Geometric responses of Quantum Hall systems

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Fractional Quantum Hall state – exotic fluid

- Two-dimensional electron gas in magnetic field forms a new type of quantum fluid
- It can be understood as quantum condensation of electrons coupled to vortices/fluxes
- Quasiparticles are gapped, have fractional charge and statistics
- The fluid is ideal – no dissipation!
- Density is proportional to vorticity!
- Transverse transport: Hall conductivity and Hall viscosity, thermal Hall effect
- Protected chiral dynamics at the boundary

Transverse transport

Signature of FQH states – quantization and robustness of Hall conductance σ_H

$$j^i = \sigma_H \epsilon^{ij} E_j \,, \qquad \sigma_H = \nu \frac{e^2}{h} \quad \text{- Hall conductivity.}$$

Are there other "universal" transverse transport coefficients?

Hall viscosity: transverse momentum transport

Thermal Hall conductivity: transverse energy/heat transport

What are the values of the corresponding kinetic coefficients for various FQH states?

Are there corresponding "protected" boundary modes?

Acknowledgments and References

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- A. Gromov, K. Jensen, and A.G. Abanov, arXiv:1506.07171 (2015).
  Boundary effective action for quantum Hall states.

Essential points of the talk

- Induced Action encodes linear responses of the system
- Coefficients of geometric terms of the induced action – universal transverse responses.
- Hall conductivity, Hall viscosity, thermal Hall conductivity.
- These coefficients are computed for various FQH states.
- Framing anomaly is crucial in obtaining the correct gravitational Chern-Simons term!

Induced action

Partition function of fermions in external e/m field A_μ is:

$$Z = \int D\psi D\psi^\dagger \; e^{iS[\psi,\psi^\dagger;A_\mu]} = e^{iS_{ind}[A_\mu]}$$

with

$$S[\psi,\psi^\dagger;A_\mu] = \int d^2x\,dt\;\psi^\dagger\left[i\hbar\partial_t + eA_0 - \frac{1}{2m}\left(-i\hbar\boldsymbol{\nabla} - \frac{e}{c}\boldsymbol{A}\right)^2\right]\psi$$

$$+ \quad \text{interactions}$$

Induced action encodes current-current correlation functions

$$\langle j_\mu\rangle = \frac{\delta S_{ind}}{\delta A_\mu}\,, \qquad \langle j_\mu j_\nu\rangle = \frac{\delta^2 S_{ind}}{\delta A_\mu \delta A_\nu}\,, \ldots$$

+ various limits $m \to 0$, $e^2/l_B \to \infty$, ...

Induced action [phenomenological]

Use general principles: gap+symmetries to find the form of S_{ind}

- Locality $\rightarrow$ expansion in gradients of A_μ
- Gauge invariance $\rightarrow$ written in terms of $\boldsymbol{E}$ and B
- Other symmetries: rotational, translational, ...

$$S_{ind} = \frac{\nu}{4\pi}\int AdA + \int d^2x\, dt\, \left[\frac{\epsilon}{2}\boldsymbol{E}^2 - \frac{1}{2\mu}B^2 + \sigma B \boldsymbol{\nabla}\boldsymbol{E} + \ldots\right]$$

Find responses in terms of phenomenological parameters $\nu, \epsilon, \mu, \sigma, \ldots$

Compute these parameters from the underlying theory.

For non-interacting particles in B with $\nu = N$ see AA, Gromov 2014.

Any functional of $\boldsymbol{E}$ and B is gauge invariant, but ...

Chern-Simons action

$$S_{CS} = \frac{\nu}{4\pi} \int AdA$$

Linear responses from the Chern-Simons action

In components

$$
\begin{aligned}
S_{CS} &= \frac{\nu}{4\pi}\int AdA \equiv \frac{\nu}{4\pi}\int d^2x\,dt\;\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda \\
&= \frac{\nu}{4\pi}\int d^2x\,dt\;\Big[A_0(\partial_1 A_2 - \partial_2 A_1) + \dots\Big]
\end{aligned}
$$

Varying over A_μ

$$
\rho = \frac{\delta S_{CS}}{\delta A_0} = \frac{\nu}{2\pi}B\,, \qquad j_1 = \frac{\delta S_{CS}}{\delta A_1} = -\frac{\nu}{2\pi}E_2\,,
$$

We have: $\sigma_H = \frac{\nu}{2\pi}$ and $\sigma_H = \frac{\partial\rho}{\partial B}$ – Streda formula.

Properties of the Chern-Simons term

- Gauge invariant in the absence of the boundary
  (allowed in the induced action)
- Not invariant in the presence of the boundary
- Leads to protected gapless edge modes
- First order in derivatives
  (more relevant than $F_{\mu\nu}F^{\mu\nu}$, B^2 or $\boldsymbol{E}^2$ at large distances)
- Relativistically invariant
  (accidentally !!!)
- Does not depend on metric $g_{\mu\nu}$
  (topological, does not contribute to the stress-energy tensor)

Elastic responses: Strain and Metric

- Deformation of solid or fluid $\boldsymbol{r} \to \boldsymbol{r} + \boldsymbol{u}(\boldsymbol{r})$
- $\boldsymbol{u}(\boldsymbol{r})$ - displacement vector
- $u_{ik} = \frac{1}{2}\left(\partial_k u_i + \partial_i u_k\right)$ - strain tensor
- u_{ik} plays a role of the deformation metric
- deformation metric $g_{ik} \approx \delta_{ik} + 2u_{ik}$ with $ds^2 = g_{ik} dx^i dx^k$
- stress tensor T_{ij} - response to the deformation metric g_{ij}

Stress tensor and induced action

Studying responses

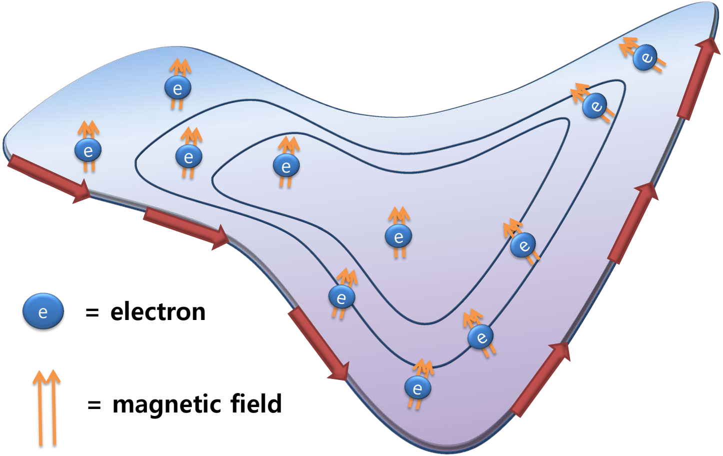
- Microscopic model $S = S[\psi]$
- Introduce gauge field and metric background $S[\psi, A, g]$
- Integrate out matter degrees of freedom and obtain and $S_{ind}[A, g]$
- Obtain E/M, elastic, and mixed responses from

$$\delta S_{ind} = \int dx\, dt\, \sqrt{g} \left(j^{\mu} \delta A_{\mu} + \frac{1}{2} T^{ij} \delta g_{ij} \right)$$

- Elastic responses = gravitational responses

Important: stress is present even in flat space!

Quantum Hall in Geometric Background (by Gil Cho)



Geometric background

- For 2+1 dimensions and spatial metric g_{ij} we introduce "spin connection" ω_μ so that

$$\begin{aligned} \frac{1}{2}\sqrt{g}R &= \partial_1\omega_2 - \partial_2\omega_1 && \text{– gravi-magnetic field,} \\ \mathcal{E}_i &= \dot{\omega}_i - \partial_i\omega_0 && \text{– gravi-electric field,} \end{aligned}$$

- For small deviations from flat space $g_{ik} = \delta_{ik} + \delta g_{ik}$ we have explicitly

$$\omega_0 = \frac{1}{2}\epsilon^{jk}\delta g_{ij}\dot{g}_{ik}\,, \qquad \omega_i = -\frac{1}{2}\epsilon^{jk}\partial_j\delta g_{ik}$$

- Close analogy with E/M fields $A_\mu \leftrightarrow \omega_\mu$!

Geometric terms of the induced action

Terms of the lowest order in derivatives

$$S_{ind} = \frac{\nu}{4\pi} \int \left[AdA + 2\bar{s}\, \omega dA + \beta'\, \omega d\omega \right].$$

Geometric terms:

- AdA – Chern-Simons term (ν: Hall conductance, filling factor)
- ωdA – Wen-Zee term ($\bar{s}$: orbital spin, Hall viscosity, Shift)
- $\omega d\omega$ – "gravitational CS term" (β': Hall viscosity - curvature, thermal Hall effect, orbital spin variance)
- In the presence of the boundary β' can be divided into chiral central charge c and $\overline{s^2}$ (Bradlyn, Read, 2014). The latter does not correspond to an anomaly. (Gromov, Jensen, AA, 2015)

The Wen-Zee term

Responses from the Wen-Zee term

$$S_{WZ} = \frac{\nu \bar{s}}{2\pi} \int \omega dA \,.$$

Emergent spin (orbital spin) $\bar{s}$

$$\frac{\nu}{4\pi}(A + \bar{s}\omega) d(A + \bar{s}\omega)$$

Wen-Zee shift for sphere $\delta N = \nu \mathcal{S}$; $\mathcal{S} = 2\bar{s}$

$$\frac{\nu \bar{s}}{2\pi} A_0 d\omega \;\to\; \delta\rho = \frac{\nu \bar{s}}{2\pi} d\omega \;\to\; \delta N = \frac{\nu \bar{s}}{4\pi} \int d^2x \sqrt{g} R = \nu \bar{s} \chi = \nu \bar{s}(2-2g)$$

Hall viscosity (per particle) $\eta_H = \frac{\bar{s}}{2} n_e$:

$$\frac{\nu \bar{s}}{2\pi} \omega dA \to \frac{B \nu \bar{s}}{2\pi} \omega_0 = n_e \bar{s} \omega_0 = n_e \frac{\bar{s}}{2} \epsilon^{jk} \delta g_{ij} \dot{g}_{ik}$$

Hall viscosity

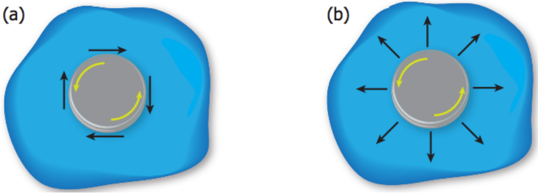
Gradient correction to the stress tensor

$$T_{ik} = \eta_H(\epsilon_{in} v_{nk} + \epsilon_{kn} v_{ni}) \,,$$

where

$$v_{ik} = \frac{1}{2}(\partial_i v_k + \partial_k v_i) = \frac{1}{2}\dot{g}_{ik} \qquad \text{– strain rate}$$

(a) Shear viscosity (b) η_H – Hall viscosity



picture from Lapa, Hughes, 2013

Avron, Seiler, Zograf, 1995

Obtaining the effective field theory for FQH states

- Reduce problem to noninteracting fermions with ν - integer interacting with statistical Abelian and non-Abelian gauge fields. Can be done, e.g., by flux attachment or parton construction (Zhang, Hansson, Kivelson, 1989; Wen, 1991; Cho, You, Fradkin, 2014)
- Integrate out fermions and obtain the effective action $S[a, A, g]$ using the results for free fermions. (AA, Gromov, 2014)
- Integrate out statistical gauge fields taking into account the framing anomaly. (Gromov et al., 2015)
- Obtain the induced action $S_{ind}^{geom}[A, g]$ and study the corresponding responses.

Flux attachment in geometric background

Cho, You, Fradkin, 2014

Flux attachment is based on the identity:

$$\int Db\,Da\,\exp\left\{-i\frac{2p}{4\pi}\int bdb - i\frac{1}{2\pi}adb + i\int_L (a+p\omega)\right\} = 1$$

Here L is a link. The last term $p\omega$ is due to the framing regularization (Polyakov, 1988).

Integrating over b (wrong!)

$$\int Da\,\exp\left\{i\frac{1}{(2p)2\pi}\int ada + i\int_L (a+p\omega)\right\} = 1$$

Integrating over b (correct!)

$$\int Da\,\exp\left\{i\frac{1}{(2p)2\pi}\int ada + i\int_L (a+p\omega) - i\frac{1}{48\pi}\int \omega d\omega\right\} = 1$$

Integrating over b (correct!)

Explicit calculation for free fermions at $\nu = N$

Starting point

$$S[\psi, \psi^\dagger; A_\mu, g_{ij}] = \int dt\, d^2x\, \sqrt{g}\, \psi^\dagger \left[i\hbar \overleftrightarrow{D_t} - \frac{\hbar^2}{2m} g^{ij} (D_i\psi)^\dagger (D_j\psi) \right] .$$

Straightforward computation at $\nu = N$ (Gromov, AA, 2014)

$$S_{\text{eff}}^{(\text{geom})} = \frac{N}{4\pi} \int \left(AdA + N\, Ad\omega + \frac{2N^2 - 1}{6} \omega d\omega \right)$$

or equivalently (as a sum over Landau levels)

$$S_{\text{eff}}^{(\text{geom})} = \sum_{n=1}^{N} \int \left[\frac{1}{4\pi} (A + \bar{s}_n \omega) d(A + \bar{s}_n \omega) - \frac{c}{48\pi} \omega d\omega \right]$$

$$\bar{s}_n = \frac{2n - 1}{2}\,, \qquad c = 1\,.$$

Geometric effective action for $\nu = 1$

$$S_{\text{eff}}^{(\text{geom})} = \int \left[\frac{1}{4\pi} \left(A + \frac{1}{2}\omega \right) d \left(A + \frac{1}{2}\omega \right) - \frac{1}{48\pi} \omega d\omega \right]$$

Is there an intuitive way to obtain this result?

The Wen-Zee construction for $\nu = 1$

Integrate out fermions but leave currents $j = -\frac{1}{2\pi} da$ (Wen, Zee, 1992)

$$S[a; A, \omega] = -\frac{1}{4\pi} \int \left[ada + 2 \left(A + \frac{1}{2}\omega \right) da \right] .$$

(Wen, Zee, 1992) + framing anomaly:

- solve for a: $a = -\left(A + \frac{1}{2}\omega\right)$
- substitute back into the action
- take into account framing anomaly (Gromov et.al., 2015)

$$S_{ind} \;=\; \int \frac{1}{4\pi} \left(A + \frac{1}{2}\omega \right) d \left(A + \frac{1}{2}\omega \right) - \frac{1}{48\pi} \omega d\omega$$

Digression: the quantum Chern-Simons theory

The partition function for Chern-Simons theory in the metric background (Witten, 1989)

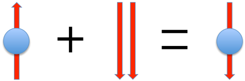
$$\int Da\, \exp\left\{-i\frac{k}{4\pi}\int ada\right\} = \exp\left\{-i\frac{c}{96\pi}\int \mathrm{tr}\left(\Gamma d\Gamma + \frac{2}{3}\Gamma^3\right)\right\}$$
$$= \exp\left\{-i\frac{c}{48\pi}\int \omega d\omega\right\},$$

where $c = \mathrm{sgn}(k)$ and the last equality is correct for our background.

- We specialized Witten's results to the Abelian CS theory
- The result is obtained from the fluctuation determinant $\det(d)$
- The dependence on metric comes from the gauge fixing $\int dV\, \phi D^\mu a_\mu$
- Action does not depend on metric, path integral does: anomaly (framing anomaly)

Consistency check (for $\nu = 1$)

Basic idea: two routes to effective action.



Route 1:

$$S_0[\psi; A, g] \to \frac{1}{4\pi}\left(A + \frac{1}{2}\omega\right) d \left(A + \frac{1}{2}\omega\right) - \frac{1}{48\pi}\omega d\omega$$

Route 2:

$$S_0[\psi; A, g] \to S_0[\psi; A + a + \omega, g] + \frac{1}{8\pi} a da - \frac{1}{48\pi}\omega d\omega$$

Introducing $\mathcal{A} = A + a + \omega$

$$\to -\frac{1}{4\pi}\left(\mathcal{A} - \frac{1}{2}\omega\right) d \left(\mathcal{A} - \frac{1}{2}\omega\right) + \frac{1}{48\pi}\omega d\omega + \frac{1}{8\pi} a da - \frac{1}{48\pi}\omega d\omega$$

$$\to \frac{1}{4\pi}\left(A + \frac{1}{2}\omega\right) d \left(A + \frac{1}{2}\omega\right) + \frac{1}{48\pi}\omega d\omega - 2\frac{1}{48\pi}\omega d\omega$$

Consistent only if the framing anomaly is taken into account (twice)!

Obtaining the effective field theory for FQH states

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Example: Laughlin's states

Flux attachment for Laughlin's states $\nu = \frac{1}{2m+1}$

$$S_0[\psi, A + a + m\omega, g] - \int \left[\frac{2m}{4\pi} bdb + \frac{1}{2\pi} adb \right]$$

Integrating out ψ, a, b

$$S_{ind}^{geom} = \int \frac{1}{4\pi} \frac{1}{2m+1} \left(A + \frac{2m+1}{2}\omega \right) d \left(A + \frac{2m+1}{2}\omega \right) - \frac{1}{48\pi}\omega d\omega$$

Coefficients

$$\nu = \frac{1}{2m+1}\,, \qquad \bar{s} = \frac{2m+1}{2}\,, \qquad c = 1\,.$$

Other states

Geometric effective actions have been obtained for:

- Free fermions at $\nu = N$
- Laughlin's states
- Jain series
- Arbitrary Abelian QH states
- Read-Rezayi non-Abelian states
- The method can be applied to other FQH states

A. Gromov et.al., PRL **114**, 016805 (2015). *Framing Anomaly in the Effective Theory of the Fractional Quantum Hall Effect.*

Consequences of the gravitational CS term

$$S_{gCS} = -\frac{c}{96\pi}\int \mathrm{tr}\left(\Gamma d\Gamma + \frac{2}{3}\Gamma^3\right) = -\frac{c}{48\pi}\int \omega d\omega\,.$$

1. From CS and WZ term [shift] (Wen, Zee, 1992)

$$n = \frac{\nu}{2\pi}B + \frac{\nu\bar{s}}{4\pi}R \quad \to \quad N = \nu(N_\phi + \bar{s}\chi)$$

From WZ and gCS term [Hall viscosity shift] (Gromov, AA, 2014 cf. Hughes, Leigh, Parrikar, 2013)

$$\eta_H = \frac{\bar{s}}{2}n - \frac{c}{24}\frac{R}{4\pi}$$

2. Thermal Hall effect [from the boundary!] (Kane, Fisher, 1996; Read, Green, 2000; Cappelli, Huerta, Zemba, 2002)

$$K_H = c\frac{\pi k_B^2 T}{6}\,.$$

Some recent closely related works

- Geometric terms from adiabatic transport and adiabatic deformations of trial FQH wave functions (Bradlyn, Read, 2015; Klevtsov, Wiegmann, 2015)
- QH wave functions in geometric backgrounds (Can, Laskin, Wiegmann, 2014; Klevtsov, Ma, Marinescu, Wiegmann, 2015)
- Newton-Cartan geometric background and Galilean invariance (Hoyos, Son, 2011; Gromov, AA, 2014; Jensen, 2014)
- Thermal transport in quantum Hall systems (Geracie, Son, Wu, Wu, 2014; Gromov, AA, 2014; Bradlyn, Read, 2014)

Main results

Response functions can be encoded in the form of the induced action for FQHE.

$$
\begin{aligned}
S_{ind} &= \frac{\nu}{4\pi}\int\Big[(A+\bar{s}\omega)d(A+\bar{s}\omega)+\beta\omega d\omega\Big] \\
&- \frac{c}{96\pi}\int \mathrm{tr}\left[\Gamma d\Gamma+\frac{2}{3}\Gamma^3\right]+\dots,
\end{aligned}
$$

where ν is the filling fraction, $\bar{s}$ is the average orbital spin, β is the *orbital spin variance*, and c is the chiral central charge.

The coefficients $\nu, \bar{s}, \beta, c$ are computed for various known Abelian and non-Abelian FQH states.

Framing anomaly is crucial in obtaining the correct gravitational Chern-Simons term!

Quantum Hall at the Edge (by Gil Cho)