\frac{d_n(n-m_n)_+^{\gamma}}{n^{\gamma}}\frac{S_n}{n}-\frac{d_n}{n^{\gamma+1}}\sum_{i=1}^{n-1}S_i\left((i+1-m_n)_+^{\gamma}-(i-m_n)_+^{\gamma}\right)$$
$$\to 0 \qquad \text{a.s.}$$

On the other hand, for suitable constants c and C, and $n\geqslant n_0$,

$$\frac{1}{n}\sum_{i=1}^{n}\left|X_i-\overline{X}_n\right|^{2+\delta}$$
$$=\frac{1}{n}\sum_{i=1}^{n}\left|e_i-\overline{e}_n+d_nn^{-\gamma}\left((i-m_n)_+^{\gamma}-\frac{1}{n}\sum_{l=1}^{n}(l-m_n)_+^{\gamma}\right)\right|^{2+\delta}$$
$$\leqslant c\,\frac{1}{n}\sum_{i=1}^{n}|e_i|^{2+\delta}+c|\overline{e}_n|^{2+\delta}+c\,d_n^{2+\delta}n^{-2\gamma-\delta\gamma-1}\sum_{i=1}^{n-m_n}i^{2\gamma+\gamma\delta}$$
$$+c\,d_n^{2+\delta}n^{-2\gamma-\delta\gamma-2-\delta}\left(\sum_{l=1}^{n-m_n}l^{\gamma}\right)^{2+\delta}$$
$$\leqslant C \qquad \text{a.s.}$$

An application of Theorem 2.1 now completes the proof. ■

3 Permutation statistics for changes of stochastic processes under strong invariance

Next we study models 2) and 3) of Section 1. For model 2), we first need to investigate the asymptotic behavior of the corresponding rank statistic:

Theorem 3.1. *Let $(R_1, \ldots, R_n)$ be a random permutation of $(1, \ldots, n)$, and $a_n(1), \ldots, a_n(n)$ be scores satisfying the following conditions:*

$$\sum_{i=1}^{n} a_n(i) = 0, \qquad \frac{1}{n}\sum_{i=1}^{n} a_n^2(i) \to 1, \tag{3.1}$$

and

$$\frac{1}{n}\max_{1\leqslant i\leqslant n} a_n^2(i) \to 0. \tag{3.2}$$

Then, as $n \to \infty$,

$$\max_{1\leqslant k\leqslant n} \frac{1}{\sqrt{n}}\Big|\sum_{i=1}^{k} a_n(R_i)\Big| \xrightarrow{\mathcal{D}} \sup_{0\leqslant t\leqslant 1} |B(t)|,$$

where $\{B(t) : 0 \leqslant t \leqslant 1\}$ denotes a Brownian bridge.

Proof. It follows from Theorem 24.2 in Billingsley [2]. ■

Lemma 3.1. *a) Let $X_{1n}, \ldots, X_{nn}$ be independent r.v.'s with $\mathrm{E}\, X_{in}^4 \leqslant D < \infty$ for all i, n. Then*

$$\frac{1}{n}\sum_{i=1}^{n}(X_{in} - \mathrm{E}\, X_{in}) \to 0 \qquad a.s. \quad (n \to \infty).$$

b) Let $\{W_n(t)\colon t \geqslant 0\}$, $n \in \mathbb{N}$, be Wiener processes and f be a function of n, then

$$W_n(f(n)) = O\left(\sqrt{f(n)\log n}\right) \qquad a.s. \quad (n \to \infty).$$

Proof. a) It follows immediately from Markov's inequality.

b) Cf. Kirch [9], Theorem 10.0.2. ■

In the sequel we assume that there is a 1-1-correspondence between N and T, which is necessary to get a countable triangular array in N, and, in turn, allows us to use the preceding lemma.
Moreover, we assume $T^* = \theta T$, $0 < \theta \leqslant 1$, and $N = o(T^{1-2/(2+\delta)})$. Let $N^* = \lfloor \frac{NT^*}{T} \rfloor = \theta N(1 + o(1))$ and

$$\Delta Y_i = \begin{cases} b\left(Y\left(i\frac{T}{N}\right) - Y\left((i-1)\frac{T}{N}\right)\right) & , \quad i \leqslant N^*, \\ b(Y(T^*) - Y(\frac{N^*T}{N})) + b^*Y^*(\frac{(N^*+1)T}{N} - T^*) & , \quad i = N^*+1, \\ b^*\left(Y^*\left(i\frac{T}{N} - T^*\right) - Y^*\left((i-1)\frac{T}{N} - T^*\right)\right) & , \quad i \geqslant N^*+2. \end{cases} \tag{3.3}$$

Lemma 3.2. *a) It holds, as $N \to \infty$,*

$$\overline{\Delta Y} = \frac{1}{N}\sum_{i=1}^{N}\Delta Y_i = O\left(\frac{\sqrt{T}\log N}{N}\right) \qquad a.s.$$

b) i) For $s = 2, 3, 4$, as $N \to \infty$,

$$\frac{N^{(s-2)/2}}{T^{s/2}}\sum_{i=1}^{N}(\Delta Y_i)^s \quad \to \quad \mathrm{E}\, W(1)^s\,(\theta b^s + (1-\theta)(b^*)^s) \qquad a.s.,$$

where $W(1)$ has a standard normal distribution.

ii) For $\nu > 0$, as $N \to \infty$,

$$\frac{N^{(\nu-2)/2}}{T^{\nu/2}}\sum_{i=1}^{N}\left|\Delta Y_i - \overline{\Delta Y}\right|^{\nu} = O(1) \qquad a.s.$$

c) For $\nu > 0$, as $N \to \infty$,

$$\frac{N^{(\nu-2)/2}}{T^{\nu/2}}\max_{1\leqslant i\leqslant N}\left|\Delta Y_i - \overline{\Delta Y}\right|^{\nu} = o(1) \qquad a.s.$$

Proof. The proof makes use of (1.3) – (1.5) in combination with Lemma 3.1 (for details confer Kirch [9], Theorem 10.0.1). ■

We are now prepared to investigate the following permutation statistics

$$M_T(\mathbf{R}) = \max_{1\leqslant k\leqslant N}\Big\{\frac{1}{\sqrt{T}}\frac{1}{\widehat{b}_T}\Big|\sum_{i=1}^{k}\big(\Delta Z_{R_i,T}-\overline{\Delta Z}_T\big)\Big|\Big\},$$

and

$$\widetilde{M}_T(\mathbf{R}) = \max_{1\leqslant k\leqslant N}\Big\{\frac{1}{\sqrt{T}}\frac{1}{\widehat{c}_T}\Big|\sum_{i=1}^{k}\Big(\widetilde{\Delta Z}^2_{R_i,T}-\overline{\widetilde{\Delta Z}}^2_T\Big)\Big|\Big\}.$$

Here again, $\mathbf{R}=(R_1,\ldots,R_n)$ denotes a random permutation of $(1,\ldots,n)$.

Theorem 3.2. *Let $\{Z(t):t\geq 0\}$ be a process according to model* (1.3). *Let $T^*=\theta T$, $0<\theta\leqslant 1$, $N=o(T^{1-2/(2+\delta)})$, and in b) also $a=a^*$. Then, for all $x\in\mathbb{R}$, as $T\to\infty$,*

a)

$$P\big(M_T(\boldsymbol{R})\leqslant x\big|Z(t),0\leqslant t\leqslant T\big)\to P\big(\sup_{0\leqslant t\leqslant 1}|B(t)|\leqslant x\big)\qquad a.s.$$

b)

$$P\big(\widetilde{M}_T(\boldsymbol{R})\leqslant x\big|Z(t),0\leqslant t\leqslant T\big)\to P\big(\sup_{0\leqslant t\leqslant 1}|B(t)|\leqslant x\big)\qquad a.s.,$$

where $\{B(t):0\leqslant t\leqslant 1\}$ is a Brownian bridge.

Proof. First note that, for the increments of $\{Z(t)\}$, we have

$$\Delta Z_{i,T}=\begin{cases} a\frac{T}{N}+\Delta Y_i & ,\quad i\leqslant N^*,\\ a\left(T^*-N^*\frac{T}{N}\right)+a^*\left((N^*+1)\frac{T}{N}-T^*\right)+\Delta Y_{N^*+1} & ,\quad i=N^*+1,\\ a^*\frac{T}{N}+\Delta Y_i^* & ,\quad i\geqslant N^*+2,\end{cases}$$

with ΔY_i as in (3.3).

Now, for the proof of a), consider the scores $a_N(i)=\frac{1}{\hat{b}_T}\sqrt{\frac{N}{T}}(\Delta Z_{i,T}-\overline{\Delta Z_{i,T}})$, $i=1,\ldots,N$. Obviously, $\sum_{i=1}^N a_N(i)=0$ and $\frac{1}{N}\sum_{i=1}^N a_N^2(i)=1$, which means that it is sufficient to verify assumption (3.2) of Theorem 3.1.

In the sequel, c and C denote suitable constants which may be different in different places. We first consider the case $\theta<1$ and $a\neq a^*$. Here, for sufficiently large T,

$$\begin{aligned}\widehat{b}^2_T &= \frac{1}{T}\sum_{i=1}^{N}\Delta Z^2_{i,T}-\frac{N}{T}\overline{\Delta_T Z}^2\\ &=\frac{1}{T}\sum_{i=1}^{N}\Delta a_i^2-\frac{N}{T}\overline{\Delta a}^2+\frac{1}{T}\sum_{i=1}^{N}(\Delta Y_i)^2-\frac{N}{T}(\overline{\Delta Y})^2\\ &\quad -2\frac{1}{T}(aT^*+a^*(T-T^*))\overline{\Delta Y}\\ &\quad +\frac{2ab}{N}Y\left(N^*\frac{T}{N}\right)+\frac{2a^*b^*}{N}\left(Y^*(T-T^*)-Y^*\left((N^*+1)\frac{T}{N}-T^*\right)\right)\\ &\quad +\frac{2}{T}\left(a\left(T^*-N^*\frac{T}{N}\right)+a^*\left((N^*+1)\frac{T}{N}-T^*\right)\right)\Delta Y_{N^*+1}\\ &\geqslant c\frac{T}{N}\qquad \text{a.s.,}\end{aligned}\tag{3.4}$$

where

$$\Delta a_i = \begin{cases} a\frac{T}{N} & , \quad i \leqslant N^*, \\ a\left(T^* - N^*\frac{T}{N}\right) + a^*\left((N^*+1)\frac{T}{N} - T^*\right) & , \quad i = N^*+1, \\ a^*\frac{T}{N} & , \quad i \geqslant N^*+2, \end{cases}$$

and $\overline{\Delta a} = \frac{1}{N}\sum_{i=1}^N \Delta a_i = \frac{1}{N}\left(aT^* + a^*(T-T^*)\right)$. The last inequality in (3.4) follows from the fact that the first terms are the dominating ones. Indeed, since $\theta < 1$, $a \neq a^*$, for T sufficiently large,

$$\begin{aligned} &\frac{1}{T}\sum_{i=1}^N \Delta a_i^2 - \frac{N}{T}\overline{\Delta a}^2 \\ &\geqslant a^2\frac{T}{N^2}N^* + a^*\frac{T}{N^2}(N-N^*-1) \\ &\quad - a^2\frac{(T^*)^2}{TN} - (a^*)^2\frac{(T-T^*)^2}{TN} - 2aa^*\frac{T^*(T-T^*)}{TN} \\ &= (1+o(1))\left(a^2\frac{T}{N}\theta(1-\theta) + (a^*)^2\frac{T}{N}\theta(1-\theta) - 2aa^*\frac{T}{N}\theta(1-\theta)\right) - \frac{(a^*)^2T}{N^2} \\ &= (1+o(1))\left(\frac{T}{N}\theta(1-\theta)(a-a^*)^2\right) - (a^*)^2\frac{T}{N^2} \\ &\geqslant c\frac{T}{N} \qquad \text{a.s.} \end{aligned} \tag{3.5}$$

Next we prove that the other terms are of smaller order and hence are negligible. Lemma 3.1 b) gives

$$\begin{aligned} &\frac{2ab}{N}Y\left(N^*\frac{T}{N}\right) + \frac{2a^*b^*}{N}\left(Y(T-T^*) - Y\left((N^*+1)\frac{T}{N} - T^*\right)\right) \\ &= \frac{2ab}{N}W_T\left(N^*\frac{T}{N}\right) + \frac{2a^*b^*}{N}\left(W^*(T-T^*) - W^*\left((N^*+1)\frac{T}{N} - T^*\right)\right) + O\left(\frac{T^{1/(2+\delta)}}{N}\right) \\ &= O\left(\frac{\sqrt{T\log N}}{N}\right) \qquad \text{a.s.} \end{aligned} \tag{3.6}$$

Since $T^* - N^*\frac{T}{N} \leqslant \frac{T}{N}$ and $(N^*+1)\frac{T}{N} - T^* \leqslant \frac{T}{N}$, we also get

$$\begin{aligned} &\left|\frac{2}{T}\left(a\left(T^* - N^*\frac{T}{N}\right) + a^*\left((N^*+1)\frac{T}{N} - T^*\right)\right)\Delta Y_{N^*+1}\right| \\ &\leqslant \frac{2}{N}\left((|a|+|a^*|)|b|\left|W(T^*) - W\left(N^*\frac{T}{N}\right)\right| + (|a|+|a^*|)|b^*|\left|W^*\left((N^*+1)\frac{T}{N} - T^*\right)\right|\right) \\ &\quad + O\left(\frac{T^{1/(2+\delta)}}{N}\right) \\ &= O\left(\frac{\sqrt{T\log N}}{N}\right) \qquad \text{a.s.} \end{aligned} \tag{3.7}$$

Lemma 3.2 further implies

$$\begin{aligned} &\frac{1}{T}\sum_{i=1}^N (\Delta Y_i)^2 - \frac{N}{T}(\overline{\Delta Y})^2 - 2\frac{1}{T}(aT^* + a^*(T-T^*))\overline{\Delta Y} \\ &= O\left(1 + \frac{\log N}{N} + \frac{\sqrt{T\log N}}{N}\right) \qquad \text{a.s.,} \end{aligned}$$

which proves (3.4). Note that

$$\Delta a_i - \overline{\Delta a} = \begin{cases} (a-a^*)\frac{T-T^*}{N} & , \quad i \leqslant N^*, \\ (a-a^*)\frac{\vartheta T - T^*}{N} & , \quad i = N^*+1, \\ (a^*-a)\frac{T^*}{N} & , \quad i \geqslant N^*+2, \end{cases}$$

for some $0 \leqslant \vartheta \leqslant 1$, hence

$$\max_{1\leqslant i\leqslant N}(\Delta a_i - \overline{\Delta a})^2 = \begin{cases} \left(\frac{T-T^*}{N}(a-a^*)\right)^2 & , \quad T^* \leqslant T/2, \\ \left(\frac{T^*}{N}(a-a^*)\right)^2 & , \quad T^* > T/2. \end{cases}$$

On combining (3.4), Lemma 3.2 a) and Lemma 3.1 b) (i) we finally get (3.2), since

$$\begin{aligned}\frac{1}{N}\max_{1\leqslant i\leqslant N} a_N^2(i) &\leqslant 2\frac{1}{T\widehat{b}_T^2}\max_{1\leqslant i\leqslant N}(\Delta a_i-\overline{\Delta a})^2 \quad +2\frac{1}{T\widehat{b}_T^2}\max_{1\leqslant i\leqslant N}(\Delta Y_i-\overline{\Delta Y})^2\\ &\leqslant \frac{2}{c}\frac{1}{N}(a-a^*)^2+\frac{2}{c}\frac{N}{T}\left(\frac{1}{T}\sum_{i=1}^{N}(\Delta Y_i)^2-\frac{N}{T}(\overline{\Delta Y})^2\right)\\ &\to 0 \qquad \text{a.s.}\end{aligned} \tag{3.8}$$

On the other hand, if $\theta=1$ or $a=a^*$, we obtain from Lemma 3.2,

$$\begin{aligned}\widehat{b}_T^2 =&\frac{1}{T}\sum_{i=1}^{N}(\Delta Z_{i,T}-\overline{\Delta_T Z})^2=\frac{1}{T}\sum_{i=1}^{N}(\Delta Y_i)^2-\frac{N}{T}(\overline{\Delta Y})^2\\ &\to \theta b^2+(1-\theta)(b^*)^2\geqslant c>0 \qquad \text{a.s.},\end{aligned} \tag{3.9}$$

for T sufficiently large. Using Lemma 3.2 c), we arrive at (3.2), i.e.

$$\begin{aligned}\frac{1}{N}\max_{1\leqslant i\leqslant N} a_N^2(i) &= \frac{1}{\widehat{b}_T^2 T}\max_{1\leqslant i\leqslant N}(\Delta Y_i-\overline{\Delta Y})^2\\ &\to 0 \qquad \text{a.s.},\end{aligned} \tag{3.10}$$

which completes the proof of a).

For the proof of b), consider $a_N(i)=\frac{1}{\sqrt{T}\widehat{c}_T}\left(\left(\Delta Y_i-\overline{\Delta Y}\right)^2-\frac{1}{N}\sum_{l=1}^{N}\left(\Delta Y_l-\overline{\Delta Y}\right)^2\right)$. It suffices again to verify the assumptions of Theorem 3.1.

Since $a=a^*$, we get $\frac{1}{N}\sum_{i=1}^{N}a_N^2(i)=1$. Similarly as above, Lemma 3.2 gives

$$\frac{N}{T^2}\sum_{i=1}^{N}\left(\Delta Y_i-\overline{\Delta Y}\right)^4\to 3\left(\theta b^4+(1-\theta)(b^*)^4\right) \qquad \text{a.s.},$$

and

$$(\widehat{b}_T)^2=\left(\frac{1}{T}\sum_{i=1}^{N}(\Delta Y_i)^2-\frac{N}{T}\overline{\Delta Y}^2\right)^2\to\left(\theta b^2+(1-\theta)(b^*)^2\right)^2 \qquad \text{a.s.} \tag{3.11}$$

¿From Jensen's inequality we conclude

$$\begin{aligned}\lim_{T\to\infty}\frac{N}{T}\widehat{c}_T^2 &= \lim_{T\to\infty}\left(\frac{N}{T^2}\sum_{i=1}^{N}\left(\Delta Y_i-\overline{\Delta Y}\right)^4-(\widehat{b}_T^2)^2\right)\\ &=3(\theta b^4+(1-\theta)(b^*)^4)-(\theta b^2+(1-\theta)(b^*)^2)^2\\ &\geqslant 2(\theta b^4+(1-\theta)(b^*)^4)>0 \qquad \text{a.s.}\end{aligned}$$

So, an application of Lemma 3.2 results in

$$\begin{aligned}\frac{1}{N}\max_{1\leqslant k\leqslant N} a_N^2(k) &= \frac{1}{T\widehat{c}_T^2}\max_{1\leqslant k\leqslant N}\left(\left(\Delta Y_k-\overline{\Delta Y}\right)^2-\frac{1}{N}\sum_{i=1}^{N}\left(\Delta Y_i-\overline{\Delta Y}\right)^2\right)^2\\ &\leqslant C\left(\frac{N}{T^2}\max_{1\leqslant k\leqslant N}(\Delta Y_k-\overline{\Delta Y})^4+\frac{1}{N}\left(\frac{1}{T}\sum_{i=1}^{N}(\Delta Y_i-\overline{\Delta Y})^2\right)^2\right)\\ &\to 0 \qquad \text{a.s.},\end{aligned}$$

which completes the proof of b). ■

Finally we turn to model 3) of Section 1 and investigate the permutation analogue of (1.10), i.e. the statistic

$$T_N^{(2)}(\mathbf{R})=\sqrt{\frac{N}{T\,\widehat{b}_T^2}}\max_{1\leqslant k<N}\left\{\frac{\left|\sum_{i=1}^{N}(i-k)_+^\gamma(\Delta S_{R_i,T}-\overline{\Delta S}_N)\right|}{\left(\sum_{i=1}^{N-k}i^{2\gamma}-\frac{1}{N}\left(\sum_{i=1}^{N-k}i^\gamma\right)^2\right)^{1/2}}\right\}.$$

The following asymptotic applies:

Theorem 3.3. *Let $\{S(t) : t \geq 0\}$ be a process according to model* (1.8)*. Assume $T^* = \theta T$, $0 < \theta \leqslant 1$, and $N\sqrt{\log N} = o(\min(T^{1-2/(2+\delta)}, T^{1/2+\gamma}))$. Then, for all $x \in \mathbb{R}$, as $T \to \infty$,*

$$P\big(\alpha_N T_N^{(2)}(\mathbf{R}) - \beta_N \leqslant x \big| S(t), 0 \leqslant t \leqslant T\big) \to \exp(-2e^{-x}) \qquad a.s.,$$

where α_N, $\beta_N = \beta_N(\gamma)$ are as in Theorem 1.1 *(with N replacing n).*

Proof. First note that, for the increments of $\{S(t)\}$, we have

$$\Delta S_{i,T} = \begin{cases} \Delta Y_i & , \ i \leqslant N^*, \\ \Delta Y_{N^*+1} + \tilde{d}\left(\frac{(N^*+1)T}{N} - T^*\right)^{1+\gamma} & , \ i = N^*+1, \\ \Delta Y_i^* + \tilde{d}\left(\left(\frac{iT}{N} - T^*\right)^{1+\gamma} - \left(\frac{(i-1)T}{N} - T^*\right)^{1+\gamma}\right) & , \ i \geqslant N^*+2. \end{cases}$$

In case of the null hypothesis, i.e. for $\theta = 1$, we can immediately verify the assumptions of Theorem 2.1 for $a_n(i) := \sqrt{\frac{N}{T}}\Delta S_{i,T}$ by using Lemma 3.2.

On the other hand, in case of $\theta < 1$, we use $a_n(i) := \frac{N}{T^{1+\gamma}}\Delta S_{i,T}$. First, via the mean value theorem,

$$\begin{aligned} &\frac{N^{1+\delta}}{T^{(1+\gamma)(2+\delta)}}\left(\sum_{i=N^*+2}^{N}\left|\left(i\frac{T}{N} - T^*\right)^{1+\gamma} - \left((i-1)\frac{T}{N} - T^*\right)^{1+\gamma}\right|^{2+\delta} + \left|(N^*+1)\frac{T}{N} - T^*\right|^{(1+\gamma)(2+\delta)}\right) \\ &= O\Big(\frac{N^{1+\delta}}{T^{(1+\gamma)(2+\delta)}} N \frac{T^{(1+\gamma)(2+\delta)}}{N^{2+\delta}}\Big) = O(1), \end{aligned}$$

which, together with Lemma 3.2, gives

$$\frac{1}{N}\sum_{i=1}^{N}\left|\frac{N}{T^{1+\gamma}}\Delta S_{i,T} - \frac{N}{T^{1+\gamma}}\overline{\Delta S}_n\right|^{2+\delta} = O(1) \qquad \text{a.s.}$$

In order to verify the second assumption of Theorem 2.1, we first realize, by using partial summation, the mean value theorem and Lemmas 3.1 resp. 3.2, that

$$\begin{aligned} &\frac{N}{T^{2+2\gamma}}\left(\sum_{i=N^*+2}^{N}\Delta Y_i\left(\left(\frac{iT}{N} - T^*\right)^{1+\gamma} - \left(\frac{(i-1)T}{N} - T^*\right)^{1+\gamma}\right)\right. \\ &\qquad\qquad \left. + \Delta Y_{N^*+1}\left(\frac{(N^*+1)T}{N} - T^*\right)^{1+\gamma}\right) \\ &= \frac{N^2}{T^{2+2\gamma}}\overline{\Delta Y}\left((T - T^*)^{1+\gamma} - \left(\frac{N-1}{N}T - T^*\right)^{1+\gamma}\right) \\ &\quad - \frac{N}{T^{2+2\gamma}}\sum_{k=N^*+2}^{N-1}\left(bY(T^*) + b^*Y^*\left(k\frac{T}{N} - T^*\right)\right) \\ &\qquad\qquad \cdot\left(\left((k+1)\frac{T}{N} - T^*\right)^{1+\gamma} - 2\left(k\frac{T}{N} - T^*\right)^{1+\gamma} + \left((k-1)\frac{T}{N} - T^*\right)^{1+\gamma}\right) \qquad (3.12) \\ &\quad - \frac{N}{T^{2+2\gamma}}\left(bY(T^*) + b^*Y^*\left((N^*+1)\frac{T}{N} - T^*\right)\right) \\ &\qquad \cdot\left(\left((N^*+2)\frac{T}{N} - T^*\right)^{1+\gamma} - 2\left((N^*+1)\frac{T}{N} - T^*\right)^{1+\gamma}\right) \\ &= o(1) + O\left(\frac{1}{T^{1+\gamma}}\sum_{k=N^*+1}^{N}\left|bY(T^*) + b^*Y^*\left(k\frac{T}{N} - T^*\right)\right|\right) \\ &= o(1) + O\left(\frac{N\sqrt{\log N}}{T^{1/2+\gamma}}\right) + O\left(\frac{NT^{1/(2+\delta)}}{T^{1+\gamma}}\right) \\ &= o(1) \qquad \text{a.s.} \end{aligned}$$

Next we have

$$\frac{N}{T^{2+2\gamma}}\left(\sum_{i=N^*+2}^{N}\left(\left(i\frac{T}{N}-T^*\right)^{1+\gamma}-\left((i-1)\frac{T}{N}-T^*\right)^{1+\gamma}\right)^2+\left((N^*+1)\frac{T}{N}-T^*\right)^{2(1+\gamma)}\right)$$
$$\geqslant \frac{1}{N}(1+\gamma)^2\sum_{i=N^*+1}^{N-1}\left(\frac{i}{N}-\theta\right)^{2\gamma}$$
$$\geqslant \frac{1}{N}(1+\gamma)^2\int_{N^*}^{N-1}\left(\frac{x}{N}-\theta\right)^{2\gamma}dx$$
$$=(1+o(1))\frac{(1+\gamma)^2}{2\gamma+1}(1-\theta)^{2\gamma+1},$$

which shows that

$$\begin{aligned}&\frac{\tilde{d}^2N}{T^{2+2\gamma}}\left(\sum_{i=N^*+2}^{N}\left(\left(i\frac{T}{N}-T^*\right)^{1+\gamma}-\left((i-1)\frac{T}{N}-T^*\right)^{1+\gamma}\right)^2+\left((N^*+1)\frac{T}{N}-T^*\right)^{2(1+\gamma)}\right)\\&-\frac{\tilde{d}^2N^2}{T^{2+2\gamma}}\left(\frac{1}{N}(T-T^*)^{1+\gamma}\right)^2\\&\geqslant(1+o(1))\frac{(1-\theta)^{2\gamma+1}}{2\gamma+1}\left(\gamma^2+\theta(2\gamma+1)\right).\end{aligned}\tag{3.13}$$

On combining (3.12), (3.13) and Lemma 3.2, we get indeed, for large T,

$$\frac{N}{T^{2+2\gamma}}\sum_{i=1}^{N}\left(\Delta S_{i,T}-\overline{\Delta S}_n\right)^2\geqslant c(\theta)$$

with some $c(\theta)>0$, which completes the proof. ■

4 Simulations

So far, we have only proven that the permutation principle is asymptotically applicable for processes satisfying models 1) to 3). Now we want to describe the results of some simulation studies to get an idea, how good the permutation method is in comparison to the original method.

4.1 Gradual changes in the mean of an i.i.d. sequence

In a first study, we generated data according to model (1.1) using the following parameters:

- $n = 100, 200$,

		Normally distributed errors				**Laplace distributed errors**			
n	γ	**90%**	**95%**	**97.5%**	**99%**	**90%**	**95%**	**97.5%**	**99%**
100	0.25	2.602	2.883	3.132	3.439	2.713	3.063	3.405	3.753
100	0.5	2.445	2.765	3.017	3.344	2.611	2.972	3.312	3.746
100	1	2.338	2.626	2.887	3.229	2.439	2.801	3.158	3.663
100	2	2.226	2.506	2.768	3.093	2.332	2.741	3.092	3.585
200	0.25	2.630	2.911	3.160	3.431	2.731	3.091	3.412	3.865
200	0.5	2.483	2.778	3.007	3.253	2.565	2.923	3.244	3.694
200	1	2.361	2.643	2.911	3.218	2.458	2.837	3.172	3.585
200	2	2.274	2.567	2.817	3.113	2.359	2.743	3.146	3.600

Table 4.1.1: Simulated critical values (under the null hypothesis)

				Normally distributed errors				Laplace distributed errors						Normally distributed errors				Laplace distributed errors			
γ	**d**	**n**	**m**	**90%**	**95%**	**97.5%**	**99%**	**90%**	**95%**	**97.5%**	**99%**	**n**	**m**	**90%**	**95%**	**97.5%**	**99%**	**90%**	**95%**	**97.5%**	**99%**
0.25	0	100		2.667	2.921	3.157	3.432	2.355	2.639	2.875	3.059	200		2.646	2.912	3.117	3.380	2.547	2.853	3.150	3.332
0.25	0.25	100	25	2.676	2.953	3.191	3.452	2.325	2.620	2.857	3.070	200	50	2.648	2.917	3.097	3.388	2.529	2.822	3.076	3.291
0.25	0.5	100	25	2.713	2.961	3.186	3.474	2.306	2.578	2.846	3.045	200	50	2.636	2.888	3.088	3.383	2.483	2.774	2.969	3.239
0.25	1	100	25	2.678	2.894	3.132	3.372	2.204	2.475	2.826	2.903	200	50	2.543	2.791	2.985	3.238	2.349	2.583	2.847	3.025
0.25	2	100	25	2.366	2.618	2.765	3.033	1.908	2.169	2.406	2.524	200	50	2.234	2.427	2.609	2.844	2.002	2.214	2.412	2.597
0.25	4	100	25	1.637	1.782	1.940	2.159	1.396	1.537	1.701	1.813	200	50	1.542	1.698	1.845	1.988	1.408	1.561	1.680	1.869
0.25	0.25	100	50	2.664	2.938	3.182	3.436	2.347	2.631	2.899	3.133	200	100	2.643	2.898	3.093	3.372	2.543	2.830	3.098	3.269
0.25	0.5	100	50	2.658	2.932	3.175	3.440	2.336	2.618	2.892	3.169	200	100	2.622	2.862	3.067	3.349	2.506	2.761	2.970	3.203
0.25	1	100	50	2.590	2.845	3.091	3.360	2.234	2.552	2.820	3.126	200	100	2.521	2.751	2.941	3.217	2.349	2.565	2.805	2.991
0.25	2	100	50	2.271	2.488	2.703	2.972	2.008	2.272	2.515	2.711	200	100	2.173	2.376	2.530	2.742	1.990	2.179	2.346	2.559
0.25	4	100	50	1.566	1.730	1.884	2.065	1.468	1.624	1.725	1.887	200	100	1.499	1.635	1.774	1.929	1.386	1.527	1.644	1.788
0.25	0.25	100	75	2.696	2.929	3.167	3.432	2.344	2.633	2.883	3.042	200	150	2.660	2.920	3.114	3.400	2.548	2.843	3.126	3.300
0.25	0.5	100	75	2.683	2.931	3.158	3.437	2.340	2.629	2.884	3.029	200	150	2.668	2.907	3.116	3.417	2.528	2.809	3.086	3.232
0.25	1	100	75	2.645	2.871	3.103	3.369	2.304	2.592	2.859	2.996	200	150	2.615	2.876	3.135	3.375	2.443	2.722	2.899	3.105
0.25	2	100	75	2.443	2.643	2.852	3.127	2.216	2.451	2.577	2.802	200	150	2.449	2.720	2.916	3.204	2.223	2.435	2.613	2.807
0.25	4	100	75	1.866	2.063	2.230	2.428	1.785	1.965	2.077	2.279	200	150	1.936	2.172	2.355	2.549	1.724	1.934	2.057	2.227
0.5	0	100		2.570	2.774	3.031	3.305	2.224	2.593	2.764	3.059	200		2.490	2.772	3.057	3.266	2.421	2.744	3.028	3.321
0.5	0.25	100	25	2.605	2.812	3.051	3.343	2.222	2.577	2.803	3.090	200	50	2.518	2.802	3.020	3.275	2.397	2.709	2.987	3.280
0.5	0.5	100	25	2.614	2.817	3.063	3.330	2.185	2.548	2.848	3.091	200	50	2.528	2.821	2.969	3.261	2.360	2.679	2.919	3.206
0.5	1	100	25	2.561	2.762	3.015	3.278	2.102	2.406	2.815	3.014	200	50	2.465	2.714	2.937	3.150	2.238	2.520	2.769	2.984
0.5	2	100	25	2.306	2.557	2.737	3.018	1.911	2.162	2.556	2.635	200	50	2.216	2.439	2.642	2.830	1.962	2.199	2.400	2.575
0.5	4	100	25	1.671	1.849	2.028	2.225	1.439	1.644	1.784	1.930	200	50	1.604	1.776	1.919	2.087	1.434	1.583	1.728	1.927
0.5	0.25	100	50	2.583	2.804	3.050	3.336	2.231	2.567	2.789	3.114	200	100	2.517	2.800	3.006	3.269	2.397	2.725	3.002	3.261
0.5	0.5	100	50	2.572	2.807	3.049	3.344	2.225	2.510	2.828	3.147	200	100	2.520	2.805	2.982	3.246	2.372	2.660	2.950	3.177
0.5	1	100	50	2.542	2.778	3.019	3.290	2.157	2.457	2.844	3.147	200	100	2.467	2.718	2.972	3.178	2.287	2.607	2.779	3.000
0.5	2	100	50	2.345	2.580	2.808	3.060	2.081	2.304	2.668	2.921	200	100	2.247	2.532	2.700	2.918	2.063	2.316	2.449	2.661
0.5	4	100	50	1.794	1.989	2.171	2.390	1.670	1.842	2.039	2.181	200	100	1.731	1.902	2.101	2.247	1.568	1.740	1.879	2.050
0.5	0.25	100	75	2.556	2.789	3.047	3.321	2.234	2.587	2.769	3.053	200	150	2.504	2.804	3.038	3.265	2.417	2.735	3.013	3.275
0.5	0.5	100	75	2.551	2.790	3.042	3.338	2.237	2.584	2.780	3.039	200	150	2.522	2.810	3.000	3.275	2.400	2.718	3.000	3.220
0.5	1	100	75	2.530	2.788	3.038	3.326	2.227	2.536	2.799	2.989	200	150	2.541	2.757	3.041	3.288	2.367	2.696	2.933	3.117
0.5	2	100	75	2.465	2.713	2.968	3.217	2.211	2.438	2.756	2.897	200	150	2.462	2.734	2.958	3.224	2.280	2.577	2.731	2.928
0.5	4	100	75	2.211	2.403	2.623	2.864	2.007	2.241	2.419	2.631	200	150	2.201	2.458	2.671	3.039	1.996	2.207	2.452	2.553
1	0	100		2.462	2.667	2.907	3.209	2.118	2.451	2.731	3.059	200		2.372	2.702	2.910	3.181	2.306	2.598	2.913	3.321
1	0.25	100	25	2.480	2.684	2.925	3.219	2.097	2.448	2.791	3.097	200	50	2.369	2.709	2.924	3.176	2.286	2.585	2.872	3.265
1	0.5	100	25	2.460	2.682	2.921	3.203	2.087	2.450	2.800	3.115	200	50	2.371	2.707	2.907	3.144	2.232	2.606	2.870	3.194
1	1	100	25	2.490	2.655	2.913	3.173	2.071	2.413	2.768	3.095	200	50	2.345	2.649	2.873	3.072	2.182	2.492	2.725	3.004
1	2	100	25	2.279	2.495	2.723	2.993	1.979	2.186	2.582	2.863	200	50	2.176	2.470	2.659	2.908	1.985	2.221	2.441	2.630
1	4	100	25	1.794	1.975	2.158	2.366	1.579	1.775	2.092	2.156	200	50	1.722	1.931	2.105	2.264	1.530	1.737	1.856	2.036
1	0.25	100	50	2.469	2.699	2.921	3.216	2.121	2.457	2.767	3.086	200	100	2.373	2.709	2.921	3.182	2.294	2.590	2.887	3.273
1	0.5	100	50	2.474	2.721	2.925	3.224	2.118	2.458	2.794	3.105	200	100	2.376	2.714	2.932	3.174	2.277	2.617	2.874	3.216
1	1	100	50	2.450	2.736	2.925	3.191	2.097	2.454	2.816	3.118	200	100	2.367	2.699	2.911	3.133	2.230	2.587	2.819	3.079
1	2	100	50	2.397	2.649	2.854	3.111	2.102	2.320	2.762	3.048	200	100	2.286	2.603	2.812	3.079	2.133	2.426	2.666	2.852
1	4	100	50	2.141	2.341	2.544	2.766	1.848	2.173	2.485	2.629	200	100	2.042	2.283	2.500	2.816	1.855	2.074	2.292	2.456
1	0.25	100	75	2.467	2.682	2.907	3.213	2.130	2.454	2.744	3.059	200	150	2.370	2.706	2.917	3.187	2.302	2.597	2.911	3.303
1	0.5	100	75	2.466	2.694	2.910	3.209	2.136	2.459	2.756	3.059	200	150	2.376	2.710	2.922	3.183	2.296	2.592	2.894	3.283
1	1	100	75	2.469	2.715	2.922	3.213	2.144	2.468	2.776	3.053	200	150	2.389	2.710	2.937	3.174	2.289	2.615	2.877	3.240
1	2	100	75	2.462	2.737	2.925	3.185	2.144	2.459	2.801	3.025	200	150	2.379	2.675	2.926	3.147	2.249	2.607	2.837	3.137
1	4	100	75	2.351	2.695	2.867	3.134	2.175	2.447	2.790	2.907	200	150	2.327	2.615	2.845	3.114	2.175	2.546	2.744	2.916
2	0	100		2.336	2.629	2.817	3.113	2.029	2.362	2.731	3.059	200		2.287	2.613	2.828	3.114	2.220	2.550	2.853	3.272
2	0.25	100	25	2.351	2.640	2.828	3.134	2.031	2.375	2.753	3.083	200	50	2.291	2.616	2.841	3.097	2.197	2.550	2.857	3.231
2	0.5	100	25	2.361	2.653	2.820	3.121	2.036	2.377	2.748	3.099	200	50	2.280	2.615	2.892	3.085	2.197	2.537	2.842	3.198
2	1	100	25	2.397	2.649	2.826	3.088	2.072	2.357	2.733	3.105	200	50	2.254	2.637	2.827	3.054	2.153	2.493	2.809	3.086
2	2	100	25	2.318	2.571	2.755	3.003	2.010	2.297	2.681	3.015	200	50	2.195	2.527	2.729	3.078	2.050	2.351	2.627	2.779
2	4	100	25	2.046	2.217	2.442	2.688	1.753	2.096	2.412	2.554	200	50	1.936	2.200	2.435	2.715	1.768	2.009	2.256	2.384
2	0.25	100	50	2.340	2.632	2.823	3.120	2.031	2.367	2.746	3.066	200	100	2.286	2.619	2.840	3.098	2.216	2.552	2.853	3.266
2	0.5	100	50	2.346	2.638	2.826	3.126	2.034	2.375	2.759	3.071	200	100	2.290	2.620	2.846	3.101	2.198	2.551	2.852	3.241
2	1	100	50	2.349	2.641	2.820	3.130	2.036	2.381	2.755	3.078	200	100	2.294	2.619	2.874	3.088	2.203	2.547	2.845	3.220
2	2	100	50	2.357	2.657	2.820	3.105	2.052	2.376	2.757	3.073	200	100	2.275	2.612	2.841	3.061	2.181	2.523	2.813	3.115
2	4	100	50	2.313	2.587	2.798	3.021	2.137	2.345	2.718	2.997	200	100	2.227	2.553	2.768	3.099	2.086	2.430	2.692	2.841
2	0.25	100	75	2.336	2.630	2.816	3.118	2.029	2.364	2.735	3.060	200	150	2.288	2.613	2.827	3.106	2.220	2.549	2.852	3.269
2	0.5	100	75	2.336	2.630	2.817	3.121	2.030	2.365	2.738	3.061	200	150	2.288	2.612	2.830	3.098	2.219	2.549	2.851	3.266
2	1	100	75	2.337	2.633	2.818	3.124	2.030	2.369	2.744	3.063	200	150	2.289	2.614	2.834	3.100	2.219	2.552	2.850	3.260
2	2	100	75	2.338	2.629	2.821	3.119	2.034	2.373	2.755	3.066	200	150	2.291	2.615	2.843	3.101	2.211	2.551	2.846	3.257
2	4	100	75	2.342	2.640	2.819	3.119	2.042	2.384	2.768	3.070	200	150	2.295	2.620	2.842	3.093	2.213	2.538	2.836	3.255

Table 4.1.2: Simulated critical values using the permutation method

n	γ	90%-Quantile	95%-Quantile	97.5%-Quantile	99%-Quantile
100	0.5	1.738	2.150	2.554	3.082
100	1	2.298	2.710	3.114	3.643
100	2	2.130	2.542	2.946	3.474
200	0.5	1.868	2.263	2.649	3.155
200	1	2.353	2.747	3.134	3.640
200	2	2.192	2.586	2.973	3.479

Table 4.1.3: Asymptotic critical values

- $m = \frac{1}{4}n$, $\frac{1}{2}n$ and $\frac{3}{4}n$,
- $d = 0, \frac{1}{4}, \frac{1}{2}, 1, 2, 4$,
- $\gamma = \frac{1}{4}$, $\frac{1}{2}$, 1 and 2,
- Normally and Laplace distributed errors (each standardized).

This gives 256 combinations of the above parameters (note that we are in the case of the null hypothesis for $d = 0$).

First we compare the exact critical values with the asymptotic ones by generating $10,000$ series $X_1, \ldots, X_n$ according to the null hypothesis. The critical values we got from these series can be found in Table 4.1.1. For comparison, Table 4.1.3 shows the asymptotic ones ($\gamma = \frac{1}{4}$ is missing, since $H_{\frac{1}{4}}$ is unknown (cf. also Remark 12.2.10 in Leadbetter et al. [10])).
First note that the asymptotic critical values are rather too small (especially for $\gamma = \frac{1}{2}$).

			90%-Quantile		95%-Quantile		97.5%-Quantile		99%-Quantile	
γ	n	d	simul.	asym.	simul.	asym.	simul.	asym.	simul.	asym.
0.5	100	0	9.7	41.5	5.1	19.7	2.2	8.0	0.7	2.0
0.5	100	0.25	84.8	57.2	91.1	75.2	94.3	87.5	96.4	95.2
0.5	100	0.5	75.6	40.6	84.4	64.7	90.4	80.4	95.1	93.1
0.5	100	1	28.2	8.3	40.6	20.4	51.6	36.8	63.1	62.4
0.5	100	2	0.2	0	0.3	0	0.6	0.6	1.2	2.2
0.5	100	4	0	0	0	0	0	0	0	0
0.5	200	0	10.0	33.7	5.3	16.5	2.7	7.0	0.5	1.5
0.5	200	0.25	81.9	51.9	89.1	73.3	92.9	86.6	96.6	95.5
0.5	200	0.5	57.3	27.6	69.2	47.7	77.3	65.2	85.4	81.9
0.5	200	1	6.9	1.8	10.1	4.8	16.3	10.2	25.3	24.5
0.5	200	2	0	0	0	0	0	0	0	0
0.5	200	4	0	0	0	0	0	0	0	0
1	100	0	9.2	9.5	4.8	3.2	2.1	1.3	0.6	0.2
1	100	0.25	87.2	87.2	93.5	94.7	96.4	98.1	98.7	99.9
1	100	0.5	81.0	80.6	88.8	91.2	93.6	96.2	96.8	99.1
1	100	1	60.9	62.1	71.8	78.9	82.8	90.6	89.2	97.5
1	100	2	12.2	14.6	18.6	26.6	25.8	45.0	37.7	71.5
1	100	4	0	0	0	0	0	0.3	0.1	3.2
1	200	0	11.4	10.8	6.0	3.8	3.0	1.4	1.3	0.2
1	200	0.25	84.9	86.0	92.5	94.8	95.9	98.0	98.4	99.2
1	200	0.5	76.9	77.2	84.8	87.9	90.2	94.7	94.6	98.5
1	200	1	41.0	43.4	54.5	61.7	65.7	74.9	74.4	88.6
1	200	2	0.8	0.9	1.2	2.6	3.2	7.8	6.4	21
1	200	4	0	0	0	0	0	0	0	0

Table 4.1.4: Simulated α- resp. β-errors in % ($1,000$ repetitions, $m = \frac{n}{2}$)

N	90%	95%	97.5%	99%	90%	95%	97.5%	99%
	Partial sums				**Poisson Process**			
100	1.165	1.295	1.409	1.564	1.156	1.283	1.398	1.554
200	1.190	1.328	1.446	1.603	1.182	1.301	1.426	1.586

Table 4.2.1: Simulated critical values (under the null hypothesis)

Next we were interested in the critical values obtained via permutation, which we simulated using the following algorithm:

1. Generate a series $X_1, \dots, X_n$ according to the given parameters.
2. Generate a random permutation $\mathbf{R} = (R_1, \dots, R_n)$ of $(1, \dots, n)$ and calculate $T_n(\mathbf{R})$.
3. Repeat step 2) $10,000$ times.
4. Calculate the empirical quantiles of these $10,000$ values.

The result can be found in Table 4.1.2. It is important that, for the different combinations of parameters, we always used the same seed in step 1, which means that we also always used the same permutations for the calculation of the empirical quantiles (as long as the series had the same length n). We realize, that the quantiles are quite good, but decrease the more obvious the change (which is quite surprising considering that the test statistic increases if there is a change (cf. Hušková and Steinebach [8], Section 4)).

In addition, we approximated the α- resp. β-errors using the following simulation:

1. Generate a series $X_1, \dots, X_n$ according to the given parameters.
2. Calculate the critical values using the permutation principle (compare steps 2-4 above).
3. Calculate the value of the statistic and see, if we had rejected the null hypothesis using the quantiles from step 2 resp. the asymptotic ones.
4. Repeat steps 1)-3) $1,000$ times.
5. Calculate the empirical α- resp. β-errors from the $1,000$ simulations above.

Table 4.1.4 contains the results for normally distributed errors and a change at $\frac{n}{2}$. We realize that both methods give good results for $\gamma > \frac{1}{2}$. For $\gamma = \frac{1}{2}$ the α-error is far too high for the asymptotic method, especially with the 90%- and 95%-quantile, which is due to the fact, that the asymptotic critical value is too small (compare Tables 4.1.1 and 4.1.3).

Moreover, we were interested in the standard deviation of the critical values obtained by the permutation method. Under the null hypothesis ($n = 100$, $\gamma = \frac{1}{2}$, normally distributed errors) we got a standard deviation of 0.182 for the 90%-quantile and of 0.265 for the 99%-quantile. The result is similar for different parameters. Here we used $1,000$ trials of step 1 to 4 of the first simulation described above.

For our simulations we used the software package R, Version 1.2.3. On a Celeron with 466 MHz and 384 MB RAM the calculation of the permutation quantiles takes approximately 10 seconds in the case of 100 observations, and 30 seconds in the case of 200 (using $10,000$ permutations). This means that the method is indeed applicable.

4.2 Change in the mean of a stochastic process under strong invariance

The following simulations are based on partial sums of normally distributed random variables (with variance 1) (cf. Horváth and Steinebach [6], Example 1.1), and on a Poisson process (cf. Horváth and Steinebach [6], Example 1.2). More specifically, we simulated the increments of the partial sums as i.i.d. r.v.'s, and the increments of the Poisson process were taken at times $1, 2, \dots$ (instead of $i\frac{T}{N}$, $i = 1, \dots, N$, since this means only a scaling of the underlying r.v.'s). Other than that, we used the following parameters:

90%	95%	99%
1.224	1.358	1.628

Table 4.2.2: Asymptotic quantiles

- $N = 100,\ 200$
- $N^* = \frac{1}{4}N,\ \frac{1}{2}N,\ \frac{3}{4}N$
- $a^* - a = 0, 1, 2, 3, 4$

Here N^* is the change point, and we are in the case of the null hypothesis for $a^* - a = 0$.

Once again we generated 10,000 series of increments $\Delta Z_1, \ldots, \Delta Z_N$ for the different parameters under the null hypothesis. The resulting quantiles can be found in Table 4.2.1. The asymptotic critical values are given in Table 4.2.2 for comparison. First note that the asymptotic quantiles are slightly too large. Moreover, we realize that the exact ones are a little larger for the partial sums than for the Poisson process.

To study the critical values obtained from the permutation method, we used the same algorithm as in Section 4.1. The results can be found in Table 4.2.3.

We realize that these critical values give better estimates than the asymptotic ones. It also does not seem to be important where exactly the change point is located.

Next we simulated the α- resp. β-errors, as we did in Section 4.1, with a change at $N^* = \frac{3}{4}N$. The results can be found in Table 4.2.4. Expectedly, the α-errors are smaller for the asymptotic method, but the

N	N*	a* − a	90%	95%	97.5%	99%	90%	95%	97.5%	99%
			Partial sums				Poisson Process			
100		0	1.175	1.301	1.423	1.557	1.141	1.273	1.393	1.547
100	25	1	1.167	1.297	1.405	1.532	1.173	1.312	1.420	1.545
100	25	2	1.165	1.290	1.403	1.541	1.172	1.294	1.407	1.532
100	25	3	1.160	1.296	1.421	1.540	1.172	1.311	1.441	1.590
100	25	4	1.166	1.296	1.415	1.551	1.165	1.291	1.406	1.551
100	50	1	1.167	1.301	1.417	1.563	1.173	1.306	1.415	1.542
100	50	2	1.173	1.307	1.426	1.567	1.164	1.290	1.407	1.574
100	50	3	1.181	1.313	1.435	1.590	1.161	1.291	1.402	1.551
100	50	4	1.184	1.324	1.441	1.589	1.160	1.288	1.397	1.557
100	75	1	1.165	1.300	1.412	1.546	1.167	1.284	1.415	1.575
100	75	2	1.170	1.304	1.404	1.547	1.164	1.292	1.408	1.547
100	75	3	1.175	1.295	1.418	1.561	1.174	1.299	1.412	1.534
100	75	4	1.171	1.296	1.422	1.578	1.165	1.299	1.404	1.551
200		0	1.190	1.328	1.455	1.589	1.179	1.311	1.417	1.568
200	50	1	1.185	1.311	1.438	1.579	1.184	1.317	1.434	1.579
200	50	2	1.177	1.306	1.424	1.567	1.180	1.308	1.426	1.555
200	50	3	1.180	1.307	1.429	1.555	1.184	1.313	1.423	1.553
200	50	4	1.180	1.305	1.428	1.555	1.186	1.314	1.427	1.563
200	100	1	1.181	1.315	1.435	1.586	1.196	1.323	1.450	1.583
200	100	2	1.180	1.312	1.423	1.551	1.190	1.316	1.442	1.574
200	100	3	1.174	1.314	1.422	1.545	1.183	1.315	1.431	1.561
200	100	4	1.181	1.311	1.425	1.560	1.182	1.311	1.428	1.569
200	150	1	1.185	1.324	1.442	1.588	1.181	1.310	1.439	1.575
200	150	2	1.185	1.325	1.449	1.583	1.192	1.330	1.449	1.580
200	150	3	1.183	1.317	1.446	1.590	1.181	1.313	1.433	1.588
200	150	4	1.181	1.315	1.441	1.597	1.194	1.330	1.450	1.613

Table 4.2.3: Simulated critical values using the permutation method

N	$a^* - a$	90%-quantile		95%-quantile		99%-quantile	
		simul.	asym.	simul.	asym.	simul.	asym.
		Partial sums					
100	0	8.7	6.4	4.4	3.4	1.0	0.7
100	1	2.3	3.0	4.5	8.0	21.0	27.6
100	2	0	0	0	0	0	0
100	3	0	0	0	0	0	0
100	4	0	0	0	0	0	0
200	0	9.7	7.8	5.1	3.8	0.7	0.3
200	1	0	0	0	0	0.7	0.7
200	2	0	0	0	0	0	0
200	3	0	0	0	0	0	0
200	4	0	0	0	0	0	0
		Poisson Process					
100	0	9.9	7.2	5.0	3.1	0.9	0.4
100	1	9.6	12.4	17.2	20.6	36.1	43.4
100	2	0	0.1	0.1	0.1	0.3	0.8
100	3	0	0	0	0	0	0
100	4	0	0	0	0	0	0
200	0	10.1	8.2	5.0	3.6	0.2	0.2
200	1	0.7	0.8	1.6	2.1	5.0	6.0
200	2	0	0	0	0	0	0
200	3	0	0	0	0	0	0
200	4	0	0	0	0	0	0

Table 4.2.4: Simulated α- resp. β-errors in % ($1,000$ trials, $N^* = \frac{3}{4}N$)

β-errors are larger. Particularly, this is significant for smaller samples and in case of the Poisson process. In the latter cases we actually do get better results using the permutation method.

Moreover, we were interested in the standard deviation of the critical values obtained by the permutation method. Under the null hypothesis (Poisson process, N=100), we got a standard deviation of 0.01 for the 90%-quantile and of 0.019 for the 99%-quantile, for the partial sums the standard deviation was even smaller. The results are comparable for different parameters. As before, we used $1,000$ repetitions of step 1 to 4 of the first simulation above.

Again, computing time is not a problem here. For example, the calculation of the permutation quantiles for a series of length 100 takes approximately 3 seconds, and for length 200 approximately 5 seconds, using a Celeron with 466 MHz and 384 MB RAM and the software package R, Version 1.2.3.

4.3 Gradual change in the mean of a stochastic process under strong invariance

The following simulations are based on partial sums of normally distributed r.v. (with variance 1) (cf. Steinebach [13], Example 1.1) and on a Poisson process (cf. Steinebach [13], Example 1.2). More precisely, we simulated the increments of the partial sums as i.i.d. r.v.'s, and the increments of the Poisson process were taken at times $1, 2, \dots$ (instead of $i\frac{T}{N}$, $i = 1, \dots, N$, as above). The following parameters were chosen:

- $N = 100,\ 200$
- $N^* = \frac{1}{4}N,\ \frac{1}{2}N,\ \frac{3}{4}N$
- $\overline{d} = 0, \frac{1}{4}, \frac{1}{2}, 1, 2, 4$

Here N^* is the change point, and the null hypothesis is given for $\overline{d} = 0$. $\overline{d}$ is as in Remark 1.2 in order to be able to compare the results with those of Section 4.1. More precisely, the increments of the change

γ	$\bar{d}$	N	N*	Partial sums 90%	Partial sums 95%	Partial sums 97.5%	Partial sums 99%	Poisson process 90%	Poisson process 95%	Poisson process 97.5%	Poisson process 99%	N	N*	Partial sums 90%	Partial sums 95%	Partial sums 97.5%	Partial sums 99%	Poisson process 90%	Poisson process 95%	Poisson process 97.5%	Poisson process 99%
0.25	0	100		2.667	2.905	3.127	3.397	2.584	2.974	3.302	4.015	200		2.606	2.859	3.114	3.366	2.575	2.863	3.117	3.574
0.25	0.25	100	25	2.67	2.94	3.17	3.426	2.447	2.774	3.169	3.758	200	50	2.608	2.864	3.094	3.371	2.557	2.836	3.1	3.606
0.25	0.5	100	25	2.712	2.943	3.186	3.454	2.452	2.779	3.175	3.646	200	50	2.595	2.882	3.071	3.382	2.53	2.82	3.048	3.408
0.25	1	100	25	2.677	2.891	3.13	3.408	2.421	2.729	3.067	3.287	200	50	2.518	2.763	2.985	3.267	2.427	2.7	2.916	3.18
0.25	2	100	25	2.362	2.61	2.778	3.027	2.181	2.381	2.593	2.859	200	50	2.203	2.417	2.611	2.839	2.123	2.31	2.497	2.714
0.25	4	100	25	1.65	1.774	1.924	2.095	1.551	1.726	1.876	2.043	200	50	1.539	1.688	1.84	1.996	1.484	1.635	1.782	1.924
0.25	0.25	100	50	2.666	2.931	3.153	3.42	2.436	2.703	3.073	3.435	200	100	2.592	2.866	3.066	3.354	2.421	2.662	2.953	3.307
0.25	0.5	100	50	2.662	2.927	3.16	3.397	2.432	2.704	3.078	3.358	200	100	2.585	2.85	3.043	3.334	2.364	2.632	2.905	3.166
0.25	1	100	50	2.585	2.845	3.097	3.336	2.339	2.635	3.002	3.115	200	100	2.492	2.754	2.936	3.225	2.247	2.501	2.682	2.98
0.25	2	100	50	2.264	2.484	2.718	2.941	2.092	2.292	2.504	2.745	200	100	2.158	2.38	2.543	2.759	1.928	2.124	2.272	2.527
0.25	4	100	50	1.567	1.72	1.859	2.016	1.491	1.642	1.786	1.938	200	100	1.506	1.655	1.793	1.927	1.378	1.512	1.631	1.782
0.25	0.25	100	75	2.693	2.912	3.131	3.406	2.397	2.652	3.032	3.407	200	150	2.619	2.888	3.093	3.37	2.604	2.898	3.186	3.597
0.25	0.5	100	75	2.691	2.899	3.134	3.392	2.372	2.645	2.998	3.347	200	150	2.632	2.892	3.093	3.377	2.596	2.889	3.173	3.577
0.25	1	100	75	2.642	2.864	3.09	3.344	2.331	2.598	2.93	3.168	200	150	2.577	2.849	3.115	3.368	2.536	2.82	3.084	3.464
0.25	2	100	75	2.441	2.65	2.866	3.094	2.151	2.409	2.654	2.842	200	150	2.407	2.687	2.921	3.21	2.337	2.598	2.836	3.031
0.25	4	100	75	1.879	2.066	2.241	2.445	1.725	1.894	2.087	2.295	200	150	1.923	2.142	2.351	2.563	1.842	2.037	2.19	2.384
0.5	0	100		2.57	2.747	2.994	3.249	2.394	2.834	3.159	4.015	200		2.456	2.734	2.978	3.261	2.413	2.704	2.974	3.465
0.5	0.25	100	25	2.602	2.776	3.021	3.285	2.275	2.584	2.977	3.792	200	50	2.495	2.779	2.982	3.244	2.39	2.777	2.952	3.452
0.5	0.5	100	25	2.611	2.792	3.04	3.3	2.289	2.608	2.998	3.726	200	50	2.471	2.772	2.943	3.221	2.378	2.75	2.94	3.472
0.5	1	100	25	2.556	2.764	3.003	3.266	2.307	2.593	2.963	3.473	200	50	2.417	2.681	2.922	3.171	2.316	2.677	2.855	3.135
0.5	2	100	25	2.298	2.565	2.748	3.006	2.158	2.414	2.647	2.915	200	50	2.184	2.414	2.618	2.834	2.107	2.307	2.505	2.757
0.5	4	100	25	1.679	1.824	1.994	2.181	1.621	1.799	1.953	2.166	200	50	1.601	1.767	1.892	2.068	1.54	1.692	1.849	2.024
0.5	0.25	100	50	2.581	2.774	3.01	3.274	2.269	2.579	2.921	3.455	200	100	2.489	2.774	2.991	3.235	2.274	2.586	2.835	3.28
0.5	0.5	100	50	2.568	2.792	3.022	3.275	2.274	2.55	2.924	3.419	200	100	2.473	2.764	2.949	3.206	2.242	2.576	2.79	3.21
0.5	1	100	50	2.538	2.776	2.999	3.249	2.256	2.526	2.888	3.269	200	100	2.42	2.703	2.927	3.178	2.159	2.479	2.675	2.938
0.5	2	100	50	2.334	2.571	2.795	3.072	2.14	2.389	2.697	2.862	200	100	2.22	2.518	2.688	2.911	1.958	2.224	2.378	2.618
0.5	4	100	50	1.786	1.988	2.162	2.345	1.706	1.907	2.027	2.246	200	100	1.727	1.908	2.07	2.243	1.527	1.688	1.858	2.014
0.5	0.25	100	75	2.557	2.758	3	3.251	2.234	2.569	2.876	3.429	200	150	2.491	2.745	2.99	3.255	2.44	2.752	3.025	3.493
0.5	0.5	100	75	2.542	2.772	3.008	3.261	2.232	2.546	2.869	3.404	200	150	2.503	2.751	2.953	3.243	2.444	2.742	3.024	3.466
0.5	1	100	75	2.532	2.766	2.997	3.246	2.217	2.491	2.859	3.328	200	150	2.492	2.749	3.004	3.244	2.417	2.706	2.99	3.423
0.5	2	100	75	2.46	2.704	2.912	3.194	2.155	2.43	2.765	3.09	200	150	2.414	2.696	2.944	3.242	2.349	2.603	2.883	3.275
0.5	4	100	75	2.225	2.413	2.625	2.855	1.952	2.208	2.424	2.63	200	150	2.171	2.43	2.675	3.088	2.086	2.353	2.588	2.785
1	0	100		2.452	2.667	2.857	3.135	2.246	2.658	3.033	4.015	200		2.351	2.618	2.837	3.165	2.253	2.572	2.903	3.331
1	0.25	100	25	2.469	2.685	2.877	3.146	2.128	2.434	2.79	3.818	200	50	2.347	2.624	2.858	3.165	2.236	2.627	2.888	3.379
1	0.5	100	25	2.449	2.679	2.888	3.155	2.169	2.447	2.824	3.795	200	50	2.336	2.622	2.894	3.151	2.237	2.604	2.905	3.362
1	1	100	25	2.471	2.64	2.872	3.147	2.199	2.466	2.849	3.664	200	50	2.289	2.609	2.817	3.126	2.217	2.577	2.902	3.289
1	2	100	25	2.261	2.491	2.703	2.964	2.109	2.434	2.739	3.127	200	50	2.139	2.447	2.62	2.909	2.106	2.374	2.666	2.828
1	4	100	25	1.78	1.965	2.14	2.344	1.721	1.939	2.145	2.31	200	50	1.705	1.897	2.06	2.263	1.655	1.864	2.011	2.202
1	0.25	100	50	2.468	2.698	2.869	3.138	2.099	2.431	2.756	3.468	200	100	2.347	2.629	2.857	3.169	2.142	2.484	2.705	3.175
1	0.5	100	50	2.473	2.713	2.882	3.137	2.103	2.456	2.76	3.46	200	100	2.351	2.631	2.888	3.158	2.128	2.458	2.691	3.143
1	1	100	50	2.443	2.72	2.882	3.151	2.115	2.476	2.758	3.415	200	100	2.327	2.641	2.861	3.14	2.086	2.414	2.687	3.096
1	2	100	50	2.385	2.658	2.832	3.102	2.142	2.361	2.736	3.214	200	100	2.244	2.57	2.774	3.081	1.992	2.282	2.551	2.751
1	4	100	50	2.13	2.342	2.541	2.762	1.94	2.217	2.495	2.655	200	100	1.992	2.247	2.505	2.85	1.77	1.966	2.137	2.474
1	0.25	100	75	2.456	2.681	2.86	3.142	2.073	2.421	2.72	3.442	200	150	2.35	2.621	2.843	3.173	2.292	2.609	2.875	3.342
1	0.5	100	75	2.463	2.691	2.859	3.143	2.07	2.426	2.723	3.438	200	150	2.347	2.626	2.853	3.178	2.291	2.604	2.873	3.344
1	1	100	75	2.469	2.699	2.868	3.139	2.071	2.431	2.726	3.425	200	150	2.344	2.627	2.873	3.167	2.284	2.596	2.866	3.327
1	2	100	75	2.463	2.696	2.854	3.121	2.07	2.428	2.722	3.38	200	150	2.341	2.637	2.859	3.175	2.275	2.554	2.858	3.32
1	4	100	75	2.345	2.663	2.823	3.081	2.084	2.394	2.688	3.224	200	150	2.294	2.581	2.833	3.109	2.226	2.514	2.769	3.17
2	0	100		2.337	2.585	2.753	3.027	2.112	2.528	3.026	4.015	200		2.273	2.516	2.777	3.114	2.142	2.453	2.822	3.234
2	0.25	100	25	2.35	2.597	2.779	3.052	2.037	2.315	2.688	3.828	200	50	2.266	2.523	2.822	3.075	2.124	2.517	2.851	3.237
2	0.5	100	25	2.355	2.614	2.8	3.066	2.054	2.332	2.695	3.829	200	50	2.257	2.549	2.834	3.071	2.131	2.512	2.882	3.223
2	1	100	25	2.389	2.616	2.788	3.04	2.093	2.348	2.725	3.793	200	50	2.211	2.577	2.783	3.08	2.153	2.513	2.884	3.221
2	2	100	25	2.31	2.57	2.735	2.983	2.073	2.379	2.69	3.577	200	50	2.144	2.491	2.698	3.079	2.127	2.491	2.74	3.188
2	4	100	25	2.047	2.228	2.419	2.647	1.883	2.172	2.545	2.765	200	50	1.903	2.175	2.419	2.742	1.925	2.168	2.46	2.618
2	0.25	100	50	2.343	2.591	2.77	3.027	1.969	2.33	2.666	3.468	200	100	2.275	2.516	2.778	3.088	2.038	2.396	2.659	3.058
2	0.5	100	50	2.345	2.596	2.782	3.035	1.977	2.334	2.664	3.469	200	100	2.266	2.522	2.811	3.08	2.03	2.39	2.646	3.064
2	1	100	50	2.347	2.599	2.79	3.051	1.983	2.348	2.677	3.466	200	100	2.263	2.53	2.838	3.073	2.021	2.368	2.658	3.054
2	2	100	50	2.346	2.609	2.789	3.037	2.018	2.362	2.679	3.439	200	100	2.247	2.552	2.797	3.097	2	2.33	2.604	3.022
2	4	100	50	2.314	2.571	2.748	3.012	2.049	2.348	2.656	3.306	200	100	2.184	2.532	2.735	3.097	1.945	2.242	2.492	2.885
2	0.25	100	75	2.338	2.587	2.755	3.029	1.954	2.329	2.647	3.445	200	150	2.275	2.517	2.777	3.106	2.193	2.491	2.762	3.265
2	0.5	100	75	2.339	2.588	2.757	3.029	1.956	2.33	2.646	3.445	200	150	2.275	2.518	2.778	3.097	2.194	2.49	2.766	3.263
2	1	100	75	2.343	2.588	2.758	3.027	1.957	2.336	2.647	3.445	200	150	2.276	2.518	2.778	3.086	2.192	2.492	2.763	3.258
2	2	100	75	2.341	2.592	2.768	3.03	1.957	2.337	2.646	3.445	200	150	2.275	2.52	2.781	3.079	2.184	2.492	2.763	3.247
2	4	100	75	2.341	2.591	2.774	3.033	1.959	2.339	2.649	3.441	200	150	2.272	2.522	2.815	3.072	2.168	2.476	2.766	3.255

Table 4.3.1: Simulated critical values using the permutation method

N	γ	Partial sums				Poisson Process			
		90%	**95%**	**97.5%**	**99%**	**90%**	**95%**	**97.5%**	**99%**
100	0.25	2.589	2.858	3.134	3.435	2.66	2.978	3.249	3.647
100	0.5	2.46	2.742	3.011	3.281	2.496	2.82	3.162	3.543
100	1	2.307	2.613	2.866	3.168	2.336	2.679	3.048	3.454
100	2	2.225	2.527	2.809	3.14	2.226	2.584	2.885	3.385
200	0.25	2.625	2.898	3.154	3.441	2.659	2.968	3.286	3.671
200	0.5	2.481	2.787	3.009	3.319	2.505	2.824	3.167	3.499
200	1	2.355	2.625	2.864	3.215	2.355	2.692	3.042	3.502
200	2	2.247	2.527	2.806	3.09	2.282	2.664	3.008	3.439

Table 4.3.2: Simulated critical values (under the null hypothesis)

were chosen as $\overline{d}\,\frac{1}{(1+\gamma)N^\gamma}\left((i-N^*)_+^{1+\gamma}-((i-1)-N^*)_+^{1+\gamma}\right)$. Note that the latter expression does not depend on T, but only on N.

Once again, we generated $10,000$ series of increments $\Delta S_1,\ldots,\Delta S_N$ under the null hypothesis for the various choices of parameters. The resulting quantiles can be found in Table 4.3.2. The asymptotic critical values are the same as in Section 4.1 and are given in Table 4.1.3. First we realize that the asymptotic quantiles are again too small (this time even more significantly).

For comparison, we simulated the critical values obtained through the permutation method as before. The results can be found in Table 4.3.1. As in Section 4.1, the critical values are quite good, but decline as the change becomes more obvious.

Note that here the consistency of the test is not guaranteed, since the estimator for b is unbounded under the alternative (which violates condition (2.4) of Steinebach [13]).

Next we simulated the α- resp. β-errors, as we did in Section 4.1, with a change at $N^*=\frac{1}{2}N$. The results can be found in Table 4.3.3. The α-errors are far too high for the asymptotic method for $\gamma=0.5$, which is due to the fact, that the asymptotic critical values are too small (compare Tables 4.3.2 and 4.1.3). The permutation method, however, gives good results. For $\gamma=1$ both methods give comparable results.

When we used $\tilde{d}$, instead of $\overline{d}$, and $T=N$ (which changes $\tilde{d}$ slightly), the critical values decreased significantly. Nevertheless, this did not seem to affect the permutation method at all – apparently the permutation quantiles were still smaller than the value of the test statistic for the unpermuted observations. With the asymptotic method, however, we only obtained good β-errors for smaller $\tilde{d}$'s, but observed a sudden jump in the β-errors (up to 100%) as soon as $\tilde{d}$ got larger. This jump e.g. occurred at $\tilde{d}=2$ for the 90%-quantile with $\gamma=0.5$, $N=100,200$.

Again, we were also interested in the standard deviation of the critical values obtained by the permutation method. Under the null hypothesis (Poisson process, $N=100$, $\gamma=1$), we got a standard deviation of 0.28 for the 90%-quantile and of 0.96 for the 99%-quantile, for the partial sums the standard deviation was even smaller. As before, we used $1,000$ repetitions of step 1 to 4 from the first simulation.

For our simulations we used again the software package R, Version 1.2.3. On a Celeron with 466 MHz and 384 MB RAM the calculation of the permutation quantiles takes approximately 10 seconds in the case of 100 observations, and 30 seconds in the case of 200 (using $10,000$ permutations).

γ	N	$\overline{\mathbf{d}}$	**90%-Quantile**		**95%-Quantile**		**97.5%-Quantile**		**99%-Quantile**	
			simul.	**asym.**	**simul.**	**asym.**	**simul.**	**asym.**	**simul.**	**asym.**
			Partial sums							
0.5	100	0	11	41.5	5.1	20.9	2.2	9.3	0.9	1.9
0.5	100	0.25	85.8	55.5	92.3	76.2	96.7	89.4	98.5	97.4
0.5	100	0.5	70.7	38.5	80.5	58.6	86.5	75.7	93.6	91
0.5	100	1	28.4	9.2	40.7	19.9	52	37	63.1	63.5
0.5	100	2	0.3	0	0.4	0.2	0.7	0.5	1.8	3
0.5	100	3	0	0	0	0	0	0	0.1	0
0.5	200	0	8.5	32.5	4.3	15.5	1.7	5.7	0.7	1.5
0.5	200	0.25	81.1	52.6	88.5	72.5	93	86.9	96.5	94.6
0.5	200	0.5	56.8	29.3	68.7	46.4	79.1	67.1	87.2	85.2
0.5	200	1	5.2	1.2	10.4	3.3	16.6	9.9	25	26.1
0.5	200	2	0	0	0	0	0	0	0	0
0.5	200	3	0	0	0	0	0	0	0	0
1	100	0	11.9	11.7	5.1	4.1	2.6	1.4	1.2	0.2
1	100	0.25	89	88.5	94	95.2	97.5	98.2	98.6	99.8
1	100	0.5	82.9	83	91	93.1	94.9	97	96.9	99.2
1	100	1	59.3	60.6	73.4	79.3	82.6	89.8	89.3	97
1	100	2	12.2	13.6	19.2	28.9	28.1	49.9	42.1	73.1
1	100	3	0.3	0.6	1	2.7	1.6	7.9	3.7	23.2
1	200	0	10.1	10.1	4.3	3.5	2.4	1.5	0.8	0.3
1	200	0.25	85.6	85.4	92.1	94.3	95.4	97.5	98	99.7
1	200	0.5	78.3	78.3	86.7	90	91.9	95.1	95.4	98.3
1	200	1	37.5	40	49.8	56.4	61.1	73.4	72.6	89.6
1	200	2	1	1	2.3	4	3.7	8.6	6.9	20.7
1	200	3	0	0	0	0	0	0	0	0
			Poisson process							
0.5	100	0	10.5	35.6	5.3	19.2	2.1	10.2	0.5	2.6
0.5	100	0.25	84.6	58.1	91.4	75.5	94.9	86.9	98	94.8
0.5	100	0.5	70.9	41.2	82.6	60.8	90.5	74.6	96.2	91.1
0.5	100	1	27.8	9.8	41.1	20.2	54.9	36.3	70.2	60.8
0.5	100	2	0.2	0	0.4	0.3	1.4	0.8	5.1	4.5
0.5	100	3	0	0	0	0	0	0	0	0
0.5	200	0	9.9	34	5	16.5	2	7.2	0.6	2.2
0.5	200	0.25	79.9	55.7	89.1	73.3	94.4	85.3	98.4	95
0.5	200	0.5	54.1	28.5	66.7	45.1	77.3	62.2	87.1	79.6
0.5	200	1	5.9	1.2	11.1	3.7	18.7	9.9	33.2	23.7
0.5	200	2	0	0	0	0	0	0	0	0
0.5	200	3	0	0	0	0	0	0	0	0
1	100	0	9.1	9.8	4.4	4.2	3.1	2.3	0.9	1.1
1	100	0.25	87.4	85.4	93.2	94.1	96.5	97.7	98.6	99.1
1	100	0.5	82.7	82.7	90.2	91.8	95.2	96.3	98.1	99.5
1	100	1	62.5	62.8	73	76	82.3	88	91.8	95.6
1	100	2	11.6	13.7	19.1	26.3	29.5	43.1	49.1	70.2
1	100	3	0.7	0.9	1.2	4.1	2.7	8.7	9	25.1
1	200	0	9.2	8.8	4.5	4.4	2.2	2.1	1.3	1
1	200	0.25	86.7	87.6	92.9	93.6	95.8	96.7	98.3	99.1
1	200	0.5	74.9	76.8	85.7	87.4	92.4	94.1	96.8	97.8
1	200	1	41.4	44.7	56.3	60.5	68.9	74.5	80.9	88.3
1	200	2	1	1.6	2.4	4.3	4.8	8.4	9.9	19.7
1	200	3	0	0	0	0	0	0	0	0.4

Table 4.3.3: Simulated α- resp. β-errors in % (1,000 repetitions, $N^* = \frac{N}{2}$)

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