Geometric Deformations and Berry Curvature in Quantum Hall States

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Outline

- Geometric deformations and the induced action
- Berry curvature and central charge
- Example - conformal block trial states
- Conclusions

Motivation

- Can we calculate central charge directly from bulk wavefunctions?

- Way forward - couple microscopic system to background geometry

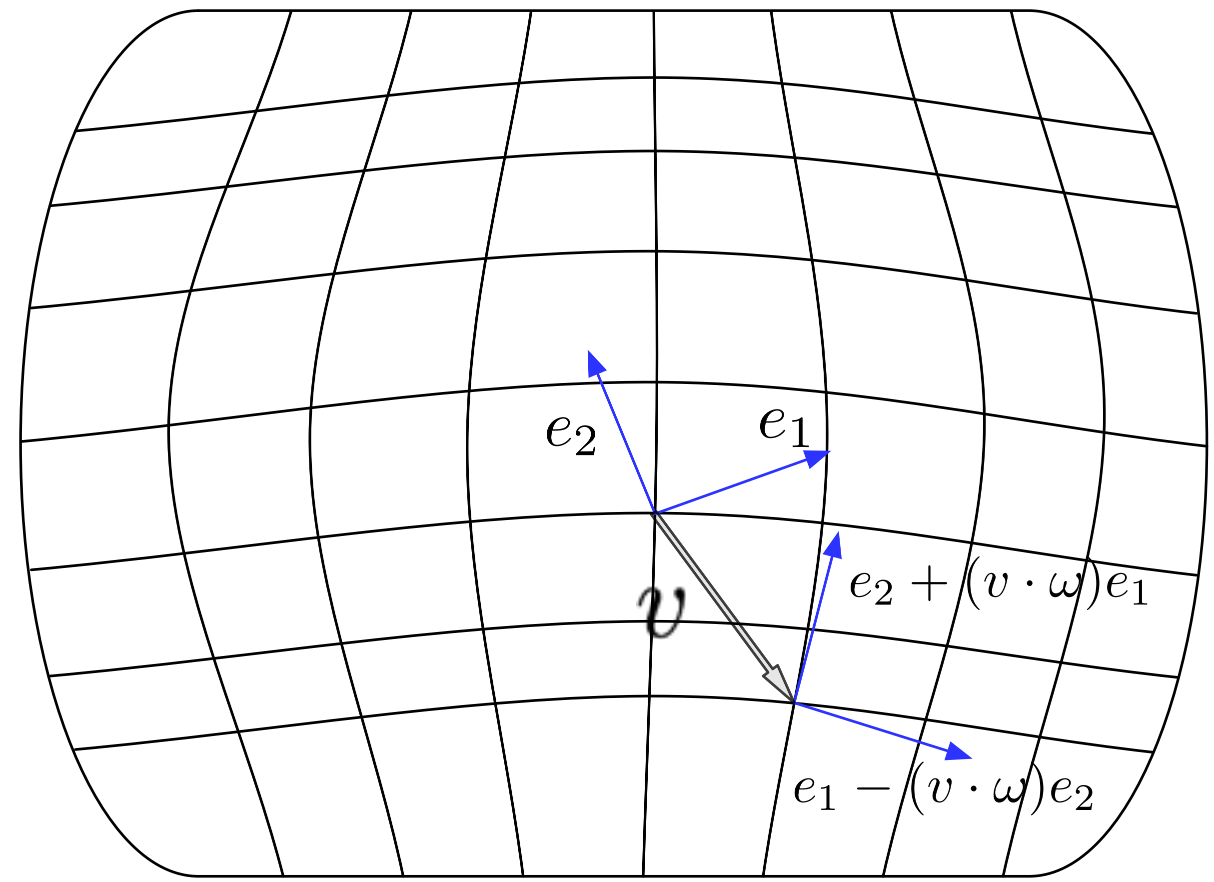
  - uniform deformations <-> viscosity

  - spatially varying deformations <-> central charge

Geometric Perturbations

- Introduce vielbeins e^{μ}_{α} - local orthonormal basis of tangent vectors
- Non-relativistic: preferred time direction $\alpha = 0$
- Spin connection ω_μ - tells us how vielbeins rotate from point to point
- (degenerate) spacetime metric

$$h_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$$



- spacetime volume element $\widehat{\sqrt{g}} = \det(\mathbf{e})$
- Torsion $T^{\alpha}_{\mu\nu} = \partial_\mu e^\alpha_\nu - \partial_\nu e^\alpha_\mu + \epsilon^{0\alpha\beta}(\omega_\mu e^\beta_\nu - \omega_\nu e^\beta_\mu)$
- Curvature $R_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu$

More on Non-relativistic geometry

- vielbein is covariantly constant

$$0 = \nabla_\mu e^\alpha_\nu = \partial_\mu e^\alpha_\nu + \omega_\mu \epsilon^a{}_b e^b_\nu \delta^\alpha_a - \Gamma^\lambda_{\mu\nu} e^\alpha_\lambda$$

- Given a choice of vielbeins, it is not in general possible to find a Levi-Civita connection

- $C^{ab} = e^\mu_0 \left(\eta^{bc} e^\nu_c T^a_{\mu\nu} + \eta^{ac} e^\nu_c T^b_{\mu\nu}\right)$ does not depend on spin connection

- => independent variables are vielbeins and “reduced torsion” $\tilde{T}^a_{\mu\nu} \equiv T^a_{\mu\nu} - \frac{1}{2}\eta_{bc} C^{ab} \left(e^0_\mu e^c_\nu - e^0_\nu e^c_\mu\right)$

Induced Action for QH States

- Gapped system -> Integrate out matter in bulk to get Induced action
$$\exp(iS_{\rm eff}) = \int \mathcal{D}\psi \exp(iS[\psi, e, A, \tilde{T}])$$
- Two types of terms

$$S_{\rm eff} = \int d^3x \widehat{\sqrt{g}\epsilon}(e,T,R,F) + \frac{\nu}{4\pi}\int d^3x\, \widehat{\epsilon^{\mu\nu\lambda}}\left(A_\mu\partial_\nu A_\lambda + 2\bar{s}\omega_\mu\partial_\nu A_\lambda + \overline{s^2}\omega_\mu\partial_\nu\omega_\lambda\right)$$
$$+ \frac{c}{96\pi}\int d^3x\, \widehat{\epsilon^{\mu\nu\lambda}}\left(\Gamma^\rho_{\mu\sigma}\partial_\nu\Gamma^\sigma_{\lambda\rho} + \frac{2}{3}\Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\theta}\Gamma^\theta_{\lambda\rho}\right)$$

- “Locally covariant”: integrand is a scalar
- “Topological”: integrand changes by boundary term under symmetries

Wen & Zee 1992, Read & Green 2000, Abanov & Gromov 2014, BB & Read 2015

Induced Action and Berry Curvature

- Consider a general induced action:

$$S_{\text{eff}}[\mathcal{Q}] = \int d^d x dt \ \left[\epsilon(\mathcal{Q}) + \mathcal{L}_{\text{top}}(\mathcal{Q})\right]$$

- gapped phase, external fields evolve along (contractible) closed path -> adiabatic theorem

$$U[\phi_i, \phi_f = \phi_i, \mathcal{Q}] = e^{i\Omega} = e^{iS_{\text{eff}}[\mathcal{Q}]}$$

- Top. terms contribute to Berry phase

$$\Omega = \gamma_D + \gamma_B, \quad \gamma_D = \int d^d x dt \, \epsilon(\mathcal{Q}), \quad \gamma_B = \int d^d x dt \, \mathcal{L}_{\text{top}}$$

Induced Action and Berry Curvature

- From $\gamma_B = \int d^d x dt \, \mathcal{L}_{\rm top}$ we can extract the Berry curvature using Stokes's theorem

- Example - U(1) Chern-Simons term $\widehat{\epsilon}^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda$

$$A_0 = 0, \quad A_i = f_i(t)/L$$

$$\Longrightarrow \int d^2 x dt \, \widehat{\epsilon}^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda = \int A_2 dA_1 - A_1 dA_2$$

$$= -2 \int dA_1 dA_2$$

This constant Berry curvature is the *Hall conductivity*

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Setup

- Take time-like vielbein trivial, $e_0^\mu = \delta_0^\mu$ and reduced torsion $\tilde{T}^a_{\mu\nu} = 0$

- degenerate metric takes the form

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & g_{ij} \end{pmatrix}$$

- Convenient parametrization for g_{ij} :

$$ds^2 = g_{ij} dx^i dx^j = e^\Phi \left| dz + \mu d\bar{z} \right|^2$$

$$dz = dx^1 + i dx^2$$

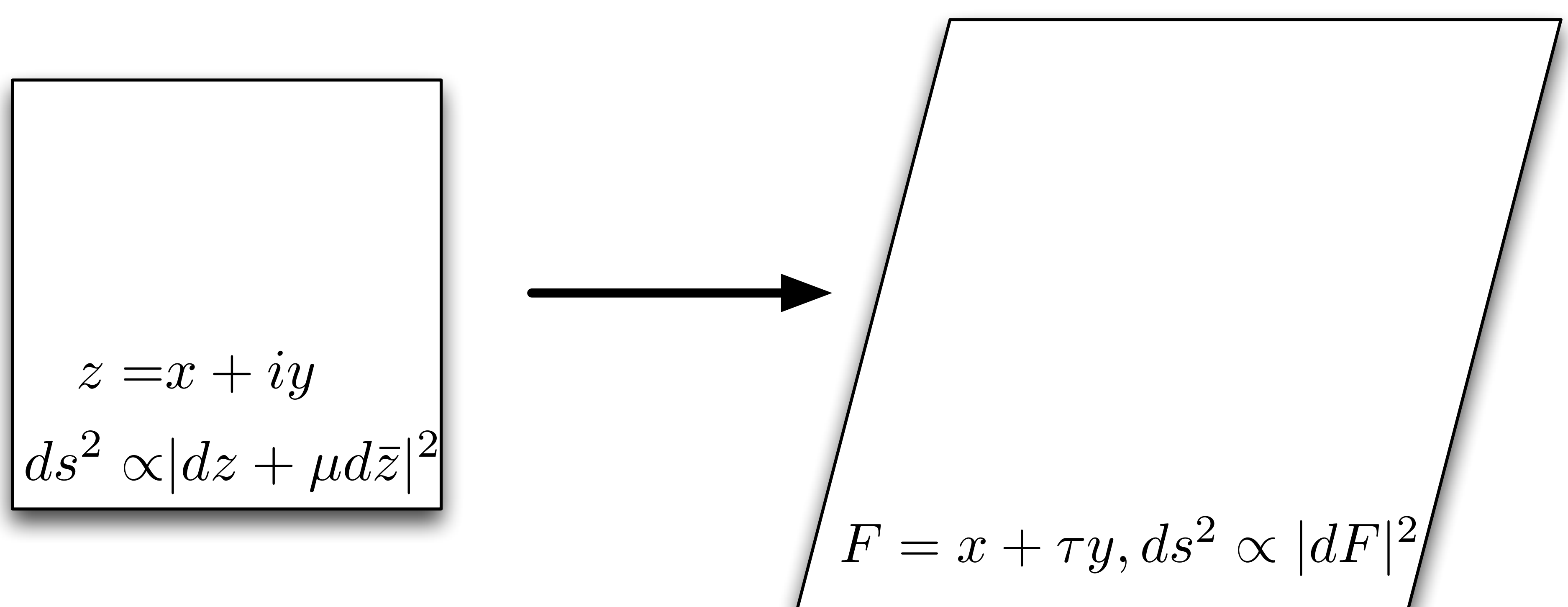
2D Complex Geometry

- μ - *Beltrami coefficient*, determines the complex structure via the Beltrami Equation

$$\bar{\partial} F = \mu \partial F$$

$$ds^2 = e^{\Phi'} \left| dF \right|^2$$

- Example: Torus, constant $\mu = \frac{i-\tau}{i+\tau}$



Berry Curvature from Induced Action

- We fix $\det g = 1 \leftrightarrow \Phi = \frac{1}{1-|\mu|^2}$
- Spin connection: $\omega_z = -i\bar{\partial}\bar{\mu} + O(\mu^2)$

$$\omega_0 = \mathrm{Im}(\mu\partial_0\bar{\mu}) + O(\mu^3)$$

- Also want to vary EM field s.t. $\delta(A_i + \bar{s}\omega_i) = 0$
- Can plug these into induced action to find Berry curvature

$$\gamma_B = \int d^2x \int d^2y \int \mathcal{F}_{\bar{\mu}\mu} d\bar{\mu}(\mathbf{x}) d\mu(\mathbf{y})$$

$$\mathcal{F}_{\bar{\mu}\mu} = -\mathcal{F}_{\mu\bar{\mu}} = i\bar{n}\bar{s}\delta(z-z') + \frac{i[c - 12\nu(\overline{s^2} - \bar{s}^2)]}{12\pi}\partial\bar{\partial}\delta(z-z')$$

Prescription

1. Obtain wavefunctions with perturbed metric and magnetic field
   - Can be done analytically using “conformal block” trial states
   - Numerics on the torus - diagnostic tool for topological order
2. Calculate the functional Berry connection

$$\mathcal{A}_{\mu}^{ee'} = i \left\langle \psi_e \left| \frac{\delta}{\delta\mu(z,\bar{z})} \right| \psi_{e'} \right\rangle, \quad \mathcal{A}_{\bar{\mu}}^{ee'} = i \left\langle \psi_e \left| \frac{\delta}{\delta\bar{\mu}(z,\bar{z})} \right| \psi_{e'} \right\rangle$$

3. Take the curl to get the curvature

$$\mathcal{F}_{\mu\bar{\mu}} = \frac{\delta \mathcal{A}_{\bar{\mu}}(z,\bar{z})}{\delta\mu(z',\bar{z}')} - \frac{\delta \mathcal{A}_{\mu}(z,\bar{z})}{\delta\bar{\mu}(z',\bar{z}')}$$

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Analytic Approah: Conformal Block Trial States

- Goal - wavefunctions for particles in the lowest-Landau-level
- These satisfy $(\bar{\partial} - i\bar{A})\psi = 0$
- Such functions can be systematically generated as chiral correlation functions in 2d CFT

$$S = \frac{1}{8\pi}\int d^2x(\nabla\phi)^2 + S(\sigma) \qquad a(z_i) = \sigma(z_i)\mathcal{O}_{1/\sqrt{\nu}}(z_i)$$

$$\psi_e(\{z_i\}) = \lim_{\alpha\to 0}\left\langle \prod_{i=1}^{N} a(z_i) \prod_{j=1}^{N/\alpha} \mathcal{O}_{-\alpha/\sqrt{\nu}}(w_j) \right\rangle_{0,e}$$

Moore, Read 1991

Trial Wavefunctions in a Curved background

- Non-trivial geometries => need $(\partial_{\bar{F}} - iA_{\bar{F}})\psi = 0$
- Naive guess - perturbed CFT correlator

$$\psi \stackrel{?}{=} \lim_{\alpha \to 0} \left\langle \prod_{i=1}^{N} a(z_i) \prod_{j=1}^{N/\alpha} \mathcal{O}_{-\alpha/\sqrt{\nu}}(w_j) e^{-\frac{1}{\pi}\int d^2x \mu T} \right\rangle_0$$

- Problem - $|\psi|^2$ not a scalar!
- Fix - we need to carefully define what we mean by "chiral correlator"
- Inspired by the usual case, we expect $\left\langle \prod_j \varphi(z_i, \bar{z}_i) \right\rangle_g = \sum_e |\Psi_e(\{z_i\}|\mu)|^2$
- **BUT** - this ignores *anomalies*

The Gravitational Anomaly

- Problem in the chiral theory: $\bar{\partial}T - \mu\partial T - 2\partial\mu T = \frac{c}{12}\partial^3\mu$
- Naive non-chiral theory: $\nabla_i T^{ij} \neq 0$
- Fix - add a *local counterterm* to the non-chiral action

$$S \to S - \frac{c}{12\pi}K[\mu, \bar{\mu}, \Phi]$$

$$K = \int d^2x\,(1-|\mu|^2)^{-1}\left(\partial\mu\bar{\partial}\bar{\mu} - \frac{1}{2}\mu(\bar{\partial}\bar{\mu})^2 - \frac{1}{2}\bar{\mu}(\partial\mu)^2\right)$$

$$+\frac{1}{4}\int d^2x(1-|\mu|^2)\left(\frac{1}{2}e^{\Phi}g^{\nu\lambda}\partial_\nu\Phi\partial_\lambda\Phi + \Phi R_*\right)$$

- Restores coordinate invariance at the expense of factorization - *holomorphic factorization anomaly*

Belavin, Knizhnik 1986, Polyakov 1987

Implications for the Wavefunction

- A general correlator can then be written

$$\left\langle \prod_j \varphi(z_i, \bar{z}_i) \right\rangle_g = e^{\frac{c}{12\pi}K - \sum_i s\Phi(z_i)} \sum_e |\Psi_e(\{z_i\}|\mu)|^2$$

$$\Psi_e = \left\langle \prod_j \varphi(z_i) e^{-\frac{1}{\pi}\int d^2x \mu T} \right\rangle_{0,e}$$

- We <u>define</u> wavefunctions

$$\psi_e(\{z_i\}) = e^{\frac{c}{24\pi}K - \sum_i \frac{s}{2}\Phi(z_i)} \lim_{\alpha \to 0} \left\langle \prod_i a(z_i) \prod_j \mathcal{O}_{-\alpha/\sqrt{\nu}}(w_j) e^{-\frac{1}{\pi}\int d^2x \mu T} \right\rangle_{0,e}$$

Verlinde, 1990

Properties of the Wavefunctions

- CFT - $T(z)T(0) \sim \frac{c}{2z^4} + \frac{2}{z^2}T(0) + \frac{1}{z}\partial T(0) + \dots$

$$T(z)a(0) \sim \frac{s}{z^2}a(0) + \frac{1}{z}\partial a(0) + \dots$$

- Lets us prove:
  - 1. $\left|\psi_e(\{\zeta_i(z_i, \bar{z}_i)\})\right|^2 = \left|\psi_e(\{z_i\})\right|^2$
  - 2. $\left(\partial_{\bar{F}} + is\omega_{\bar{F}}\right)\psi_e(z_1, ..., z, ..., z_N) = 0$
- Last step - singular gauge transformation to make ψ_e single valued -> $(\partial_{\bar{F}} + is\omega_{\bar{F}} - iA^0_{\bar{F}})\psi_e = 0, \;\; \nabla \times \mathbf{A^0} = \frac{2\pi\rho}{\nu}$ ->

$$B = \frac{2\pi\rho}{\nu} - sR$$

Last Important Fact: Generalized Screening

- Need to normalize our wavefunctions
- Recall: Laughlin's trial wavefunction for the 1/Q FQHE:

$$\psi_L = \prod_{i<j}(z_i - z_j)^Q e^{-\sum_i \frac{1}{4}|z_i|^2}$$

- $$|\psi_L|^2 = \exp\left[\frac{1}{Q}\left(\sum_{i<j} 2Q^2 \log|z_i - z_j| - \sum_i \frac{Q}{2}|z_i|^2\right)\right]$$
- => Normalization is the partition fn. for a classical OCP
- Plasma screens for small Q => short-range correlations, homogeneous fluid ground state, fractionally charged excitations, etc.

Last Important Fact: Generalized Screening

- Need to normalize our wavefunctions
- Laughlin 1983 -$\int \prod_i d^2 z_i \, |\psi_L|^2$ is the partition fn for a classical plasma
- Generalization to Moore-Read language: $\int \prod_i d^2 z_i \, |\psi_e(\mu = 0)|^2$ is the partition function for a *perturbed* CFT (considered as a 2D stat. mech. model)

$$< \psi_e | \psi_e >= \left\langle e^{\int d^2 z \lambda \bar{a}(\bar{z}) a(z)} e^{-\frac{i\sqrt{\nu}}{2\pi} \int d^2 z \phi(z, \bar{z})} \right\rangle_e$$

  - Screening<-> the theory is driven to a massive phase
  - We assume we are in a top. phase s.t. screening holds

Consequences of Screening

1. Particle density = background charge density $\bar{n} = \rho$

2. $\int \prod_i \left(\sqrt{\det g_i} d^2 z_i\right) \psi_e^* \psi_{e'} = N(g)\delta_{ee'}$

3. Normalization has a local expansion

$$N(g) = \exp\left[\int d^2 z(a_0\sqrt{\det g} + a_1 R + \dots)\right]$$

4. Fixed area + Gauss-Bonnet theorem: first two terms are *constants*

Berry Curvature from the Wavefunction

- $\int \prod_i dz_i \, |\psi|^2$ is independent of $\mu, \bar{\mu}$ through 2nd order in derivatives using the screening hypothesis

- LLL wavefunctions in magnetic field $B = \frac{2\pi}{\nu}\bar{n} - sR$

- ψ depends on $\bar{\mu}$ only via the local counterterms to this order

- Using these properties, we can vary $\mu, \bar{\mu}$ with ρ fixed to find:

$$\mathcal{F}_{\mu\bar{\mu}} = -2\mathrm{Im}\left(\left\langle \frac{\delta\psi_e}{\delta\mu(z')} \middle| \frac{\delta\psi_e}{\delta\bar{\mu}(z)} \right\rangle\right) = i\bar{n}\delta(z-z') + \frac{ic}{12\pi}\partial\bar{\partial}\delta(z-z')$$

- Conclusion: $\overline{s^2} = \overline{s}^2$ for CFT trial states

Conclusion

- Central charge computed as a Berry curvature

- c is robust to breaking translation & rotational symmetry, but this is not manifest in our approach

- References: BB & N. Read Phys. Rev. B **91** 125303 (2015)
  BB & N. Read Phys. Rev. B **91** 165306 (2015)

- c.f. related work by: Klevtsov et. al., Klevtsov & Wiegmann, Can Laskin & Wiegmann, Gromov et. al., and many others