Gravitational response of the Quantum Hall effect

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arXiv:1309.7333, JHEP01(2014)133

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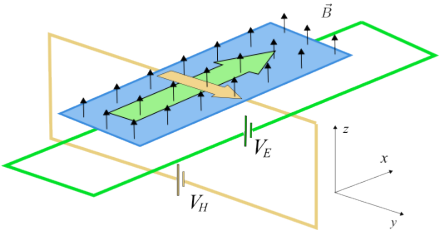
(+ discussions with P. Wiegmann, A. Abanov, R. Bondesan, J. Dubail ...)

Observed in two-dimensional electron systems subjected to low temperatures and strong magnetic fields. Hall conductance is quantized $\sigma_H = I/V_H = \nu$, where ν is integer for integer QHE, or a fraction for fractional QHE. Involves many ($N \sim 10^6$) electrons on lowest Landau level, described by a collective (Laughlin) state.

$$\Psi(\{z_i\}) = \prod_{i<j}^{N} (z_i - z_j)^{\beta} e^{-\frac{B}{2}\sum_i |z_i|^2}, \quad \beta = 1/\nu \in \mathbb{Z}_+$$



Motivation

In physics one can sometimes obtain important information about the system by putting it on a curved manifold.
Prototypical example: a CFT partition function on a compact Riemann surface (M, g_0) has the following behavior under the transformation of the reference metric g_0 to a new metric $g = e^{2\sigma} g_0$

$$\log \frac{Z^{CFT}(g)}{Z^{CFT}(g_0)} = \frac{c}{12\pi} S_L(g_0, \sigma),$$

where $c \in \mathbb{R}$ is a central charge of the CFT, and the Liouville action functional is

$$S_L(g_0, \sigma) = \int_M (\partial\sigma\bar{\partial}\sigma + R_0\sigma) d^2 z,$$

and R_0 is the scalar curvature of g_0.

Our main goals:
1) Consider the partition function of the integer Quantum Hall effect on a compact Riemann surface (M, g_0). How does it transform under the change of metric $g_0 \to g$? Can we derive an analog of CFT formula for QHE ($\beta = 1$):

$$\log \frac{Z^{QHE}(g)}{Z^{QHE}(g_0)} =?$$

2) Same question for the fractional Quantum Hall effect ($\beta \neq 1$):

$$\log \frac{Z^{FQHE}(g)}{Z^{FQHE}(g_0)} =?$$

As a byproduct: The problem naturally generalizes to Kähler (M, g) of $\dim_{\mathbb{C}} M = n$, equipped with holomorphic line bundle L. Relation to modern Kähler geometry.

Motivation

Our main goals:
1) Consider the partition function of the integer Quantum Hall effect on a compact Riemann surface (M, g_0). How does it transform under the change of metric $g_0 \to g$? Can we derive an analog of CFT formula for QHE ($\beta = 1$):

$$\log \frac{Z^{QHE}(g)}{Z^{QHE}(g_0)} = (\textit{main result})$$

2) Same question for the fractional Quantum Hall effect ($\beta \neq 1$):

$$\log \frac{Z^{FQHE}(g)}{Z^{FQHE}(g_0)} = (\textit{status report})$$

As a byproduct: The problem naturally generalizes to Kähler (M, g) of $\dim_{\mathbb{C}} M = n$, equipped with holomorphic line bundle L. Relation to modern Kähler geometry.

Main result

Let k be the flux of the magnetic field. Then we have large k expansion

$$\log \frac{Z^{QHE}(g)}{Z^{QHE}(g_0)} = -\frac{k^2}{2\pi} S_{AY}(g_0,\phi) + \frac{k}{4\pi} S_M(g_0,\phi) + \frac{1}{6\pi} S_L(g_0,\phi) + \text{remainder terms } \mathcal{O}(1/k)$$

Instead of conformal form the metric $g = e^{2\sigma} g_0$, it is natural to use Kähler form $g = g_0 + \partial\bar{\partial}\phi$ ($g := g_{z\bar{z}} = \sqrt{\det g_{ij}}$). New action functionals (unknown in physics - coming from Kähler geometry):

$$S_{AY}(g_0,\phi) = \int_M \left(\frac{1}{2}\phi\partial\bar{\partial}\phi + \phi g_0\right) d^2 z \qquad \text{Aubin} - \text{Yau}$$

$$S_M(g_0,\phi) = \int_M \left(-\phi R_0 + g \log \frac{g}{g_0}\right) d^2 z \qquad \text{Mabuchi}$$

Lowest Landau level on Riemann surface

On the plane LLL wavefunctions are $\psi_i = z^i e^{-\frac{k}{2}|z|^2}$. What is the analog of LLL on (M, g_0)? Constant magnetic field: $F = dA = kg_0$. Shrödinger equation for the lowest energy level reduces to

$$(\bar\partial + A_{\bar z})\psi = 0, \quad \text{where } A_{\bar z} = -k\bar\partial \log h_0$$

with many solutions $\psi_i(z,\bar z) = s_i(z) h_0^k(z,\bar z)$. Mathematically, magnetic field is described by the holomorphic line bundle $L^k \to M$, $s_i(z)$ is the basis of holomorphic sections ($i = 1, .., N_k$) and constant magnetic field $F = kg_0$ means choice of "polarization", implying positivity of the line bundle ($F > 0$ since the metric shall be positive definite everywhere on M). Examples:

- S^2, $s_i(z) = z^{i-1}$, $i = 1, .., k+1$.
- T^2, $s_i(z) = \theta_{\frac{i}{k},0}(kz, k\tau)$, $i = 1, .., k$
- on M of genus h there are $N_k = k - h + 1$ sections.

Laughlin wave function on Riemann surface

The Laughlin wave function of N_k non-interacting fermions (integer QHE) is given by Slater determinant

$$\Psi(z_1, \dots, z_{N_k}) = \frac{1}{\sqrt{N_k!}} \det \psi_i(z_j)$$

For the metric g_0, the (normalized) wavefunctions are $\psi_i^0(z) = s_i(z) h_0^k$. Then for the new metric $g = g_0 + \partial\bar{\partial}\phi$ the wavefunctions are

$$\psi_i(z) = \psi_i^0(z) e^{-k\phi}$$

The Laughlin wave function for the fractional QHE is

$$\Psi_\beta(z_1, \dots, z_{N_k}) = \frac{1}{\sqrt{N_k!}} \left(\det \psi_i(z_j)\right)^\beta$$

Partition function for integer QHE

Partition function (generating functional)

$$Z^{QHE}(g_0, g) = \int_{M^{\otimes N_k}} |\Psi(z_1, \dots, z_{N_k})|^2 \prod_{i=1}^{N_k} g(z_i) d^2 z_i =$$

$$= \frac{1}{N_k!} \int_{M^{\otimes N_k}} |\det s_i(z_j)|^2 e^{-k \sum_i \phi(z_i)} \prod_{i=1}^{N_k} h_0^k(z_i) g(z_i) d^2 z_i.$$

Varying $Z^{QHE}(g_0, g)$ with respect to $\delta\phi(z)$ allows to define density correlation functions $\rho(z) = \frac{1}{k} \sum_i \delta(z - z_i)$:

$$\langle \rho(x) \rho(y) \dots \rho(z) \rangle.$$

Example of S^2, or $\mathbb{C}$ compactified, Wiegmann-Zabrodin (2006):

$$Z^{FQHE, S^2}(W) = \frac{1}{N_k!} \int_{\mathbb{C}^{\otimes N_k}} |\Delta(z)|^{2\beta} e^{-k \sum_i W(z_i)} \prod_{i=1}^{N_k} d^2 z_i,$$

They derived large k expansion, using loop equation, for arbitrary W.

Derivation

The integer QHE partition function enjoys determinantal representation

$$Z^{QHE}(g_0,g)=\det_{ij}\int_M \bar{s}_i s_j h_0^k e^{-k\phi} g d^2 z,$$

studied by Donaldson (2004). Variation of the free energy wrt $\delta\phi(z)$

$$\delta\log Z^{QHE}(g_0,g)=\int_M(-k\rho_k+\Delta\rho_k)\delta\phi\, gd^2z. \qquad (1)$$

is controlled by the density of states function

$$\rho_k(z)=\sum_{i=1}^{N_k}\bar{\psi}_i(z)\psi_i(z)=\sum_{i=1}^{N_k}|s_i(z)|^2 h_0^k e^{-k\phi}.$$

In math this is known as Bergman kernel on the diagonal. Number of states $N_k=\int_M\rho_k(z)gd^2z$.

Density of states (Bergman kernel)

On Kähler (M, g) of $\dim_{\mathbb{C}} = n$ with L^k there is asymptotic expansion

$$\rho_k(z) = \sum_{i=1}^{N_k} |s_i(z)|^2 h_0^k e^{-k\phi} = k^n + k^{n-1}\frac{1}{2}R + k^{n-2}\frac{1}{3}\Delta R + ...$$

Tian-Yau-Zelditch expansion (Zelditch 1998). Check: number of states is topological, for $n = 1$, $N_k = \int_M \rho_k(z) g d^2 z = k + \frac{1}{2}(2 - 2h)$.
In physics can be derived from path integral representation of the density of LLL states (Douglas - SK 2008)

$$\rho_k(z) = lim_{T\to\infty} \int_{z=x(0)}^{z=x(T)} e^{-\frac{1}{\hbar}\int_0^T (g_{ij}\dot{x}^i\dot{x}^j + A_i\dot{x}^i)dt} Dx(t). \quad (2)$$

All-order closed formula (Hao Xu 2011) valid in normal coordinate system

$$\rho_k(z) = \sum_{s=0}^{\infty} k^{n-s} c_s \partial..\partial\bar{\partial}..\bar{\partial} g \quad (3)$$

c_s are known - full control over the expansion.

Derivation, cont'd

Using the expansion of ρ_k we can integrate out the free energy order by order in k (in principle to all orders)

$$
\begin{aligned}
\delta \log Z^{QHE}(g_0, g) &= \int_M (-k\rho_k + \Delta\rho_k)\delta\phi \, g d^2 z \\
\log Z^{QHE}(g_0, g) &= -\frac{k^2}{2\pi} S_{AY}(g_0, \phi) + \frac{k}{4\pi} S_M(g_0, \phi) + \frac{1}{6\pi} S_L(g_0, \phi) \\
&\quad - \frac{5}{96\pi k} \left(\int_M R^2 g d^2 z - \int_M R_0^2 g_0 d^2 z \right) + \mathcal{O}(1/k^2)
\end{aligned}
$$

The action functionals here satisfy one-cocycle condition on the space of metrics: $S(g_0, g_2) = S(g_0, g_1) + S(g_1, g_2)$. Starting from order $1/k$ this becomes easy, since $S(g_0, g) = S(g) - S(g_0)$ (exact one-cocycle). Conjecture: remainder term is exact one-cocycle.

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Check to the order $1/k^2$:

$$\log Z^{QHE}(g_0, g) = -\frac{k^2}{2\pi} S_{AY}(g_0, \phi) + \frac{k}{4\pi} S_M(g_0, \phi) + \frac{1}{6\pi} S_L(g_0, \phi)$$
$$-\frac{5}{96\pi k}\left(\int_M R^2 g d^2 z - \int_M R_0^2 g_0 d^2 z\right) +$$
$$+\frac{1}{2880\pi k^2}\left(\int_M (29 R^3 - 66 R\Delta R)\, g d^2 z - \right.$$
$$\left. - \int_M (29 R_0^3 - 66 R_0 \Delta_0 R_0)\, g_0 d^2 z\right) + \mathcal{O}(1/k^3)$$

Conjecture check

Conjecture: remainder term is exact one-cocycle.
Check to the order $1/k^2$:

$$\log Z^{QHE}(g_0, g) = -\frac{k^2}{2\pi} S_{AY}(g_0, \phi) + \frac{k}{4\pi} S_M(g_0, \phi) + \frac{1}{6\pi} S_L(g_0, \phi)$$
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$$\left.-\int_M (29R_0^3 - 66R_0\Delta_0 R_0)\, g_0 d^2 z\right) + \mathcal{O}(1/k^3)$$

✓

Result for FQHE on S^2 (Can-Laskin-Wiegmann, 1402.1531)

$$\log Z^{FQHE,\beta}(g_0,g) =$$
$$= -\beta\frac{k^2}{2\pi}S_{AY}(g_0,\phi) + \beta\frac{k}{4\pi}S_M(g_0,\phi) + (\frac{1}{3} + \frac{\beta-1}{2})\frac{1}{2\pi}S_L(g_0,\phi) + ...$$

using loop equation method. T^2 and higher genus (SK-Wiegmann et al, on the way). Nonperturbative relation:

$$\mathcal{Z}^\beta(g) = \frac{Z^{FQHE,\beta}(g_0,g)}{(Z^{QHE}(g_0,g))^\beta}$$

$\mathcal{Z}^\beta(g)$ is independent of g_0 (background independent)

$$\log\frac{\mathcal{Z}^\beta(g)}{\mathcal{Z}^\beta(g_0)} = \frac{\beta-1}{12\pi}S_L(g_0,\sigma)$$

exact relation at $k=\infty$. Compare to the CFT partition function

$$\log\frac{Z^{CFT}(g)}{Z^{CFT}(g_0)} = \frac{c}{12\pi}S_L(g_0,\sigma),$$

There is a long-term effort to understant the effective theory of QHE systems. One interpretation is that electrons in QHE form an inconpressible "quantum Hall liquid" (Girvin-MacDonald-Platzman 1986, see also recent papers by Hoyos, Son, Wiegmann, Abanov etc). The Bergman kernel is the density profile of the liquid.

$$\log Z^{FQHE} = -\beta\frac{k^2}{2\pi}S_{AY}(g_0,\phi)+\beta\frac{k}{4\pi}S_M(g_0,\phi)+(\frac{1}{3}+\frac{\beta-1}{2})\frac{1}{2\pi}S_L(g_0,\phi)$$

The first coefficient is inverse Hall conductance. The response to the curvature is interpreted as an anomalous Hall viscosity

$$\eta=\frac{\delta\rho_k}{\delta R} \quad \text{where} \quad (\delta S_M = R \qquad \delta S_L = \Delta R)$$

The Liouville term is related to heat transport (M.Stone, J.Cardy).

Beta-deformed Bergman kernel

Begman kernel is ubiquitous in Kähler geometry. Analogs of Z^{QHE} have been studied by Donaldson (2004) and R. Berman (2008-) on Kähler M - derived leading order term in the large k expansion of Z^{QHE}. Darboux-Cristoffel formula for the Bergman kernel

$$\rho_k(z_1) = \frac{1}{(N_k-1)!}\int_{M^{N_k-1}} |\det s_i(z_j)|^2 e^{-k\sum_i \phi(z_i)} \prod_{i=2}^{N_k} h_0^k(z_i) g^n(z_i) d^{2n} z_i$$

We define "β-deformed" Bergman kernel ⚠

$$\rho_k^{\beta}(z_1) = \frac{1}{(N_k-1)!}\int_{M^{N_k-1}} |\det s_i(z_j)|^{2\beta} e^{-\beta k\sum_i \phi(z_i)} \prod_{i=2}^{N_k} h_0^{\beta k}(z_i) g^n(z_i) d^{2n} z_i$$

This is a new object in Kähler geometry.

FQHE and off-diagonal Bergman kernel

Statement: β-deformed Bergman kernel has large k local asymptotic expansion (no rigorous proof available at the moment).
Conjecture: there exists asymptotic expansion

$$\rho_k^\beta = \beta \sum_{s=0}^{\infty} (\beta k)^{n-s} P_s(\beta)(R^s + ...),$$

where $P_s(\beta)$ is a polynomial in β of degree s, and R^s schematically denotes curvature invariants of degree s.
Another representation for the FQHE partition function $\mathcal{Z}^\beta(g)$

$$\mathcal{Z}^\beta(g) = \frac{1}{N_k!} \int_{M^{N_k}} (\det B_k(z_i, z_j))^{2\beta} \prod_{i=1}^{N_k} g^n(z_i) d^{2n} z_i$$

Where $B_k(z_i, z_j) = \sum_i \bar{\psi}(z_i)\psi(z_j)$ - off-diagonal Bergman kernel.

$$B_k(z_i, z_j) = e^{-k|z_i - z_j|^2}(k^n + ...)$$

Thank you