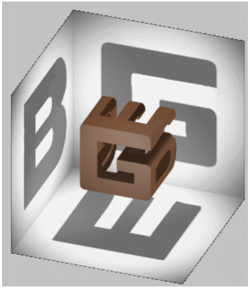
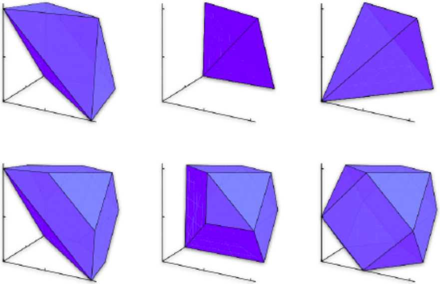
Entanglement Polytopes





David Gross

April 2017

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Disclaimer:

- This speaker has nothing to say on the quantum Hall effect, mathematical or otherwise.

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Instead: I will report on. . .

- . . . a program to find applications of non-commutative moment polytopes for quantum information.
- . . . extracting global information about a pure state from single-particle measurements alone.

[M. Walter, B. Doran, D. Gross, M. Christandl, Science '13],
[C. Schilling, D. Gross, M. Christandl, PRL '13].

Outline

- Quantum Marginal Problem
- Entanglement Polytopes
- Generalized Pauli Constraints
- Optionally: Computational Aspects

Quantum Marginal Problems

Marginals in classical probability

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- Marginals are distributions of subsets of a number of random variables.
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- Marginals are distributions of subsets of a number of random variables.
- If these overlap ⇒ non-trivial compatibility conditions.
- Compatible subsets are *convex polytopes* (in QM, known as *Bell polytopes*)
- In general, membership problem is NP-hard.

Marginals in quantum probability

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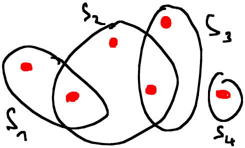

- For subset S_i specify state ρ_i.
- Q: Are these *compatible*:

$$\rho_i = \mathrm{tr}_{\setminus S_i} \rho$$

for some global ρ?

Solves *all* physical ground-state problems:

$$\min_{\rho} \mathrm{tr} H\rho = \min_{\rho} \sum_{i,j} \mathrm{tr} h_{i,j}\, \rho = \min_{\{\rho_{i,j}\}} \sum_{i,j} \mathrm{tr} h_{i,j}\, \rho_{i,j}.$$

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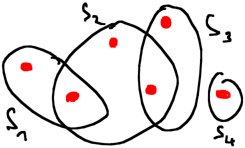

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Terms live on two systems $\Rightarrow$ simple (if marginal prob is).

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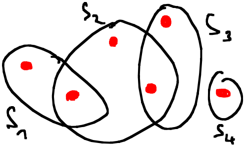

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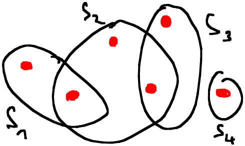

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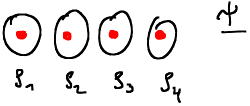
But:

- Ground state problem is *intrinsically hard*: QMA-complete.
- Convex optimization $\Rightarrow$ so is quantum marginal problem. ☹

Single-site marginal problem

Specific instance: marginals do not overlap, global state pure



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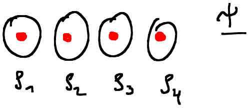

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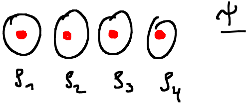

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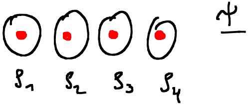
Classical version:

- Globally pure
  $\Leftrightarrow$ no global randomness
  $\Rightarrow$ no local randomness.
- . . . trivial.

Reduction to eigenvalues



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Which set of ordered local eigenvalues $\vec{\lambda}^{(i)}$ can occur?

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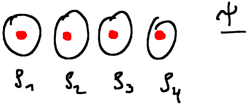

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…progress was scant for three decades …

- …until A. Klyachko identified these sets as images of *moment maps*.
- In particular: Compatible sets are convex polytopes.

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$$|\psi\rangle = \sqrt{\lambda^{(1)}}|e_1\rangle \otimes |f_1\rangle + \sqrt{\lambda^{(2)}}|e_2\rangle \otimes |f_2\rangle$$

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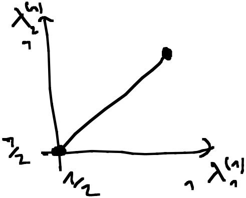
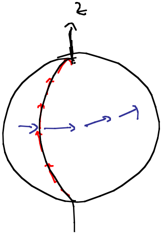
In terms of largest eigenvalue, get simple polytope:



Moment maps

Methods require detour via group actions on symplectic manifolds.



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$(dH_{\sigma_z})^*$

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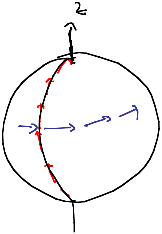

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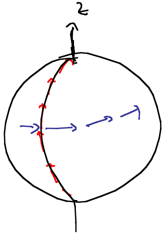

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$$\begin{aligned} \tilde{\mu} : \mathfrak{g} &\rightarrow F(M) \\ g &\mapsto \big(H_g : m \mapsto H_g(m)\big) \end{aligned}$$

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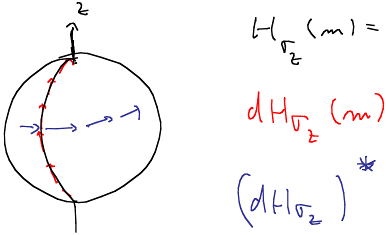

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  be a function that associates with every one-parameter group $e^{t\,g}$ a Hamiltonian H_g.

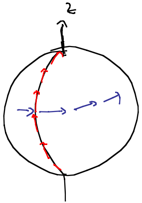
This defines a group action of G on M, where the flow generated by $e^{t\,g}$ is the Hamiltonian flow of H_g.

Moment maps

- Re-arranging parameters, one gets *moment map*

$$\begin{aligned}\mu : M &\to (\mathfrak{g}^* \simeq \mathfrak{g}) \\ m &\mapsto (g \mapsto H_g(m))\end{aligned}$$

sending points of the manifold into Lie algebra.



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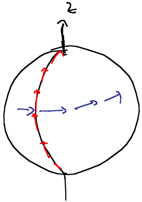

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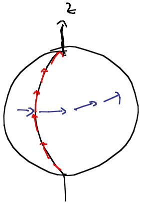
  with symplectic form: $\operatorname{Im}\langle \cdot | \cdot \rangle$.
- Specializing to local action $U(\mathbb{C}^d)^{\times n}$ on tensor products $\left(\mathbb{C}^d\right)^{\otimes n}$:

$$\mu(\psi)(g_1 \oplus \cdots \oplus g_n) = \sum_i \operatorname{tr} \rho^{(i)} g_i$$

  so that

$$\mu(\psi) \simeq \rho^{(1)} \oplus \cdots \oplus \rho^{(n)}.$$



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Convexity properties of moment map

Central theorem by Kirwan ('84):

> Image of moment map in positive Weyl chambre (here: diagonal matrices with ordered eigenvalues) is convex polytope.

Summary: Overview of Quantum Marginal Prob

- Quantum Marginal Prob originates in chemistry.
- Generally computationally intractable.
- *Single-site quantum-marginal problem* non-trivial, but seems tractable. . .
- . . . due to unexpected geometric structure.

Entanglement

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- Often referred to as *SLOCC classes*. But that sounds too unpleasant.
- Formally:

$$\psi \sim \phi \qquad \Leftrightarrow \qquad \psi = (g_1 \otimes \cdots \otimes g_n)\phi$$

  with g_i local invertible matrices (*filtering operations*).
- So we're looking at $\mathrm{SL}(\mathbb{C}^d)^{\times n}$-orbits in $\left(\mathbb{C}^d\right)^n$.

SLOCC, SLOCC! – Who's There?

- For three qubits ($d = 2, n = 3$), equivalence classes known since mid-1800s. Re-discovered in 2000 to great effect:

Three qubits can be **entangled** in **two inequivalent ways**
W Dür, G Vidal... - Arxiv preprint quant-ph/0005115, 2000 - arxiv.org
Abstract: Invertible local transformations of a multipartite system are used to define equivalence classes in the set of **entangled** states. This classification concerns the **entanglement** properties of a single copy of the state. Accordingly, we say that **two** states **...**
Cited by 1683 - Related articles - BL Direct - All 22 versions - Import into BibTeX

Four **qubits can** be **entangled** in nine different **ways**
F Verstraete, J Dehaene, B De Moor... - Physical Review A, 2002 - APS
... to the singlet state by SLOCC operations **3**. In the case of **three entangled qubits**, it was shown 2,4,5 that each state **can** be converted by SLOCC operations either to the GHZ-state (000 111)/&, or to the W-state (001 010 100)/), leading to **two inequivalent ways** of **entangling ...**
Cited by 350 - Related articles - BL Direct - All 12 versions - Import into BibTeX

Control and measurement of **three-qubit entangled** states
CF Roos, M Riebe, H Häffner, W Hänsel... - Science, 2004 - sciencemag.org
... The ions' electronic **qubit** states are initialized in the S state by optical pumping. **Three qubits can** be **entangled** in only **two inequivalent ways**, represented by the Greenberger-Horne-Zeilinger (GHZ) state, , and the W state, (17). **...**
Cited by 273 - Related articles - All 13 versions - Import into BibTeX

Examples

Classes:

- Products $\psi = \phi_1 \otimes \phi_2 \otimes \phi_3$.
- Three classes of *bi-separable* states: $\psi = \phi_1 \otimes \phi_{2,3}$.
- The W-class:

$$|W\rangle = |001\rangle + |010\rangle + |100\rangle.$$

- The GHZ-class:

$$|GHZ\rangle = |000\rangle + |111\rangle.$$

Further examples

4 qubits:

- Classification apparently first obtained in QI community [Verstraete *et al.* (2002)].
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Beyond:

- Number of parameters required to label orbits increases exponentially.
- Only sporadic facts known.

Desiderata

Can we come up with theory that

- is systematic
  (any number of particles, local dimensions, symmetry constraints),
- is efficient
  (only polynomial number of parameters have to be learned),
- experimentally feasible
  (parameters easily accessible, robust to noise)?

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Claim:

The single-site quantum marginal problem lives up to these standards.

Entanglement Polytopes

[M. Walter, B. Doran, D. Gross, M. Christandl, Science '13]

Central observation, entanglement polytopes

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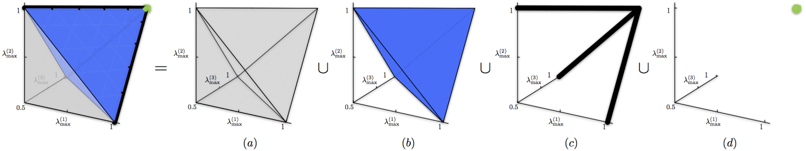
- Turns out: $\Delta_{\mathcal{C}}$ is again polytope: the *entanglement polytope* associated with $\mathcal{C}$.
- Clearly: the position of $\vec{\lambda}(\psi)$ w.r.t. the entanglement polytopes contains *all* local information about global entanglement class.

Examples re-visited: 3 qubit entanglement polytopes

For three qubits, polytopes resolve all 6 entanglement classes:



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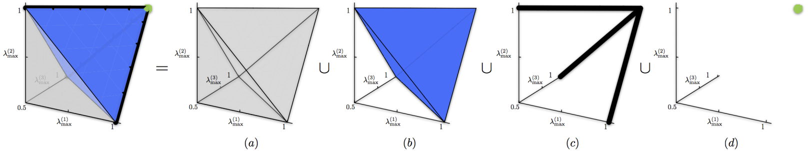

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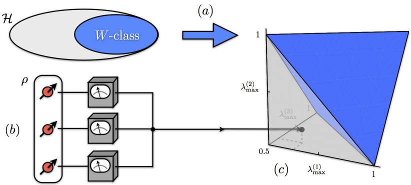

W-class corresponds to "upper pyramid":

$$\lambda_{\max}^{(1)} + \lambda_{\max}^{(2)} + \lambda_{\max}^{(3)} \geq 2.$$

Any violation of that witnesses GHZ-type entanglement.

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4 qubits:

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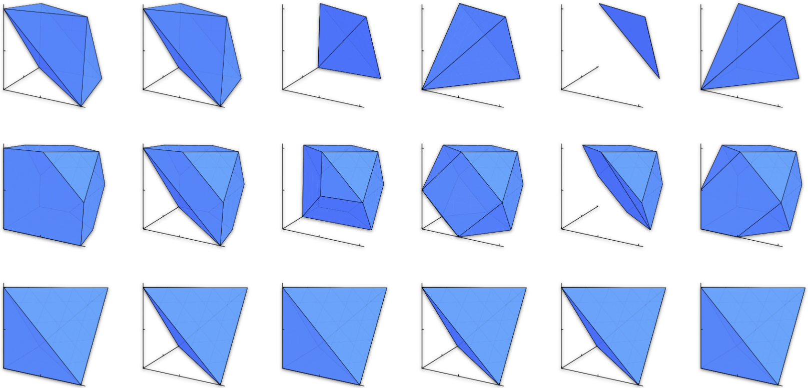
Example: 4-qubit W-class

$$\mathcal{C}_W \ni |0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle$$

again an "upper pyramid":

$$\lambda^{(1)}_{\text{max}} + \lambda^{(2)}_{\text{max}} + \lambda^{(3)}_{\text{max}} + \lambda^{(4)}_{\text{max}} \geq 3.$$

Example: 4 qubit entanglement polytopes



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- Let $(\mu_1, \ldots, \mu_n)$ be $\big(\mathrm{SL}(\mathbb{C}^d)\big)^{\times n}$ irrep in F_n (with μ_i Young frames). Note that $\frac{1}{d}\mu_i$ are formally probability distributions.

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$$\frac{1}{d}(\mu_1, \ldots, \mu_n) \in \Delta_{\mathcal{C}}.$$
- Points arising this way are dense in $\Delta_{\mathcal{C}}$.

$\Rightarrow$ Entanglement polytope corresponds to normalized irreps in the homogeneous coordinate ring over $\mathcal{C}$.

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- Let F_n be the homogeneous polynomials on $\mathcal{C}$ of degree n.
- $\mathcal{C}$ is SL-orbit $\Rightarrow$ SL acts on F_n.
- Let $(\mu_1, \ldots, \mu_n)$ be $\left(\mathrm{SL}(\mathbb{C}^d)\right)^{\times n}$ irrep in F_n (with μ_i Young frames). Note that $\frac{1}{d}\mu_i$ are formally probability distributions. Then
$$\frac{1}{d}(\mu_1, \ldots, \mu_n) \in \Delta_{\mathcal{C}}.$$
- Points arising this way are dense in $\Delta_{\mathcal{C}}$.

$\Rightarrow$ Entanglement polytope corresponds to normalized irreps in the homogeneous coordinate ring over $\mathcal{C}$.

- ... we use computer algebra system to reduce out coordinate ring.

Example: Marginal polytope for all bi-partite states

Q: which $(\mathsf{SL} \times \mathsf{SL})$-irreps occur in $\mathrm{Sym}^n(\mathbb{C}^d \otimes \mathbb{C}^d)$?

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Hence, for bi-partite pure state: $\vec{\lambda}^{(1)}=\vec{\lambda}^{(2)}$.

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Consider n bosonic qubits:

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Analyze polytopes:

- $|0,\ldots,0\rangle$ in all $\mathcal{C}$'s $\Rightarrow \Delta_{\mathcal{C}} = [\gamma_{\mathcal{C}}, 1]$.
- Turns out: Possible choices are

$$\gamma_{\mathcal{C}} \in \left\{\frac{1}{2}\right\} \cup \left\{\frac{N-k}{N} : k = 0, 1, \ldots, \lfloor N/2 \rfloor\right\} \ldots$$

- ... with innermost point γ the image of W-type states.

Example: No Solipsism

- A vector is *genuinely n-partite entangled* if it does not factorize w.r.t. any bi-partition:

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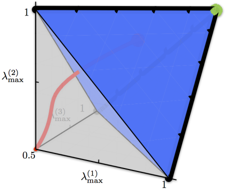
Interpretation:

- *no solipsism*: love needs a partner!
  (And entangled qubits need their counter-parts).

Example: Distillation



Entanglement measures from local information:

- (Linear) entropy of entanglement

$$E(\psi) = 1 - \frac{1}{N}\sum_i \mathrm{tr}\rho_i^2$$

  simple function of Euclidean distance of eigenvalue point to origin.

- "Closer to origin $\Rightarrow$ more entanglement".

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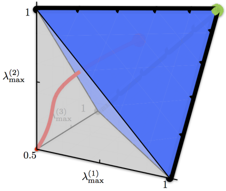

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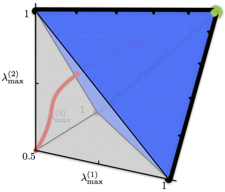

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- $\Rightarrow$ can bound *distillable* entanglement from local information!
- Can even give *distillation procedure* without need to know state beyond local densities
  (generalizing [Verstraete *et al.* 2002]).

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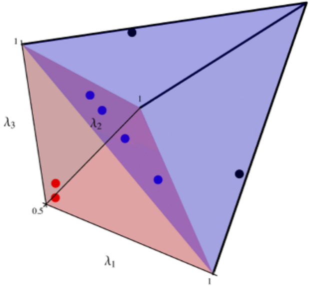
- Results are epsilonifiable: if distance d of spectrum to a polytope Δ exceeds

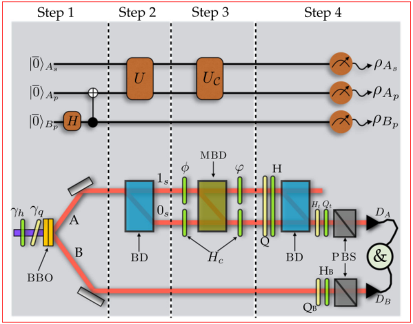
$$4N\sqrt{1-p},$$

  then $\rho \notin \mathrm{conv}(\Delta)$.
- $p = \mathrm{tr}\,\rho^2$ is purity, which an be lower-bounded from local information alone.

Experiments

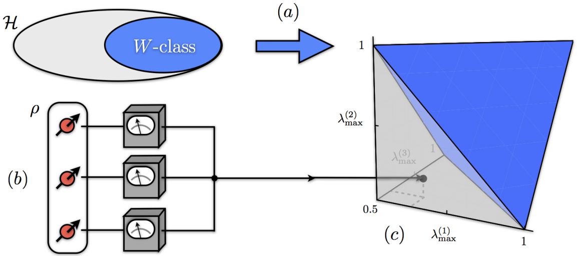
Recently, two experimental implentations.





[Aguilar, *et al.*, PRX '15]

Summary of Entanglement Polytopes



- Locally accessible info about global entanglement encoded in *entanglement polytopes* – subpolytopes of the set of admissible local spectra.
- Provides a systematic and efficient way of obtaining information about entanglement classes.

Another facet:

The Pauli principle and a generalization of Hartree-Fock

Generalizing the Pauli Principle [Klyachko]

Consider Fermionic wave function

$$\psi \in \wedge^n(\mathbb{C}^d).$$

- Eigenvalues of 1-RDM

$$\rho^{(1)}_{i,j} = \langle \psi | a_i^\dagger a_j | \psi \rangle.$$

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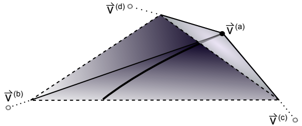
Questions:

- Are these additional constraints saturated in "typical" physical systems?
- Do they have an effect on e.g. ground state wave functions?

Motivation: Klyachko's "super-selection rules"

Vectors ψ that map to a facet of the polytope are "simple":

- Take the set of weights that lie on the affine hull of the facet,
- then ψ has non-zero coefficients only w.r.t. these weight vectors.
- I.e., such ψ's have a sparse representation.



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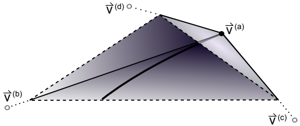

- Klyachko presented numerical evidence that certain few-electron atoms show "pinned" spectra.

Super-selection rules

Extensive quantum chemistry calculations [Schilling *et al.*]:

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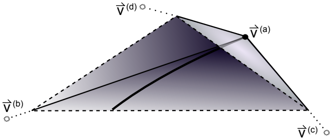

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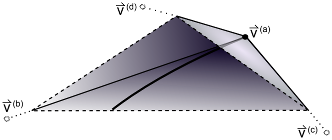
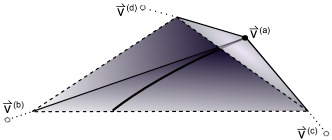


Very recent [Schilling, Benavides, Vrana, '17]:

- Super-selection rules are *stable*:
- "Quasi-pinned" ⇒ "quasi-sparse".
- Physical mechanism responsible for quasi-pinning?
- Generalize theory on structure of quasi-pinned wave functions.

Physics?

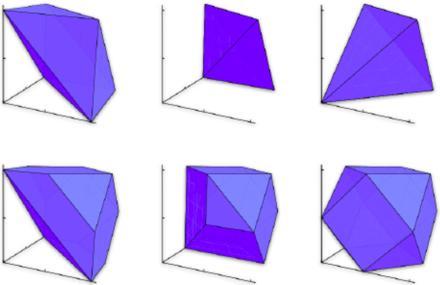


To be done:

- Physical mechanism responsible for quasi-pinning?
- Applications?

Some words on computational aspects?

Thank you for your attention!



David Gross

April 2017