Single particle Green's functions, topological invariants and quantum Hall effect

Victor Gurarie

based on work with A. Essin

Cologne, 2015

Contents

- Bulk-boundary correspondence of the topological invariants in 2D: IQHE (a repetition in a different language of the bulk-boundary introduced by Hermann Schulz-Baldes).

- Bulk-boundary correspondence of the topological invariants in 2D: FQHE

- Wen-Zee Shift and conformal spin of the electron operator (and Hall viscosity)

A. Essin and VG, PRB 2012

VG and A. Essin, JETP Lett., 2013

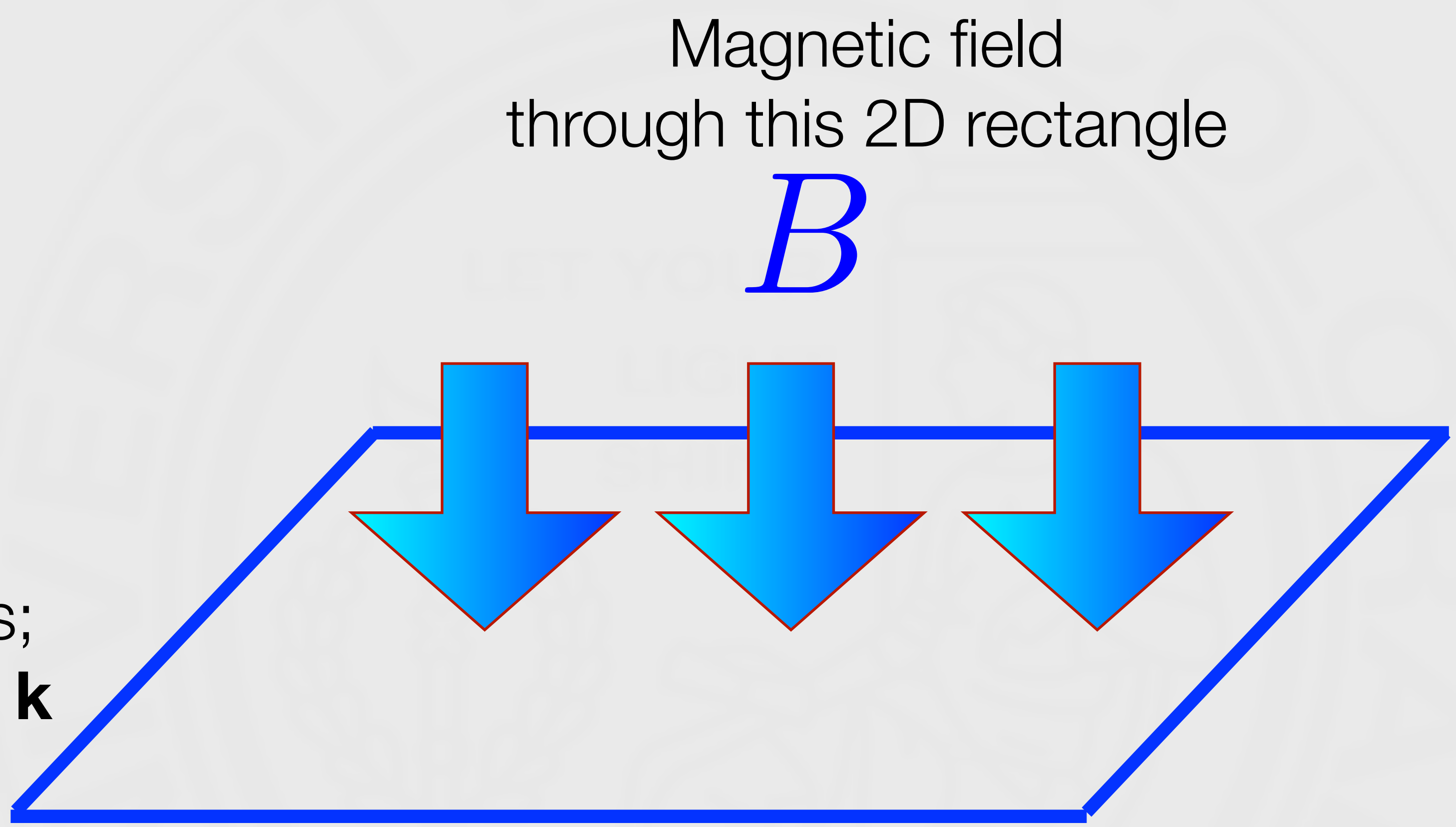
VG, J. Phys. Cond. Matt, 2015

Bulk-boundary correspondence of topological invariants in 2D: IQHE

TKNN Invariant, IQHE

Electrons hopping in a lattice with a magnetic field

$$\hat{H} = \sum_{xy} \mathcal{H}_{xy}\, \hat{a}_x^\dagger \hat{a}_y$$



Electron eigenstates and energy levels; bands labeled by *n;* quasi-momentum **k**

$$\sum_b \mathcal{H}_{ab}(\mathbf{k})\, \psi_b^{(n)}(\mathbf{k}) = \epsilon_n(\mathbf{k})\, \psi_a^{(n)}(\mathbf{k})$$

Thouless et al (TKNN), 1982:

$$\sigma_{xy} = \frac{e^2}{h} N = \frac{ie^2}{2\pi h} \sum_{n,\ \epsilon_n<0} \sum_a \int d^2k \left(\frac{\partial \psi_a^{*(n)}}{\partial k_1} \frac{\partial \psi_a^{(n)}}{\partial k_2} - \frac{\partial \psi_a^{*(n)}}{\partial k_2} \frac{\partial \psi_a^{(n)}}{\partial k_1} \right)$$

integer

TKNN Invariant, IQHE

Electrons hopping in a lattice with a magnetic field

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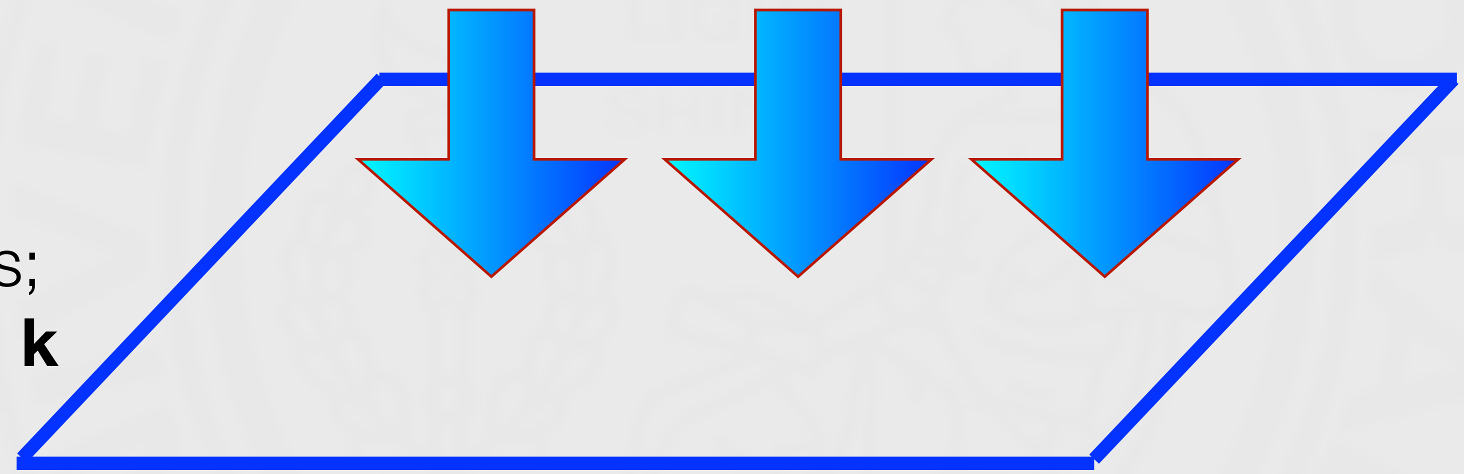
Magnetic field through this 2D rectangle

B



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integer

Niu, Thouless, Wu (1985): $G_{ab}(\omega, \mathbf{k}) = [i\omega - \mathcal{H}(\mathbf{k})]^{-1}_{ab}$

<u>Green's function</u>

Alternative expression for the TKNN invariant

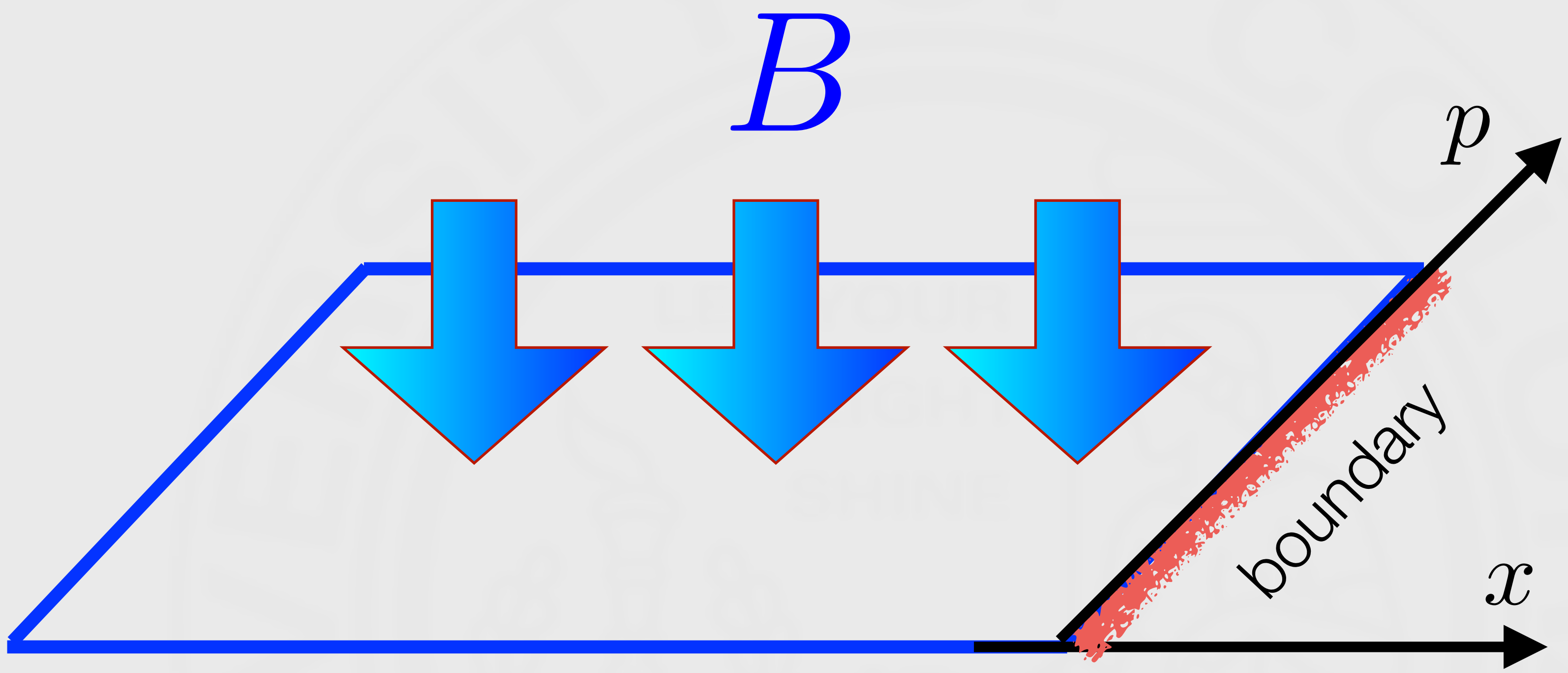
$$N = \frac{1}{24\pi^2} \sum_{\alpha\beta\gamma} \epsilon_{\alpha\beta\gamma} \int d\omega d^2k\, \mathrm{tr}\, \left[G^{-1}\partial_\alpha G G^{-1}\partial_\beta G G^{-1}\partial_\gamma G\right]$$

$\alpha, \beta, \gamma = \omega, k_1, k_2$

From TKNN to the boundary, IQHE

Green's function in the presence of the boundary

$$G_{ab}(x, x'; \omega, p)$$



$$\sum_{b,\, x'} G_{ab}(x, x'; \omega, p)\, \psi_b^{(n)}(x', p) = g_n(\omega, p)\, \psi_a^{(n)}(x, p)$$

Green's function of the mode *n*

$$g_n(\omega, p) = \frac{1}{i\omega - \epsilon_n(p)}$$

This quantity is also a topological invariant; can't be changed by small perturbations of *G*

From TKNN to the boundary, IQHE

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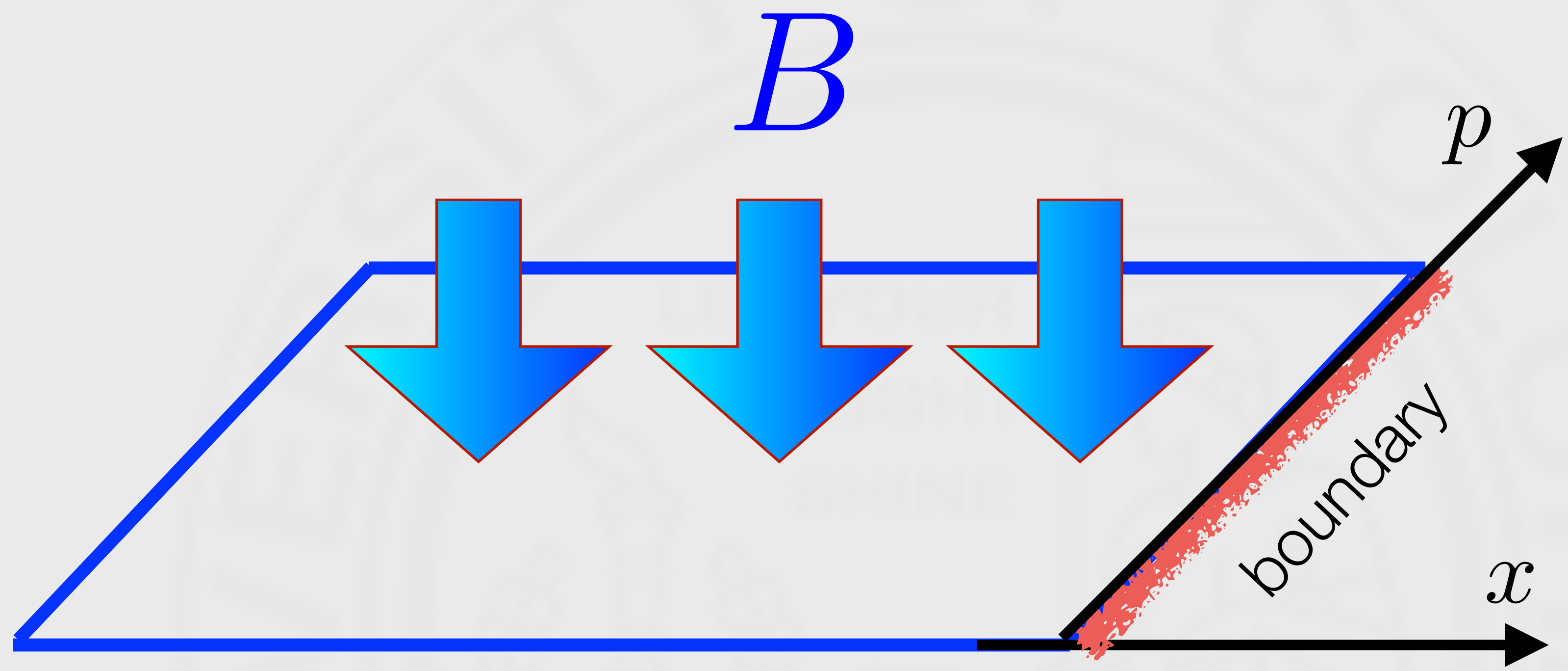

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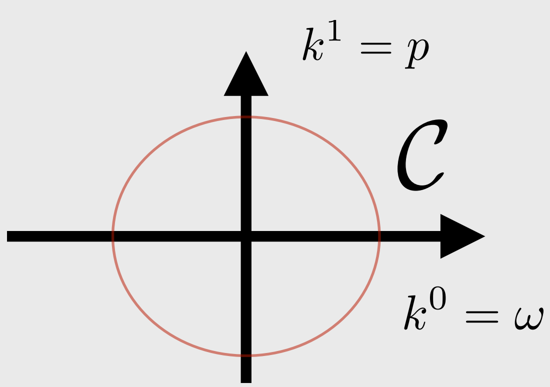

$$g_n(\omega, p) = \frac{1}{i\omega - \epsilon_n(p)}$$

$$N_{\rm edge} = \sum_\mu \oint_{\mathcal{C}} \frac{dk^\mu}{2\pi i}\, \mathrm{tr} \int dx dx'\, G^{-1}(x, x') \partial_\mu G(x', x) = \sum_n \oint \frac{dk^\mu}{2\pi i}\, \partial_\mu \ln g_n$$

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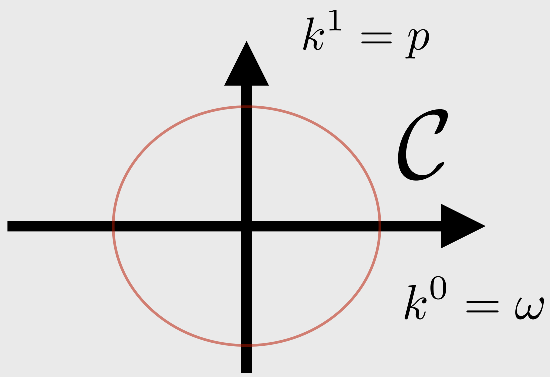
Chiral edge states & winding of g_n

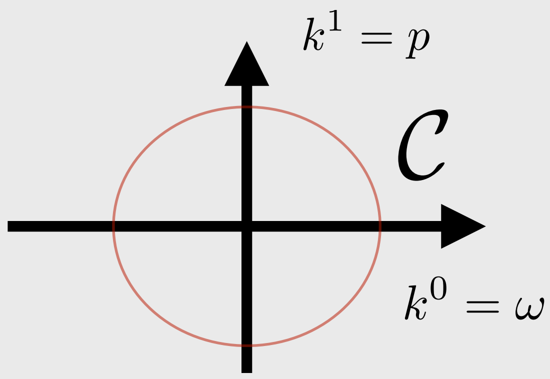


$$g_n(\omega, p) = \frac{1}{i\omega - \epsilon_n(p)}$$

Chiral edge state:

$$\epsilon_{\rm chiral} \approx vp \qquad g_{\rm chiral} \approx \frac{1}{i\omega - vp} \qquad \oint_{\mathcal{C}} \frac{dk^\mu}{2\pi i} \partial_\mu \ln g_{\rm chiral} = 1$$

Chiral edge states & winding of g_n



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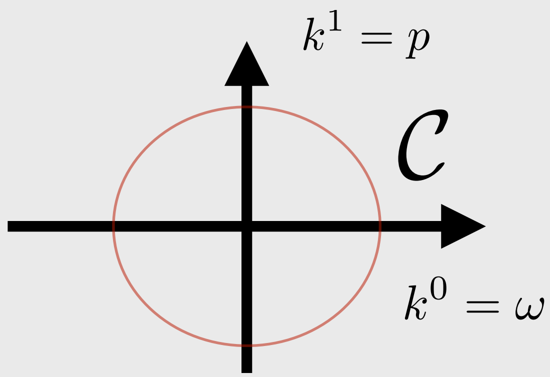
Other (bulk) states

$$\epsilon(p) \approx \text{const} \qquad g \approx \frac{1}{i\omega - \text{const}} \qquad \oint_{\mathcal{C}} \frac{dk^\mu}{2\pi i} \partial_\mu \ln g = 0$$

Chiral edge states & winding of g_n



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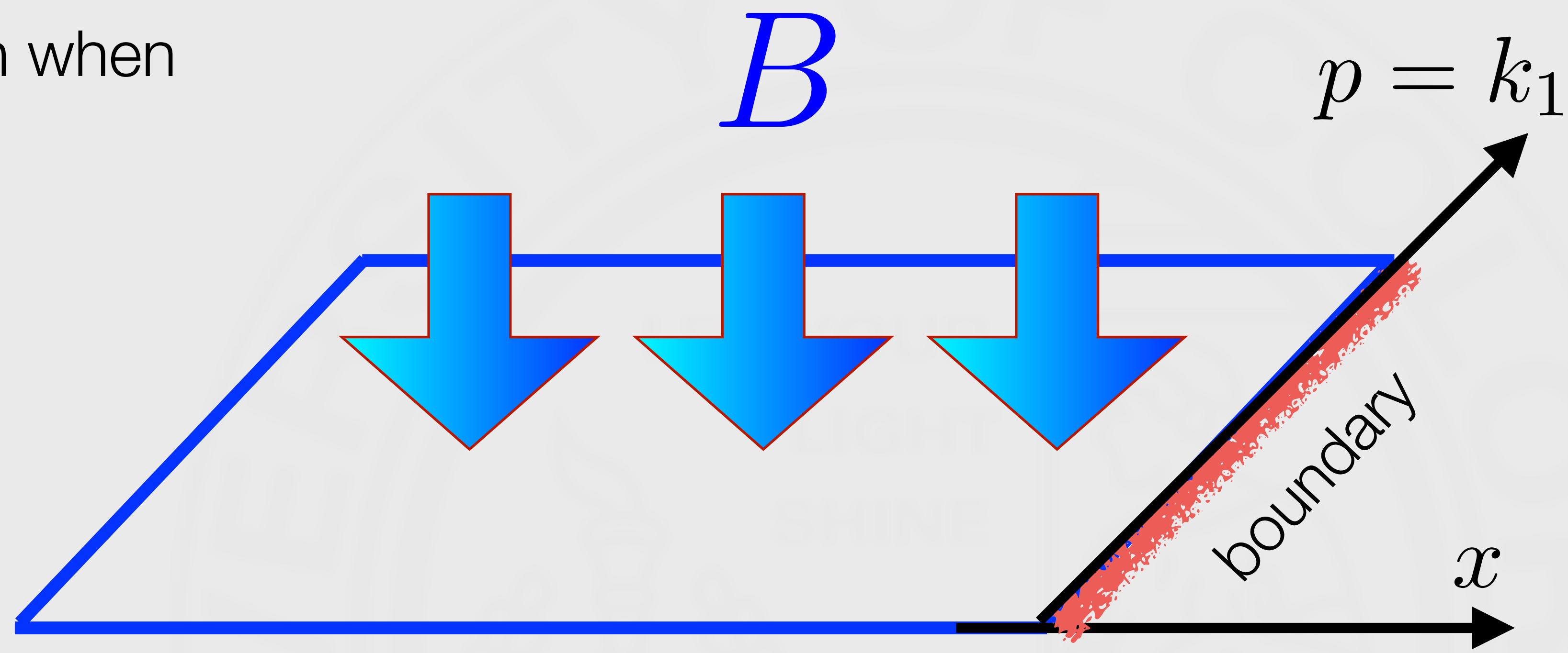
Conclusion:

$$N_{\text{edge}} = \#\,\text{chiral edge states} \qquad N_{\text{edge}} = \sum_n \oint \frac{dk^\mu}{2\pi i} \partial_\mu \ln g_n$$

Bulk-edge correspondence

We have the Green's function when boundary is present

$$G_{ab}(x, x'; \omega, p)$$



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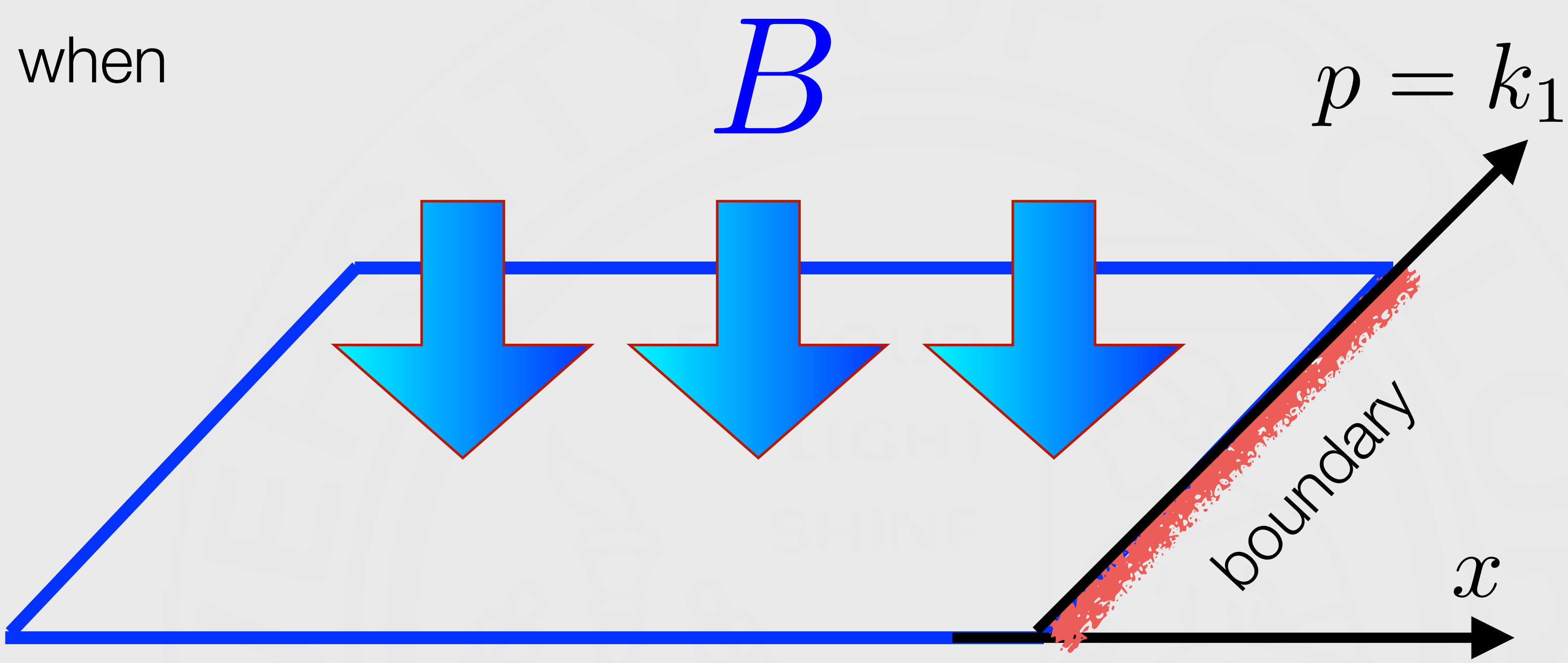

We have N_{edge}

$$N_{\text{edge}} = \sum_\mu \oint_{\mathcal{C}} \frac{dk^\mu}{2\pi i} \,\mathrm{tr} \int dx dx' \, G^{-1}(x, x') \partial_\mu G(x', x)$$

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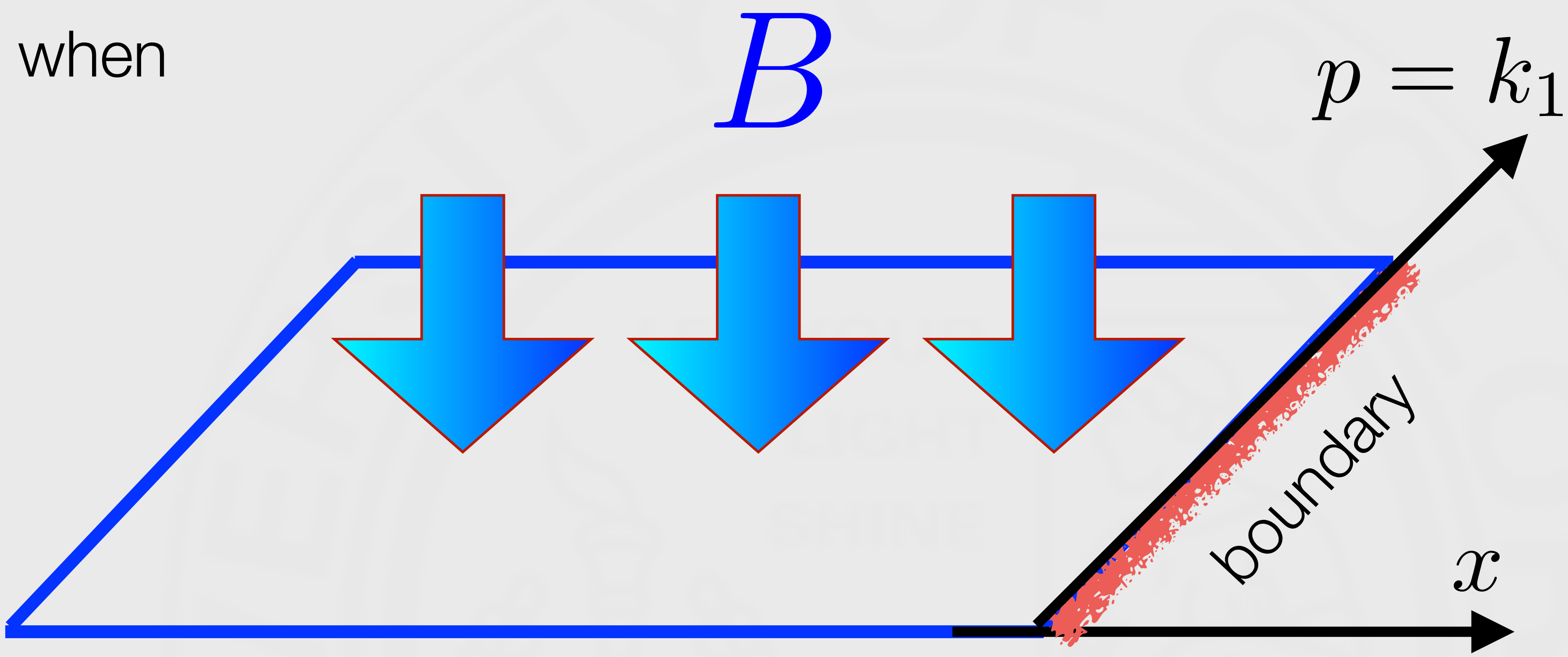

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We have the bulk Green's function far away from the edge

$$G_{\text{bulk}}(\omega, k_1, k_2) = \lim_{R \to -\infty} \int dr \, e^{-irk_2} \, G\left(R + \frac{r}{2}, R - \frac{r}{2}; \omega, k_1\right)$$

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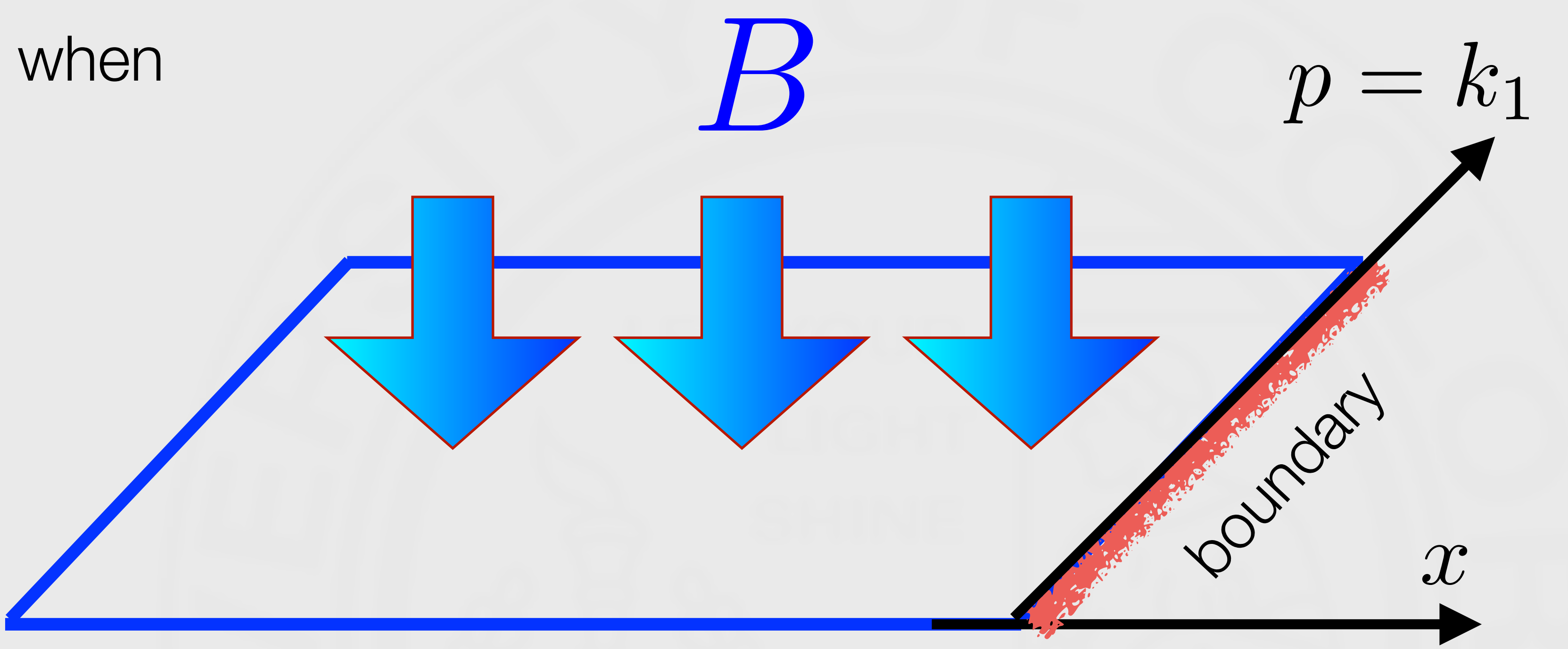

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We can construct the TKNN number with this bulk Green's function

$$N = \frac{1}{24\pi^2} \sum_{\alpha\beta\gamma} \epsilon_{\alpha\beta\gamma} \int d\omega d^2k \,\mathrm{tr}\, \left[G_{\mathrm{bulk}}^{-1} \partial_{\alpha} G_{\mathrm{bulk}} G_{\mathrm{bulk}}^{-1} \partial_{\beta} G_{\mathrm{bulk}} G_{\mathrm{bulk}}^{-1} \partial_{\gamma} G_{\mathrm{bulk}}\right]$$

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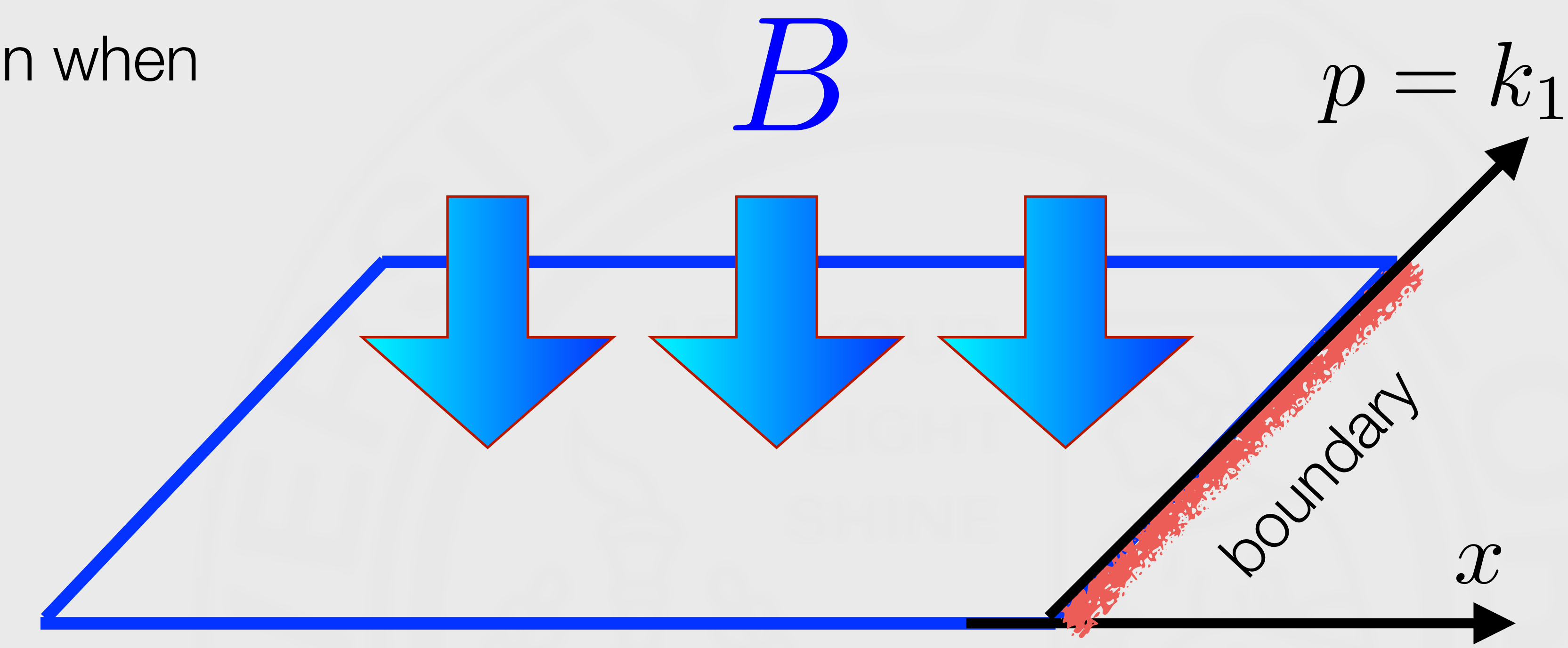

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"Theorem":

$$\boxed{N = N_{\rm edge}}$$

Implication for IQHE: Hall conductance = # of chiral edge modes

Cf: 10am talk by Hermann Schulz-Baldes

Bulk-boundary correspondence of topological invariants in 2D: FQHE

Interacting Green's functions

$$\cancel{G_{ab}(\omega, \mathbf{k}) = [i\omega - \mathcal{H}(\mathbf{k})]^{-1}_{ab}} \longrightarrow G_{ab}(\omega, \mathbf{k}) = \int d\tau e^{i\omega\tau} \left\langle \mathcal{T} \hat{a}_a(\mathbf{k}, \tau)\, \hat{a}^\dagger_b(\mathbf{k}, 0) \right\rangle$$

Idea: use interacting Green's functions to compute the same invariants and learn about the behavior of the boundary of a system.

Works very well in 1D: S. Manmana, A. Essin, R. Noack, VG (2012)

How about 2D.

Interactions and Green's functions

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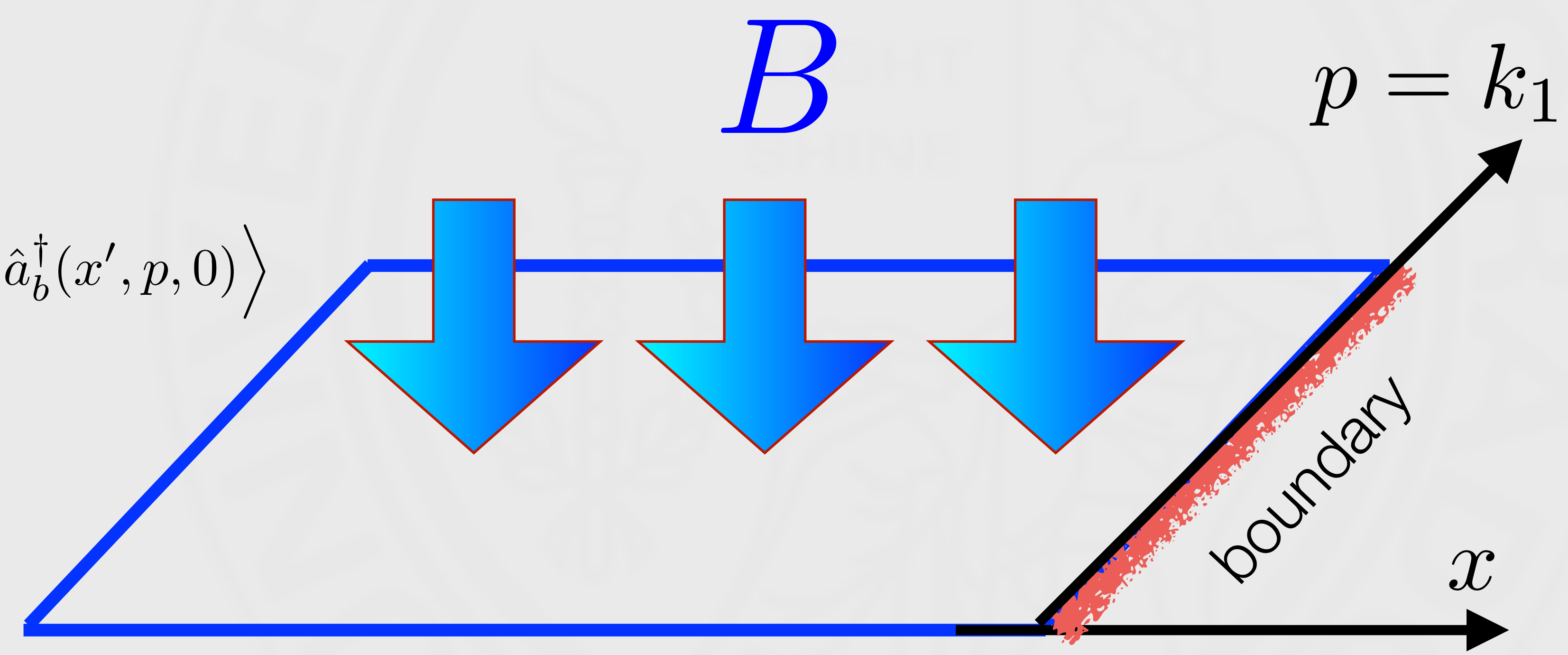
What if there's a boundary?

$$G_{ab}(x, x'; \omega, p) = \int d\tau e^{i\omega\tau} \left\langle \mathcal{T} \hat{a}_a(x, p, \tau)\, \hat{a}_b^\dagger(x', p, 0) \right\rangle$$



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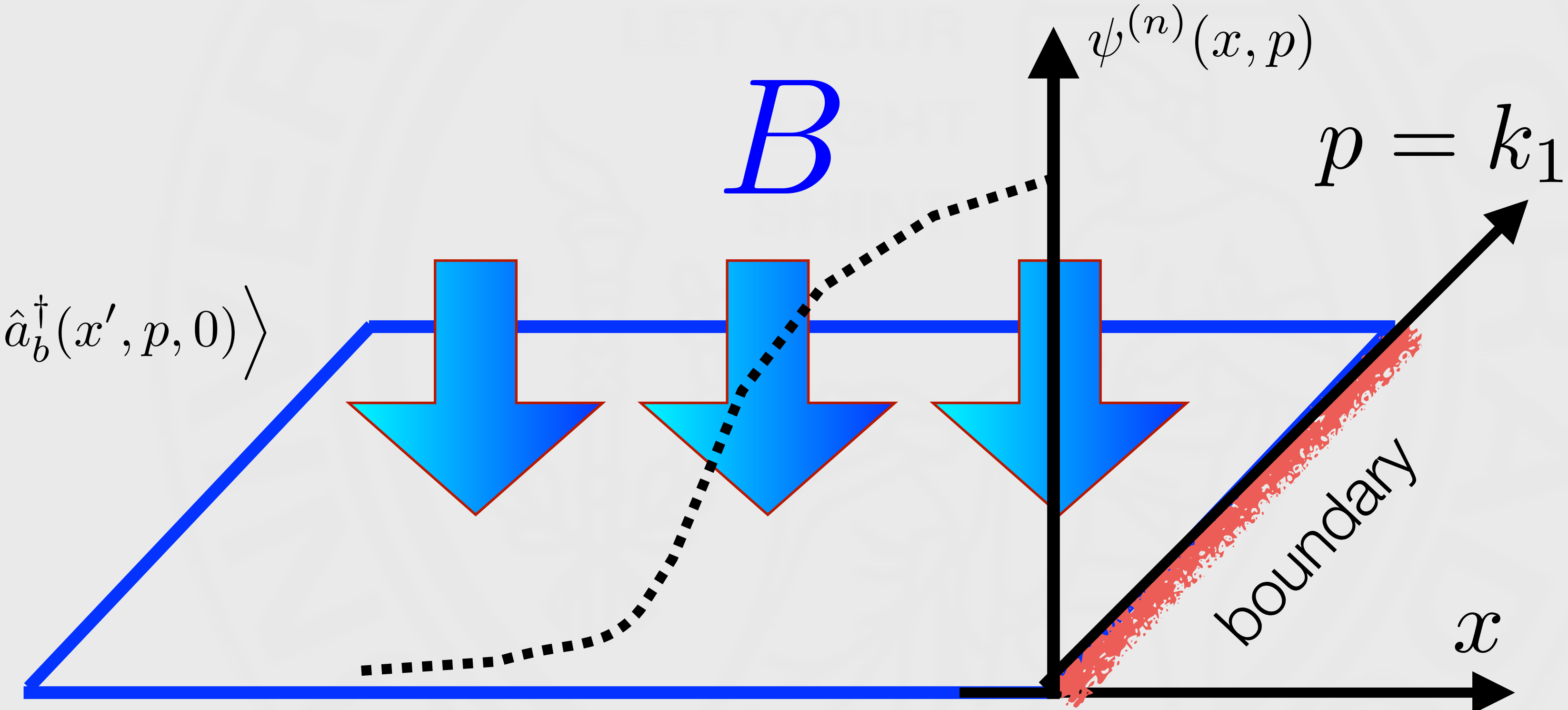

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What is the meaning of g_n?

$$\sum_{b,\,x'} G_{ab}(x,x';\omega,p)\,\psi^{(n)}_b(x',p) = g_n(\omega,p)\,\psi^{(n)}_a(x,p)$$

$$g_n(\omega,p) = \sum_{ab}\int dx dx' \int d\tau\, e^{i\omega\tau}\, \underbrace{\psi^{*(n)}_a(x,p)\left\langle \mathcal{T}\hat{a}_a(x,p,\tau)}_{\hat{a}_n(p)}\, \underbrace{\hat{a}^\dagger_b(x',p,0)\right\rangle \psi^{(n)}_b(x',p)}_{\hat{a}^\dagger_n(p)}$$

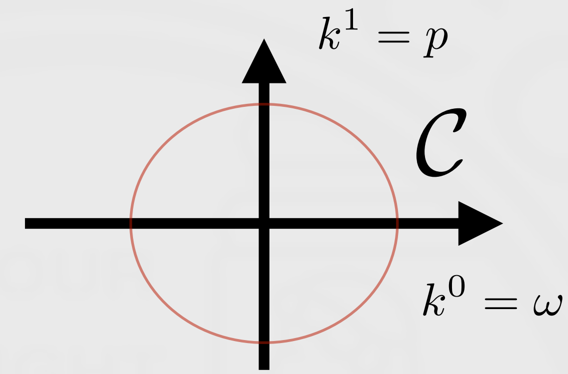
Annihilating a particle with momentum *p* along the boundary in a state *n*

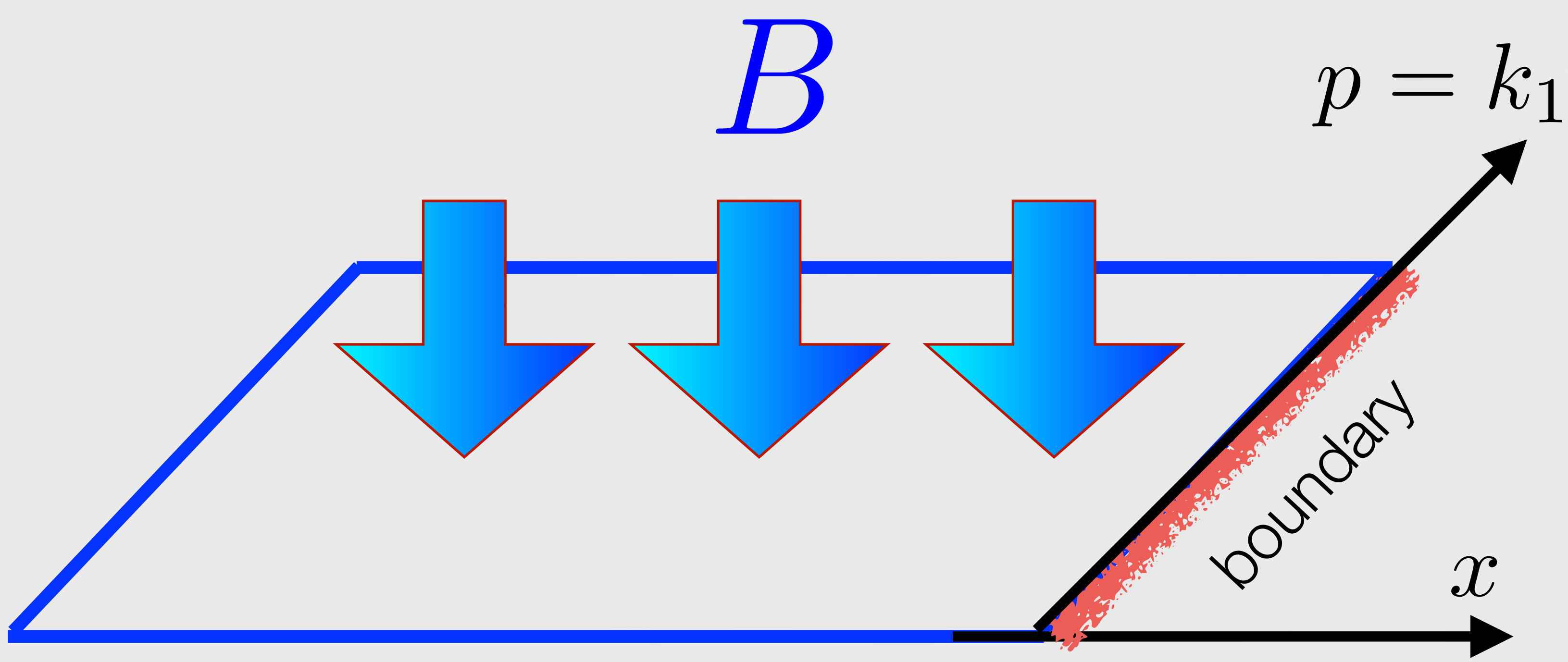
Creating a particle with momentum *p* along the boundary in a state *n*

$$g_n = \int d\tau e^{i\omega\tau} \left\langle \hat{a}_n(p,\tau)\,\hat{a}^\dagger_n(p,0)\right\rangle$$

FQHE: boundary topological invariant

$$N_{\text{edge}} = \sum_{\mu} \oint_{\mathcal{C}} \frac{dk^{\mu}}{2\pi i} \operatorname{tr} \int dx dx' \, G^{-1}(x,x') \partial_{\mu} G(x',x)$$
$$= \sum_{n} \oint \frac{dk^{\mu}}{2\pi i} \partial_{\mu} \ln g_n$$





This quantity is also a topological invariant; can't be changed by small perturbations of *G.*

At least some of g_n have got to be the boundary electron Green's functions, described by chiral Luttinger liquid theory.

<u>*Question:*</u> what is this invariant equal to for the chiral Luttinger liquid?

New types of invariants for bulk FQHE

Given bulk Green's function G for a FQHE, we can now define a new type of an invariant.
For IQHE, it's a Chern number, so it is not really new.
However, what is it for FQHE?

$$N = \frac{1}{24\pi^2} \sum_{\alpha\beta\gamma} \epsilon_{\alpha\beta\gamma} \int d\omega d^2 k \, \mathrm{tr} \, \left[G^{-1} \partial_\alpha G G^{-1} \partial_\beta G G^{-1} \partial_\gamma G \right]$$

$$\alpha, \beta, \gamma = \omega, k_1, k_2$$

By Volovik's argument, it has got to be equal to the boundary invariant computed within the chiral Luttinger liquid theory.

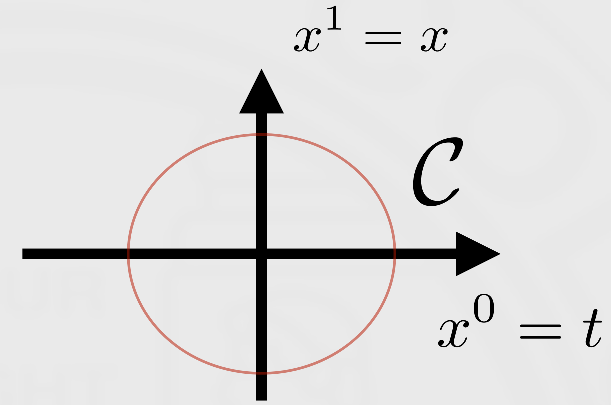
Perhaps we can compute the bulk invariant numerically within some simulation on the torus to relate its value to the known value of a proposed Luttinger liquid theory description of the boundary of the state we are attempting to study numerically.

Attempts to calculate boundary invariant for FQHE

Boundary Green's function

Among these g_n there must be one (or a few) which correspond to the electron's boundary Green's function.

In the context of FQHE, these are usually constructed via conformal field theory (especially in case of the non-Abelian FQ states - abelian ones can be found using a simpler language of bosonization).



$$\oint \frac{dx^\mu}{2\pi i} \partial_\mu \ln g = 2\Delta$$

$$x^0 = \tau, \ x^1 = x$$

$$g \sim \frac{1}{(x - iv\tau)^{2\Delta}}$$

scaling dimension of the electron operator

time-space invariant
(as opposed to
freq-momentum invariant)

<u>More generally,</u> there might be left moving and right moving branches

$$g \sim \prod_i \frac{1}{(x - iv_i\tau)^{2\Delta_i}} \prod_j \frac{1}{(x + iv_j\tau)^{2\bar{\Delta}_j}}$$

$$\oint \frac{dx^\mu}{2\pi i} \partial_\mu \ln g = 2\sum_i \Delta_i - 2\sum_j \bar{\Delta}_j$$

This is called the conformal spin of the electron operator

Conjecture: N_{edge} measures the conformal spin of the electron operator

CFT arguments: electron spin = shift

CFT description of the quantum Hall wave functions

$$\Psi(z_1, z_2, \ldots, z_M) = \langle \mathcal{O}(z_1)\mathcal{O}(z_2)\ldots\mathcal{O}(z_N)\rangle \left\langle e^{i\alpha\phi(z_1)} e^{i\alpha\phi(z_2)} \ldots e^{i\alpha\phi(z_N)} \right\rangle e^{-\frac{1}{4}\sum_i |z_i|^2}$$

$$= \mathcal{F}(z_1, z_2, \ldots, z_N) \prod_{i<j} (z_i - z_j)^{\frac{1}{\nu}} e^{-\frac{1}{4}\sum_i |z_i|^2}$$

Importantly $\mathcal{F} \sim \frac{1}{z_1^{2\Delta_{\mathcal{O}}}}, \; z_1 \to \infty$

$\mathcal{O}(z)e^{i\alpha\phi(z)}$ electron operator both at the boundary and in the bulk

If z_1 is taken around a big contour encircling all other points z_i, this accumulates a phase $2\pi \left[-2s_{\mathcal{O}} + \frac{1}{\nu}(N-1) \right]$

Conformal spin of the operator $\mathcal{O}$

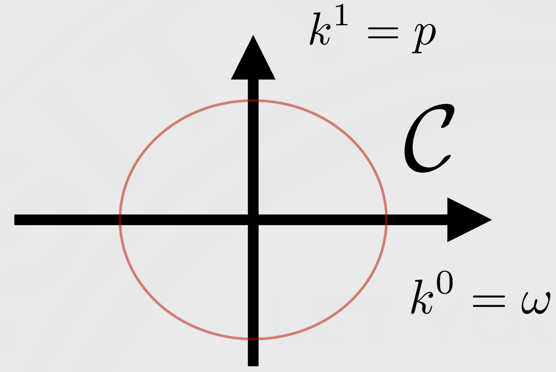
The phase measures the number of flux quanta $N_\phi = \frac{N}{\nu} - \underbrace{2s_{\mathcal{O}} - \frac{1}{\nu}}_{\mathcal{S} = 2s_{\mathcal{O}} + \frac{1}{\nu}}$

shift of the quantum Hall state

$$2s_{\mathcal{O}} + \frac{1}{\nu} =$$ twice the total spin of the electron operator = shift

Green's functions in momentum space

$$N_{\mathrm{edge}} = \sum_n \oint \frac{dk^\mu}{2\pi i} \partial_\mu \ln g_n$$



$$g \sim \frac{1}{(x - iv\tau)^{2\Delta}}$$

position space

$$g(\omega, p) = \int dx d\tau e^{ipx + i\omega\tau} \frac{1}{(x - iv\tau)^{2\Delta}}$$

strong UV divergence

How can we Fourier transform this?

Rotation-invariant cutoff!

$$g(\omega, p) = \int_{x^2 + v^2\tau^2 > a^2} dx d\tau e^{ipx + i\omega\tau} \frac{1}{(x - iv\tau)^{2\Delta}} \sim \frac{(i\omega + vp)^{2\Delta - 1}}{i\omega - vp}$$

$$N_{\mathrm{edge}} = 2s_{\mathcal{O}}$$

However: This is a Green's function which grows with ω. Cannot be fully physical because Green's functions must go as 1/ω at large ω.

Green’s functions wind at most once

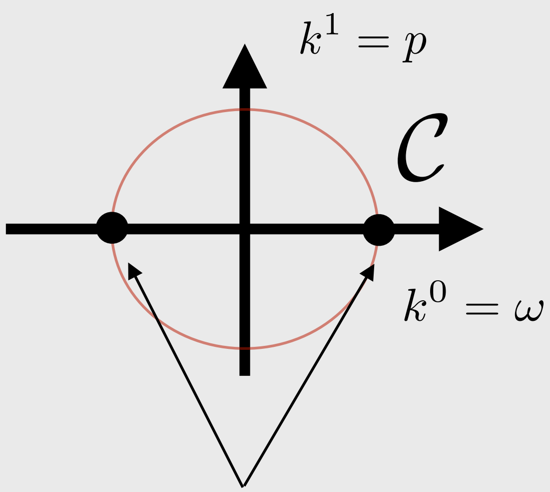
spectral decomposition:

$$G(\omega) = \sum_n \frac{\rho_n}{i\omega - \epsilon_n}$$

$$\rho_n > 0$$

$$\operatorname{Im} G(\omega) = -i\omega \sum_n \frac{\rho_n}{\omega^2 + \epsilon_n^2}$$

$$\operatorname{Im} G(\omega) = 0 \ \rightarrow \omega = 0$$



The only two points where Im G = 0.

As we go around this contour, Green’s function cannot wind more than once.

The function on the previous slide then is some kind of a regularized low-energy function.

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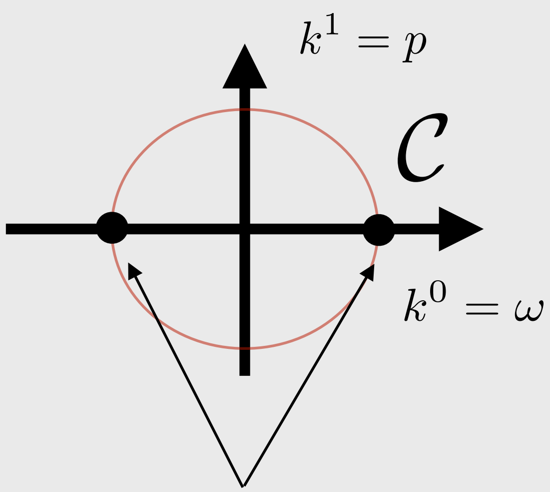

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Bosonic Green's functions

$$G(\omega) = \sum_n \frac{\rho_n}{i\omega - \epsilon_n}$$

$$\rho_n \epsilon_n > 0$$

$$\operatorname{Re} G = -\sum_n \frac{\epsilon_n \rho_n}{\omega^2 + \epsilon_n^2} < 0$$

This function does not wind at all.

Further elucidation of the bulk invariant

$$N = \frac{1}{24\pi^2} \sum_{\alpha\beta\gamma} \epsilon_{\alpha\beta\gamma} \int d\omega d^2k \, \mathrm{tr} \, \left[G^{-1}\partial_\alpha G G^{-1}\partial_\beta G G^{-1}\partial_\gamma G\right]$$

Dropping subscript *r* to avoid cluttering

Might not be easy to calculate generally; depends on the matrix structure of *G* which describes physics of higher Landau levels which should not be relevant for the FQHE state in the lowest LL.

Simplifying *N*

1. Work in basis of the wave functions $\Psi_n(k_x, k_y; x, y)$ spanning the n-th Landau level

$$\Psi_n(x+a, y) = e^{ip_x a}\Psi_n(x,y) \qquad \Psi_n(x, y+a) = e^{ip_y a + i\frac{2\pi}{a}x}\Psi_n(x,y)$$

2. In this basis we expect G to be diagonal (its eigenfunctions discussed throughout are just Ψ_n):

$$G_{00} = G_{LL}(\omega, \mathbf{k})$$

Non-trivial bulk Green's function in the lowest LL

$$G_{nn} \approx \frac{1}{i\omega - (n+1/2)\omega_0 + \mu}, \quad n = 1, 2, \ldots$$

Trivial Green's function in the higher Landau levels; ω_0 is the Larmor frequency.

3. Rewriting trace in the expression for *N* in terms of of the states Ψ_n

Requires some algebra; end result

$W =$ winding of $G_{LL}(\omega, \mathbf{k})$ as ω is taken from -∞ to ∞ in multiples of π.

N = Chern number of the lowest Landau level $\times \; (W+1)/2$

Bottom line: evaluating *W* numerically gives *N*.

Integer Hall:

$$G_{LL} = \frac{1}{i\omega - \omega_0/2 + \mu} \longrightarrow W = 1$$

Rectified Green's functions

$$g_r(\omega, p) = \int_{x^2+v^2\tau^2>a^2} dx d\tau e^{ixp+i\omega\tau} g(x,\tau) = \int dp' d\omega' K(p-p', \omega-\omega') g(p', \omega') \quad \equiv \hat{K} g$$

Imposed fairly arbitrary cutoff

$$K(p,\omega) = \delta(p)\delta(\omega) - \frac{a}{2\pi\sqrt{\omega^2+(pv)^2}} J_1\left(a\sqrt{\omega^2+(pv)^2}\right)$$

This rectified Green's function
winds appropriately

$$\sum_{b,\, x'} G_{ab}(x, x'; \omega, p)\, \psi_b^{(n)}(x', p) = g_n(\omega, p)\, \psi_a^{(n)}(x, p)$$

$$G_{r\,ab}(x, x'; \omega, p) = \psi_a^{*(n)}(x, p) \hat{K} g_n \psi_b(x, p)$$

Full rectified Green's function which
satisfies bulk-boundary correspondence

Bulk rectified Green's function $\qquad G_r(\omega, k_1, k_2) = \int d\omega' dk_1'\, K(k_1 - k_1', \omega - \omega')\, G(\omega', k_1', k_2)$

$$N = \frac{1}{24\pi^2} \sum_{\alpha\beta\gamma} \epsilon_{\alpha\beta\gamma} \int d\omega d^2k \,\mathrm{tr}\, \left[G_r^{-1}\partial_\alpha G_r G_r^{-1}\partial_\beta G_r G_r^{-1}\partial_\gamma G_r\right]$$

= conformal spin of the electron operator
= shift of the FQH state

Rectified Green's functions

$$K(p,\omega) = \delta(p)\delta(\omega) - \frac{a}{2\pi\sqrt{\omega^2+(pv)^2}} J_1\left(a\sqrt{\omega^2+(pv)^2}\right)$$

$$G_r(\omega,k_1,k_2) = \int d\omega' dk_1'\, K(k_1-k_1',\omega-\omega')\, G(\omega',k_1',k_2)$$

$$N = \frac{1}{24\pi^2}\sum_{\alpha\beta\gamma}\epsilon_{\alpha\beta\gamma}\int d\omega d^2k\,\mathrm{tr}\,\left[G_r^{-1}\partial_\alpha G_r G_r^{-1}\partial_\beta G_r G_r^{-1}\partial_\gamma G_r\right]$$

Identifying "electron operator"

Consider a general Abelian quantum Hall state described in terms of a *K*-matrix. Its boundary action is

$$S = \frac{1}{2}\sum_{IJ}\int dxdt\left[K_{IJ}\partial_t\varphi_I\partial_x\varphi_J - V_{IJ}\partial_x\varphi\partial_x\varphi\right].$$

Electron operators can be taken as: $V_I = e^{i\sum_J K_{IJ}\varphi_J}$

The total conformal spin of this operator $\mathrm{tr}\, K$

However, this theory is supposed to be invariant under

$$K \to W^T K W$$
$$W \in \mathrm{SL}(2,\mathbb{Z})$$

But **tr K** is not

More general electron operators: $V_n = e^{i\sum_{IJ} n_I K_{IJ}\varphi_J}$

But which ones to use for the Green's function?

Summary

Non-interacting systems such as IQHE: Chern number can be mapped into a boundary invariant counting chiral edge states.

Interacting systems such as FQHE: a bulk and an edge invariants can be defined, generalizing the Chern-number and the IQHE boundary invariant.

However, computing it for interacting boundary described by a Luttinger liquid theory remains a challenge.

The end