Emergent dynamical metric of the FQHE

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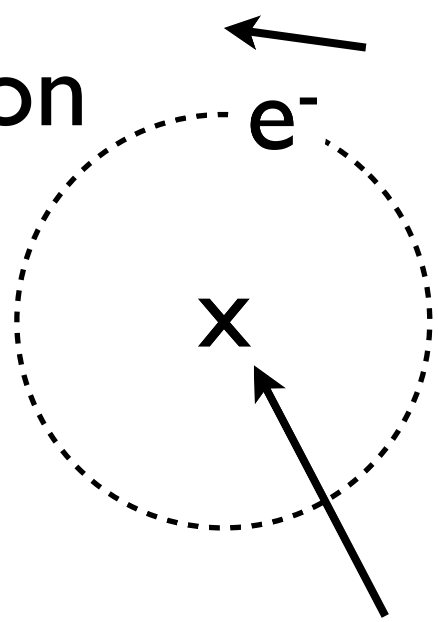
- An effective field theory for the Fractional quantum Hall effect, as a dynamical emergent spatial metric describing flux attachment
- Quantum geometry and analogies to gravity

- electron in 2D Landau orbit (bound to 2D surface)

Becomes a "fuzzy object" after kinetic energy is quantized

cyclotron orbit



guiding center R

magnetic flux density B normal to 2D surface

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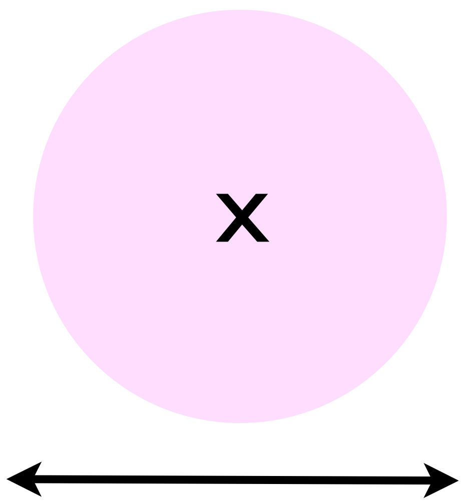
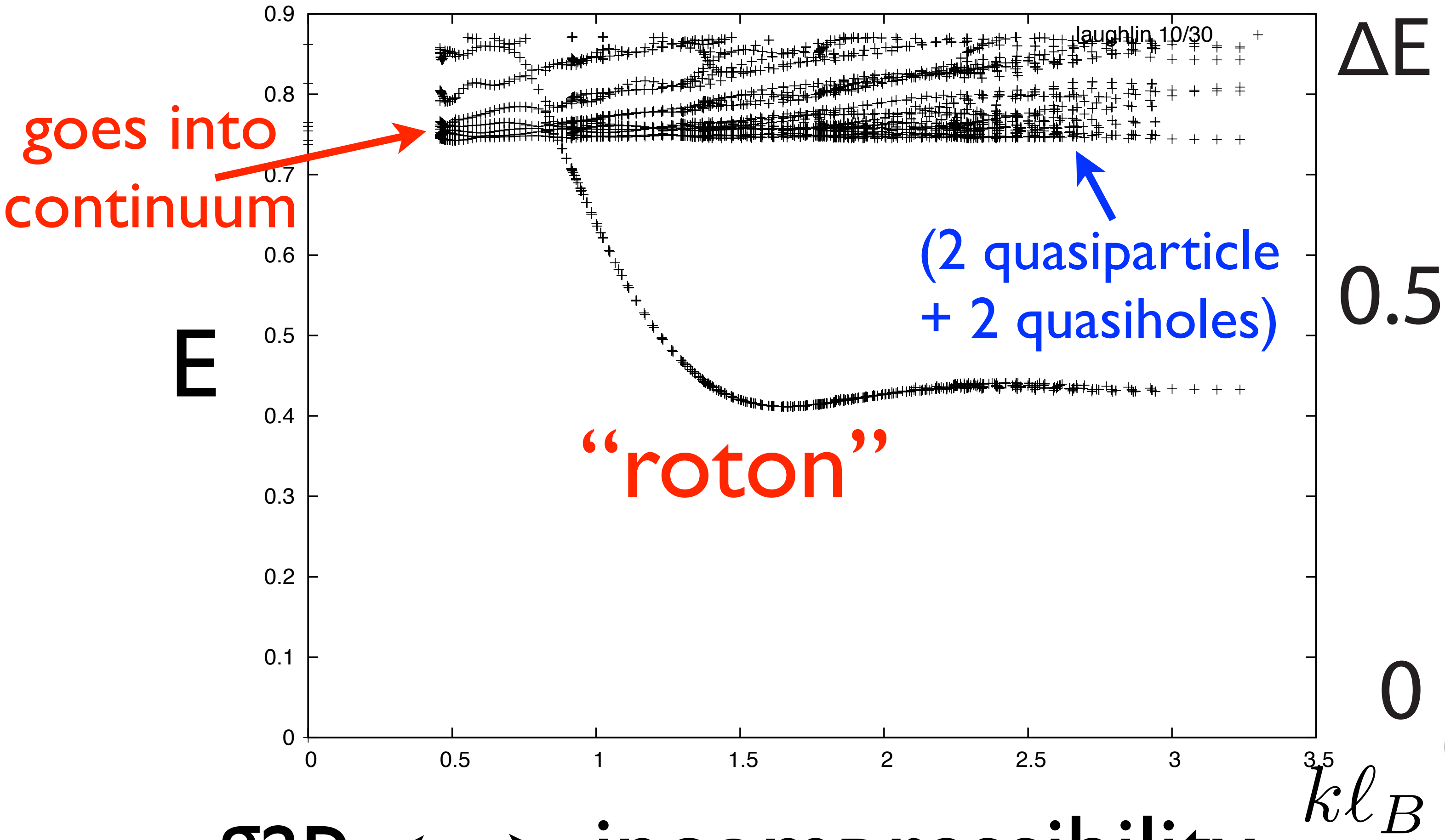
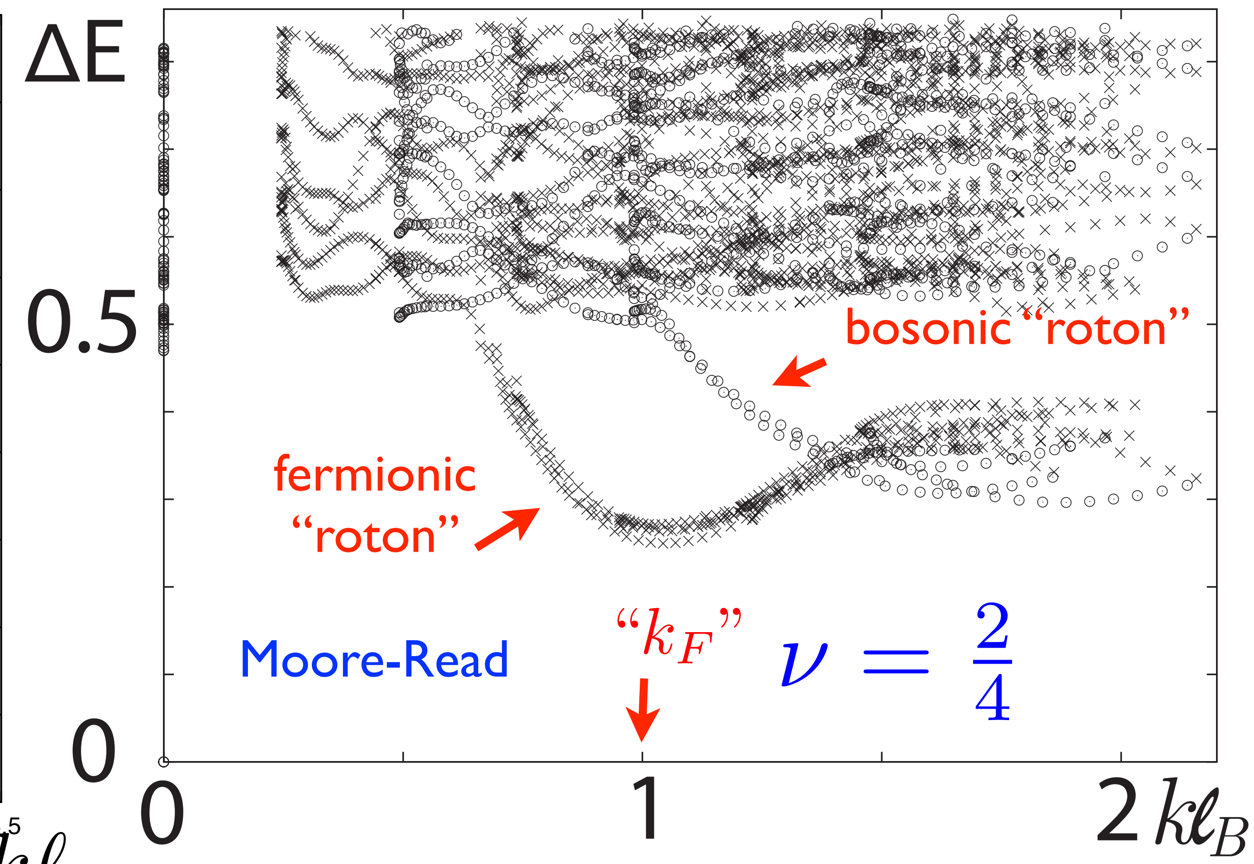


$$\ell_B = \left(\frac{\hbar}{|eB|}\right)^{\frac{1}{2}}$$

$$\boxed{[R^x, R^y] = -i\ell_B^2}$$

non-commutative geometry





gap $\longleftrightarrow$ incompressibility

Collective mode with short-range V_1 pseudopotential, 1/3 filling (Laughlin state is exact ground state in that case)

Collective mode with short-range three-body pseudopotential, 1/2 filling (Moore-Read state is exact ground state in that case)

- momentum $\hbar$k of a quasiparticle-quasihole pair is proportional to its **electric dipole moment $\mathbf{p_e}$** $\quad \hbar k_a = \epsilon_{ab} B p_e^b$

gap for electric dipole excitations is a MUCH stronger condition than charge gap: **doesn't transmit pressure!**

(origin of Virasoro algebra in FQHE ?)

Laughlin's model wavefunction has provided the inspiration for the modern understanding of the fractional quantum Hall effect

$$\Psi \propto \prod_{i<j}(z_i - z_j)^m \prod_i e^{-\frac{1}{4}z_i^* z_i/\ell_B^2}$$

- It has a striking holomorphic form that is generally attributed to "Lowest Landau Level physics"
- It has a natural interpretation in terms of "flux attachment"
- It involves a "complex structure" $z = x+iy$ that defines a unimodular metric on a Riemann surface
- It has the rotational symmetry of this metric, and has been recognized to be mathematically equivalent to a "conformal block" of a 2D conformal field theory

I will give a somewhat heretical reinterpretation of the Laughlin state

- Despite what Laughlin told us, its holomorphic structure has nothing to do with the electrons being in the "Lowest Landau Level"
- It should not be regarded as a "wavefunction", but as a Heisenberg state of guiding centers, which obey a "quantum geometry"
- It was proposed as a "trial wavefunction" with no apparent variational parameter: it does in fact have such a parameter: its metric.

- Perhaps one of the most surprising (and very fruitful) aspects of the Laughlin state is its connection to conformal field theory.
- Its "conformal block" property was noticed as an empirical observation, but has never really been explained.
- Incompressible (bulk) FQHE states are **essentially unlike** gapless cft's (the conformal group here is the "(2+0)d" conformal orthogonal group, not the "(1+1)d" Lorentz variant)

- The conformal orthogonal group CO(2) is a profound local extension of the global SO(2) rotation group (that can be regarded as “the rotation group on steroids” !)

- non-generic “Toy models” with CFT properties are particularly simple to treat, because the CFT makes their generic topological properties easy to expose, but the topological properties do not require conformal invariance

- I will argue that SO(2) rotational invariance is a “toy model” feature that should not be part of a fundamental theory of the FQHE, just as the SO(3) and Galilean invariance of the free electron gas should not be part of the theory of metals.

- The "standard model" for the QHE is usually taken to be the Galileian-invariant Newtonian-dynamics model

$$p_a = -i\hbar\frac{\partial}{\partial x^a} - eA_a(\boldsymbol{x})$$

$$H = \sum_i \frac{1}{2m}\delta^{ab}p_{ia}p_{ib} + \sum_{i<j}\frac{e^2}{4\pi\epsilon_0\epsilon}\frac{1}{d(\boldsymbol{x}_i,\boldsymbol{x}_j)}$$

$$d(\boldsymbol{x}_1,\boldsymbol{x}_2)^2 = \delta_{ab}(x_1^a - x_2^a)(x_1^b - x_2^b)$$

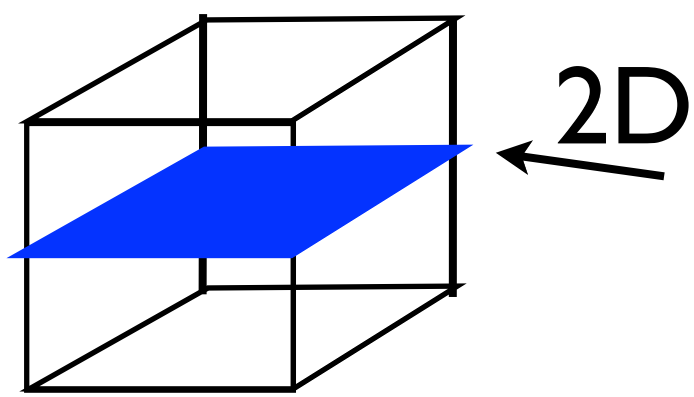
Euclidean metric of 2D plane

(derived from the spatial metric of an inertial frame in which the plane on which the electrons move non-relativistically is embedded)

Cartesian coordinates $\boldsymbol{x} = x^a\boldsymbol{e}_a \quad \boldsymbol{e}_a \cdot \boldsymbol{e}_b = \delta_{ab}$

- However, the continuous translational symmetry "plane" on which the electrons move is an emergent symmetry of a low-density of electrons moving on a crystal lattice plane, and generically does NOT have the rotational invariance of Newtonian dynamics

- The only generic point symmetry of a crystal plane is 2D inversion (180° rotation in plane)



2D plane of epitaxial quantum well embedded in 3D crystal

- The effective continuum Hamiltonian is

$$H = \sum_i \varepsilon(\boldsymbol{p}_i) + \sum_{i<j} V(\boldsymbol{x}_i - \boldsymbol{x}_j)$$

- The model has 2D inversion symmetry if

$$\varepsilon(\boldsymbol{p}) = \varepsilon(-\boldsymbol{p})$$

- The only role played by the Euclidean metric of the inertial background frame is the non-relativistic criterion

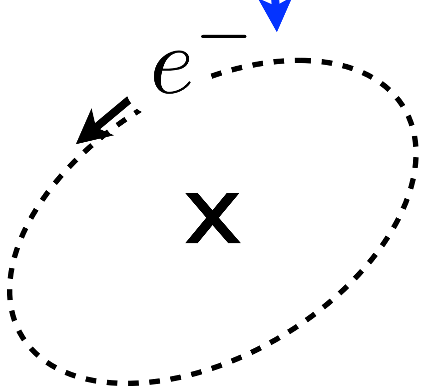
$$\delta_{ab} v^a v^b \ll c^2 \qquad\qquad v^a(\boldsymbol{p}) = \frac{\partial \varepsilon}{\partial p_a}$$

Generic model with translation and inversion symmetry only, no rotational symmetry

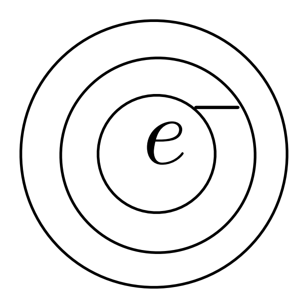
$$H = \sum_i \varepsilon(\boldsymbol{p}_i) + \sum_{i<j} V(\boldsymbol{x}_i - \boldsymbol{x}_j) \qquad \varepsilon(\boldsymbol{p}) = \varepsilon(-\boldsymbol{p})$$

affected by elastic degrees of freedom

- two distinct unrelated sources of geometry



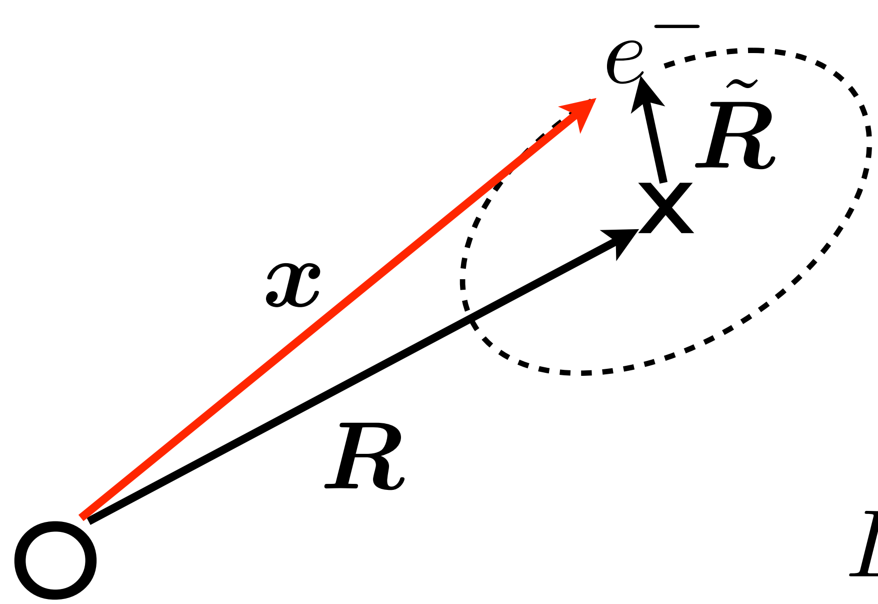
shape of Landau orbit around guiding center



equipotentials around point charge (from 3D dielectric tensor)

- The "holomorphic lowest Landau level wavefunction" is a property of a SO(2) rotationally-invariant system:

$$\boldsymbol{x} = \boldsymbol{R} + \tilde{\boldsymbol{R}}$$



$$[R^a, R^b] = -i\ell_B^2 \epsilon^{ab}$$
$$[\tilde{R}^a, \tilde{R}^b] = i\ell_B^2 \epsilon^{ab}$$
$$[R^a, \tilde{R}^b] = 0$$

angular momentum

$$L = \frac{\hbar}{2\ell_B^2}\delta_{ab}(R^a R^b - \tilde{R}^a \tilde{R}^b)$$
$$= \tfrac{1}{2}\hbar\left(a^\dagger a - b^\dagger b\right)$$

guiding center   Landau level

Two sets of ladder operators:

- Now write the Laughlin state as a Heisenberg state, not a Schrödinger wavefunction:

$$|\Psi_L\rangle \propto \prod_{i<j}(a_i^\dagger - a_j^\dagger)^m|0\rangle \quad a_i|0\rangle = 0 \qquad a^\dagger = \frac{R^x + iR^y}{\sqrt{2}\ell_B}$$

$\boxed{b_i|0\rangle = 0 \quad \text{lowest Landau level condition}}$

In the Heisenberg form, we see that the LLL condition is quite incidental to the Laughlin state, which involves guiding-center correlations

- The fundamental form of the Laughlin state does not reference the details of the Landau level in any way:

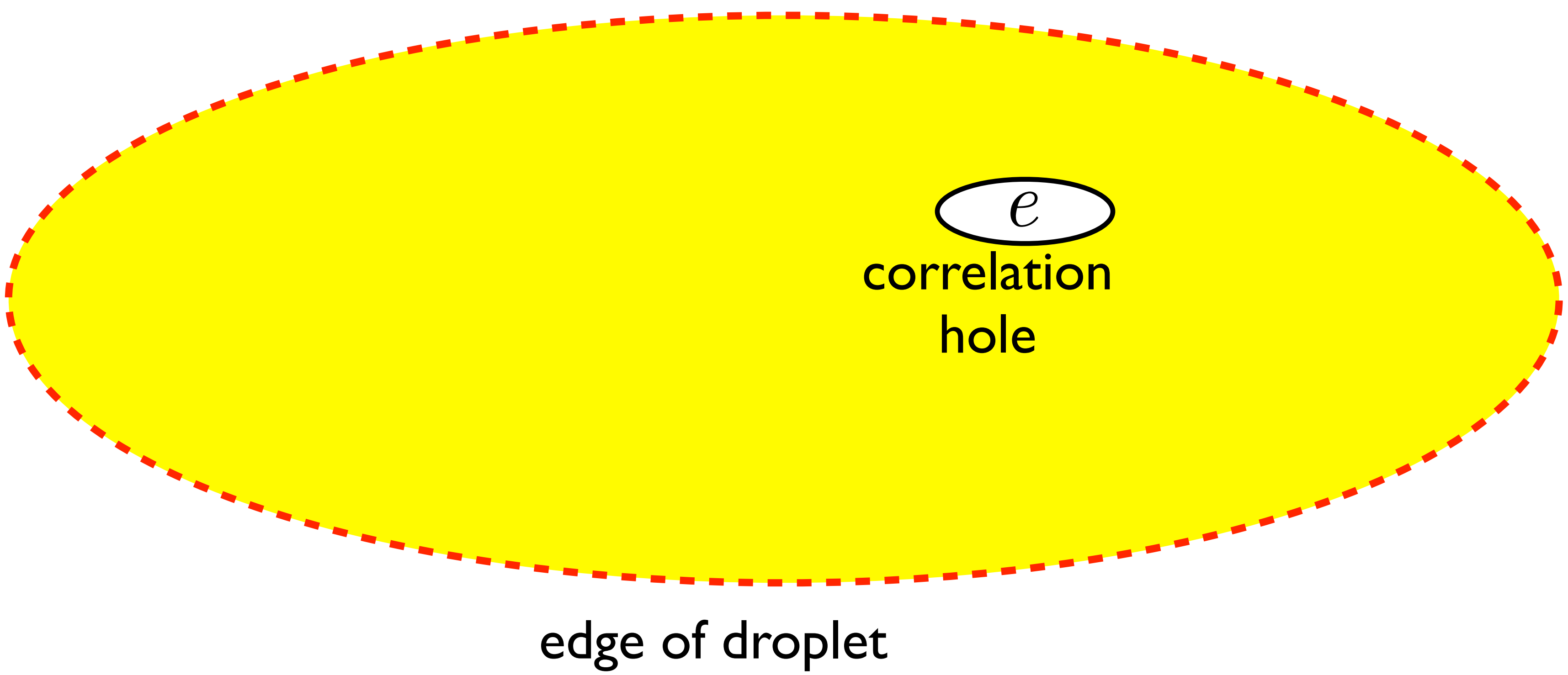
$$|\Psi_L(\tilde{g})\rangle \propto \prod_{i<j}(a_i^\dagger - a_j^\dagger)^m|0\rangle \quad a_i|0\rangle = 0 \qquad a^\dagger = \frac{\omega_a R^a}{\sqrt{2}\ell_B}$$

$$\omega_a^*\omega_b = \tfrac{1}{2}\left(\tilde{g}_{ab} + i\epsilon_{ab}\right) \qquad \det\tilde{g} = 1$$

a unimodular Euclidean-signature metric that <u>parameterizes</u> the Laughlin state

- The historical identification of this metric with the Euclidean metric is unnecessary unless there is SO(2) symmetry.

- The original form of the Laughlin state is a finite-size droplet of N particles on the infinite plane.
- Somewhat confusingly, in this droplet state the metric parameter fixes both the shape of the droplet state **<u>and</u>** the shape of the correlation hole around each particle formed by "flux attachment":



- to remove the edge, compactify on the torus with N_Φ flux quanta:
- An unnormalized holomorphic single-particle state has the form

$$|\psi\rangle = \prod_{i=1}^{N_\Phi} \sigma(a^\dagger - w_i)|0\rangle, \qquad \sum_{i=1}^{N_\Phi} w_i = 0$$

Weierstrass sigma function

$$\sigma(z) = z \prod_{L\neq 0} (1 - \tfrac{z}{L}) \exp(\tfrac{z}{L} + \tfrac{1}{2}(\tfrac{z}{L})^2)$$

Filled Landau level $N = N_\Phi$

$$|\Psi_{\text{filledLL}}\rangle = \sigma(\textstyle\sum_i a_i^\dagger) \displaystyle\prod_{i<j} \sigma(a_i^\dagger - a_j^\dagger)|0\rangle$$

independent of choice of metric, after normalization

- Laughlin state on torus $(\nu = 1/m, \quad m > 1)$

$$|\Psi_L^m(\tilde{g})\rangle \propto \left(\prod_{j=1}^{m} \sigma(\textstyle\sum_i a_i^\dagger - w_j)\right) \prod_{i<j} \sigma(a_i^\dagger - a_j^\dagger)^m |0\rangle$$

Topological degeneracy parametrized by w_j $\quad \sum_{j=1}^{m} w_j = 0$

- Unlike the filled LL state, the Laughlin state does depend on the metric, which characterizes the shape of the correlation hole (flux attachment).

- The Laughlin state is indeed a variational trial state, we must choose its metric to minimize the correlation energy

$$H = \frac{1}{N_\Phi}\sum_{\boldsymbol{q}}\left(\frac{\tilde{V}(\boldsymbol{q})|f_n(\boldsymbol{q})|^2}{2\pi\ell_B^2}\right)\sum_{i<j} e^{i\boldsymbol{q}\cdot(\boldsymbol{R}_i-\boldsymbol{R}_j)}$$

Fourier transform of interaction

Landau-level form-factor

reciprocal vector compatible with pbc

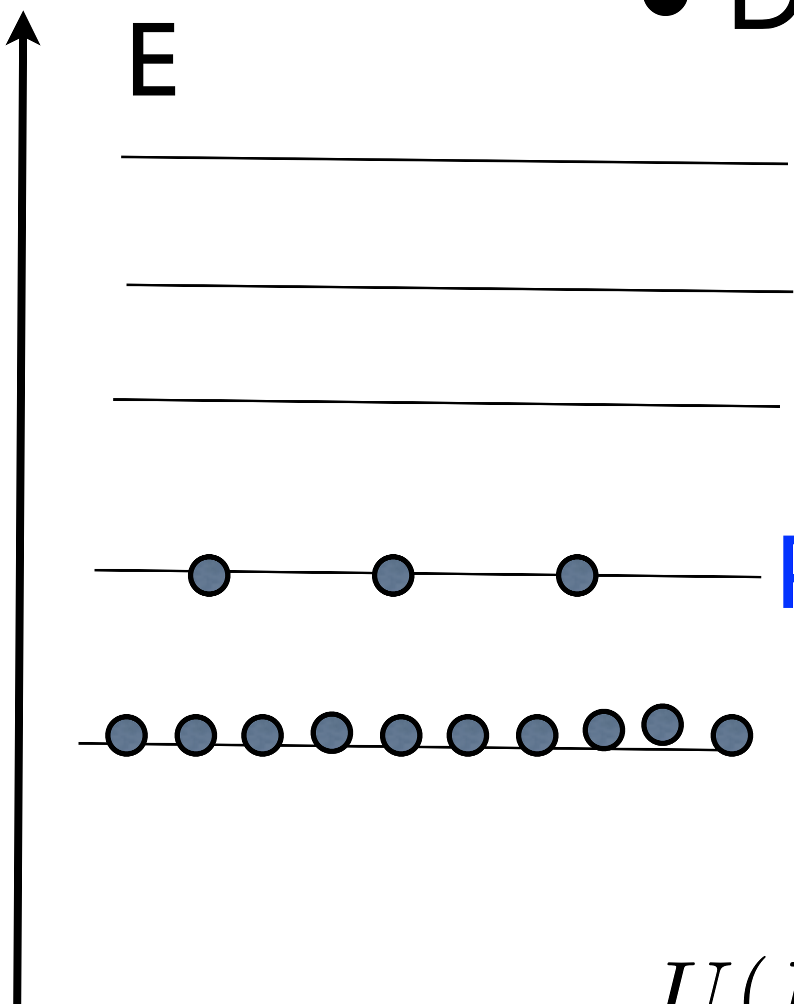
- Note that the residual two-body interaction between guiding centers always has 2D inversion symmetry.

- The Laughlin states are also the exact zero-energy ground states of the metric-dependent "pseudopotential" interaction

$$H(\tilde{g}) = \frac{1}{N_\Phi} \sum_{\boldsymbol{q}} \left(\sum_{m'<m} V_{m'} L_{m'}(q^2 \ell_B^2) e^{-\frac{1}{2} q^2 \ell_B^2} \right) \sum_{i<j} e^{i\boldsymbol{q}\cdot(\boldsymbol{R}_i - \boldsymbol{R}_j)}$$

$$q^2 \equiv \tilde{g}^{ab} q_a q_b$$

- Degenerate (flat) Landau levels



empty

partially occupied

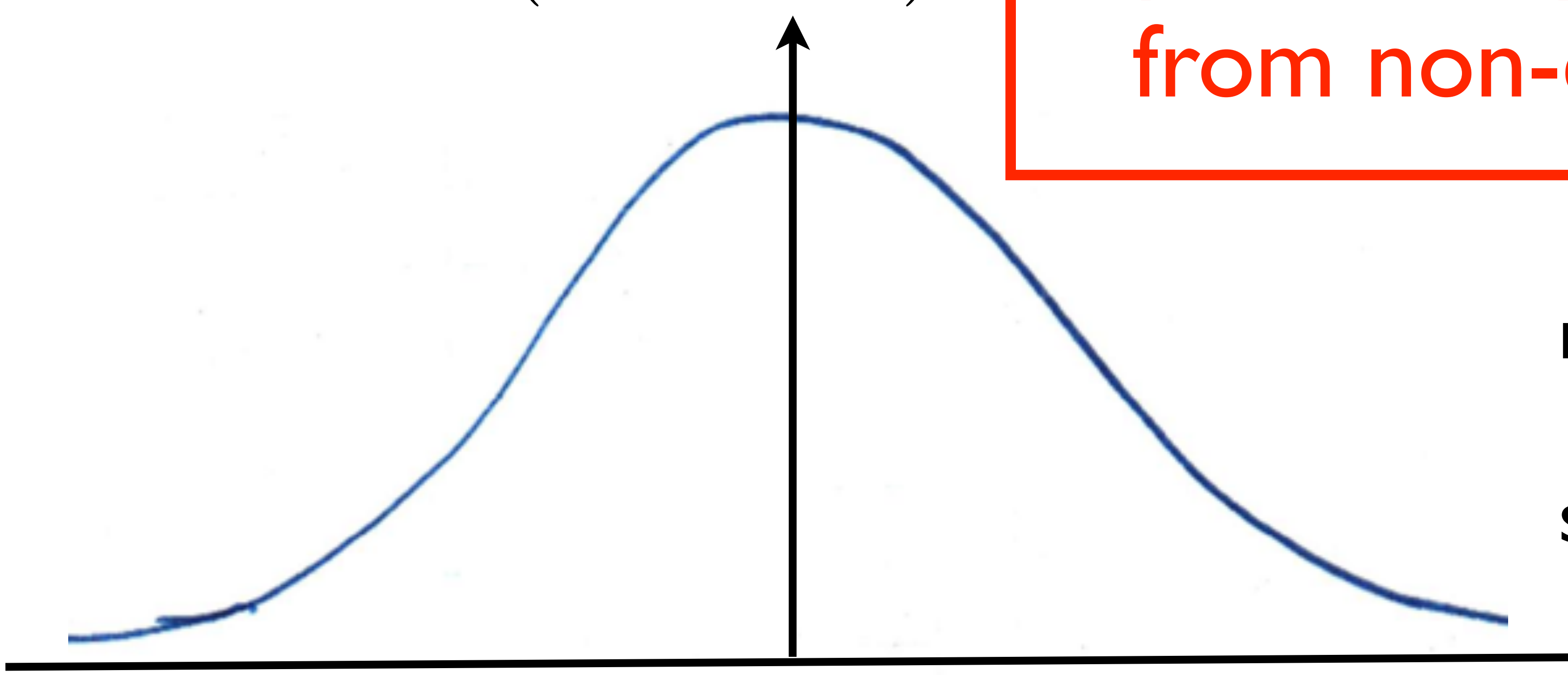
filled

$$H = \sum_{i<j} U(\boldsymbol{R}_i - \boldsymbol{R}_j)$$

$$[R^x, R^y] = -i\ell_B^2$$

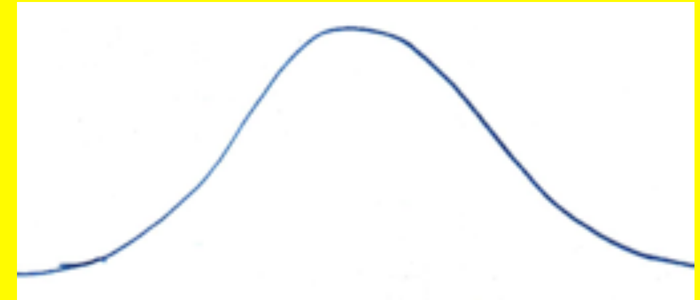
quantum dynamics comes from non-commutativity

$U(\boldsymbol{R}_1 - \boldsymbol{R}_2)$



effective Coulomb repulsion is analytic at origin because of smoothing by Landau-orbit form factor

This is the **entire** problem: nothing other than this matters!



$$H = \sum_{i<j} U(\boldsymbol{R}_i - \boldsymbol{R}_j)$$

$$[R^x, R^y] = -i\ell_B^2$$

like phase-space, has Heisenberg uncertainty principle

- H has translation and inversion symmetry

$$[(R_1^x + R_2^x), (R_1^y - R_2^y)] = 0$$

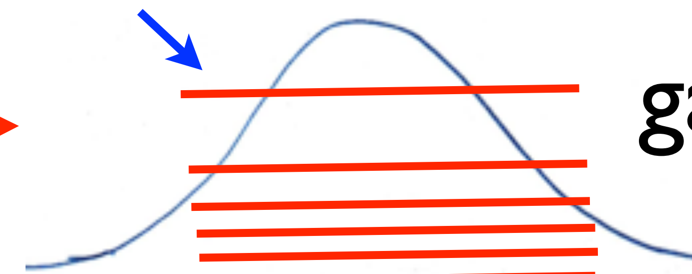
$$[H, \sum_i \boldsymbol{R}_i] = 0$$

- generator of translations and electric dipole moment!

$$[(R_1^x - R_2^x), (R_1^y - R_2^y)] = -2i\ell_B^2$$

- relative coordinate of a pair of particles behaves like a single particle

want to avoid this state

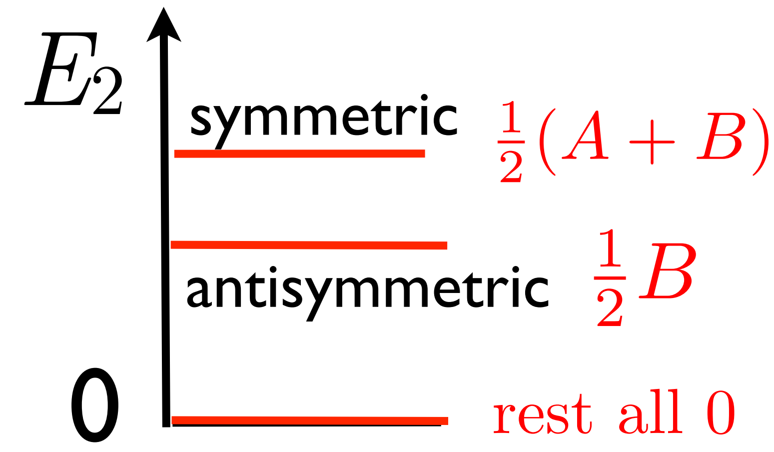


gap

two-particle energy levels

- Solvable model! ("short-range pseudopotential")

$$U(r_{12}) = \left(A + B\left(\frac{(r_{12})^2}{\ell_B^2}\right)\right) e^{-\frac{(r_{12})^2}{2\ell_B^2}}$$



- Laughlin state

$$|\Psi_L^m\rangle = \prod_{i<j}\left(a_i^\dagger - a_j^\dagger\right)^m |0\rangle$$

$$a_i|0\rangle = 0 \qquad a_i^\dagger = \frac{R^x + iR^y}{\sqrt{2}\ell_B}$$

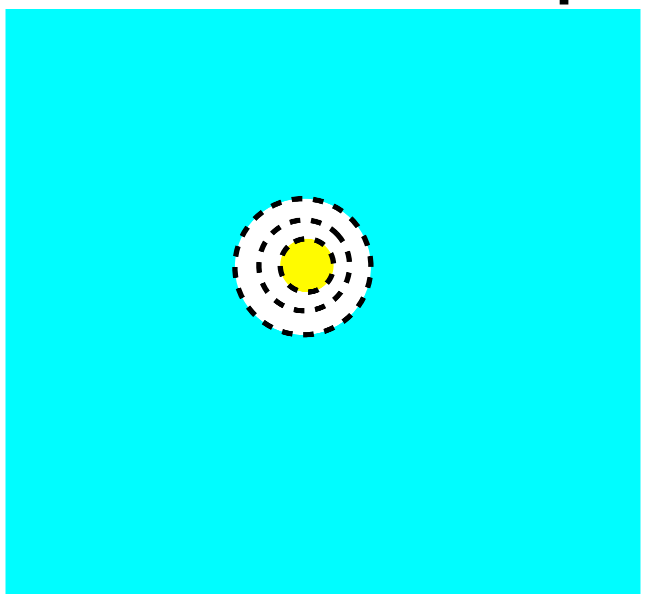
$$E_L = 0 \qquad [a_i, a_j^\dagger] = \delta_{ij}$$

maximum density null state

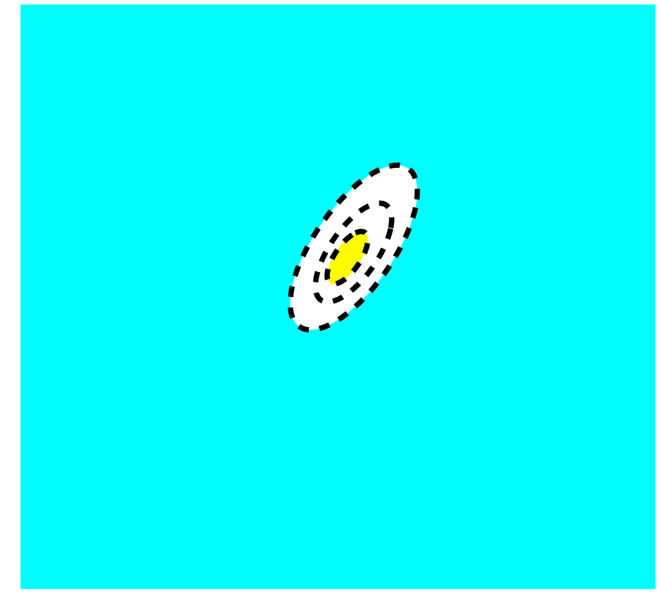
- m=2: (bosons): all pairs avoid the symmetric state $E_2 = \frac{1}{2}(A+B)$
- m=3: (fermions): all pairs avoid the antisymmetric state $E_2 = \frac{1}{2}B$

- the essential unit of the 1/3 Laughlin state is the electron bound to a correlation hole corresponding to "units of flux", or three of the available single-particle states which are exclusively occupied by the particle to which they are "attached"
- In general, the elementary unit of the FQHE fluid is a "composite boson" of p particles with q "attached flux quanta"
- This is the analog of a unit cell in a solid....

- The Laughlin state is parametrized by a unimodular metric: what is its physical meaning?



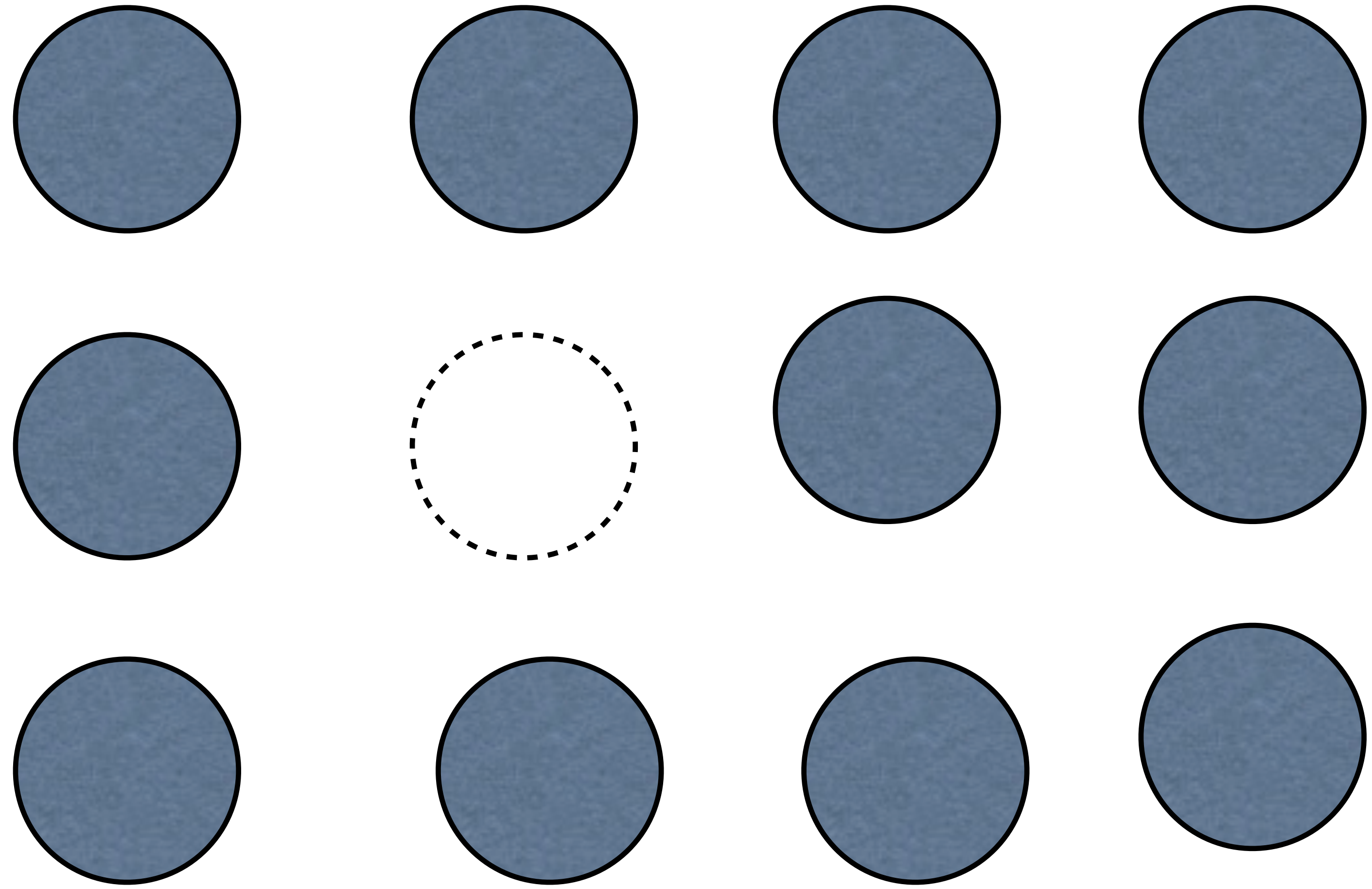
correlation holes in two states with different metrics



- In the $\nu = 1/3$ Laughlin state, each electron sits in a correlation hole with an area containing 3 flux quanta. The metric controls the *shape* of the correlation hole.

- In the $\nu = 1$ filled LL Slater-determinant state, there is <u>no</u> correlation hole (just an exchange hole), and this state does <u>**not**</u> depend on a metric

- quantum solid
- unit cell is correlation hole
- defines geometry



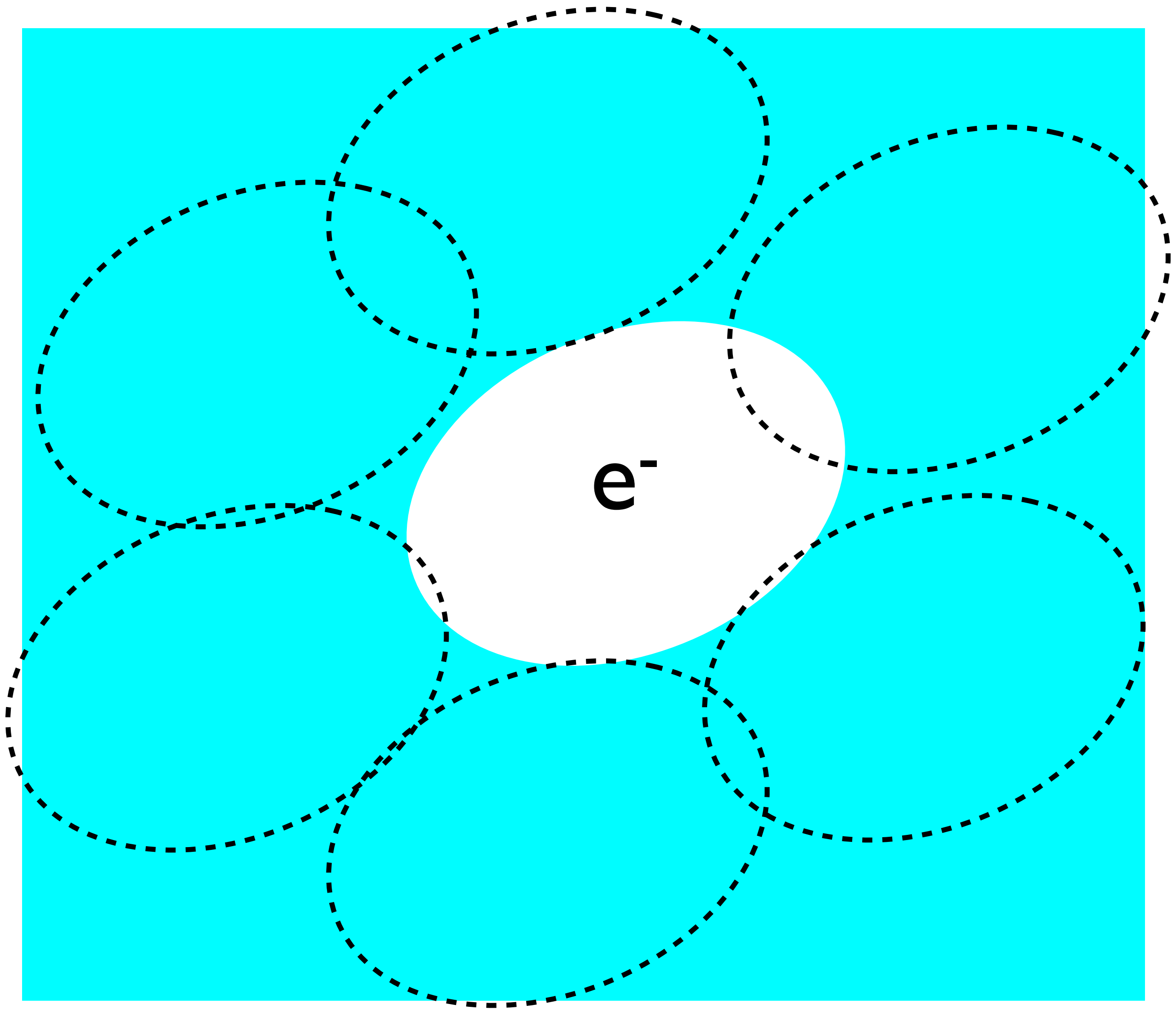
- repulsion of other particles make an attractive

potential well strong enough to bind particle

solid melts if well is not strong enough to contain zero-point motion (Helium liquids)

but no broken symmetry

- similar story in FQHE:



- "flux attachment" creates correlation hole
- defines an emergent geometry
- potential well must be strong enough to bind electron
- continuum model, but similar physics to Hubbard model
- new physics: Hall viscosity, geometry...........

- The composite boson fluid covers the plane, and provide an intrinsic dimensionless spatial distance measure on the plane, analogous to measuring distances in lattice units in the solid.
- The effective field theory should only involve a connection compatible with the intrinsic spatial metric, not the connection compatible with the Euclidean metric.

- space-time connection compatible with a time-dependent intrinsic spatial metric $g_{ab}(\boldsymbol{x}, t)$

$$\nabla_\mu f_a = \partial_\mu f_a - \Gamma^b_{\mu a} f_b$$

$$\Gamma^a_{\mu b} = \tfrac{1}{2} g^{ac} \left(\partial_\mu g_{bc} + \delta^d_\mu \left(\partial_b g_{cd} - \partial_c g_{bd}\right)\right)$$

- unusual feature, connection 1-form carries only spatial indices $\Gamma^a{}_b = \Gamma^a_{\mu b} dx^\mu$

- Geometric Chern-Simons 3-form is analog of gravitational CS form, but trace is over spatial indices

$$\Gamma^a{}_b \wedge d\Gamma^b{}_a + \tfrac{2}{3}\Gamma^a{}_b \wedge \Gamma^b{}_c \wedge \Gamma^c{}_a \qquad = 2\omega \wedge d\omega$$

spin connection

- conserved Gaussian curvature current of intrinsic metric:

$$g_{ab} = \sqrt{g}\,\tilde{g}_{ab}$$

unimodular part

$$\boxed{\begin{aligned} J^\mu_g = {} & \tfrac{1}{2}\left(\delta^\mu_a \partial_t - \delta^\mu_0 \partial_a\right)\left(\partial_b \tilde{g}^{ab} + \tilde{g}^{ab}\partial_b \ln\sqrt{g}\right) \\ & + \tfrac{1}{8}\epsilon^{\mu\nu\lambda}\epsilon_{ac}\tilde{g}_{bd}\left(\partial_\nu \tilde{g}^{ab}\right)\left(\partial_\lambda \tilde{g}^{cd}\right) \qquad \text{(Brioschi formula)} \end{aligned}}$$

$$\partial_\mu J^\mu_g = 0$$

- <u>any</u> non-singular time-dependent symmetric spatial tensor field can define a conserved Gaussian curvature current

- three dynamical ingredients $g_{ab},\ v^a, P^a$:
  - a "dynamic emergent 2D spatial metric" $g_{ab}(\boldsymbol{x}, t)$ with $g \equiv \det\ g$, and Gaussian curvature current $J^\mu_g = \epsilon^{\mu\nu\lambda}\partial_\nu\omega_\lambda(\boldsymbol{x}, t)$
  - a flow velocity field $v^a(\boldsymbol{x}, t)$
  - an electric polarization field $P^a(\boldsymbol{x}, t)$
  - a composite boson current $J^\mu_b = \sqrt{g}(\boldsymbol{x}, t)\,(\delta^\mu_0 + v^a(\boldsymbol{x}, t)\delta^\mu_a)$

here a is a 2D spatial index, and μ is a (2+1D) space-time index. The fluid motion is non-relativistic relative to the preferred inertial rest frame of the crystal background

- effective bulk action: $\sigma_H = \dfrac{(pe)^2}{2\pi\hbar K}$

$U(1)$ Chern-Simons field

$U(1)$ condensate field

"spin connection" of the metric

$$S = \int d^2x dt\, \mathcal{L}_0 - \mathcal{H}$$

$$\mathcal{L}_0 = \frac{\hbar}{4\pi}\epsilon^{\mu\nu\lambda}\left(K^{-1} b_\mu \partial_\nu b_\lambda + \beta \omega_\mu \partial_\nu \omega_\lambda\right)$$
$$+ J^\mu_b \left(\hbar(\partial_\mu \varphi - b_\mu - S\omega_\mu) + peA_\mu\right)$$

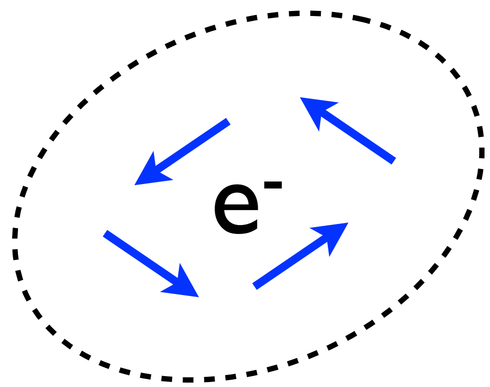
$$\mathcal{H} = \sqrt{g}\left(\varepsilon(\boldsymbol{v}, B) - U(g, B, P) - (E_a + \epsilon_{ab} v^b B) P^a\right)$$

kinetic energy of flow

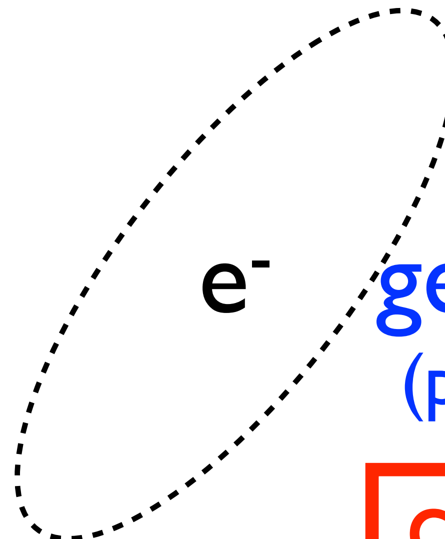
metric-dependent correlation energy

- shape of correlation hole (flux attachment) fluctuates, adapts to environment (electric field gradients)

S



new property:
"spin" couples
to curvature

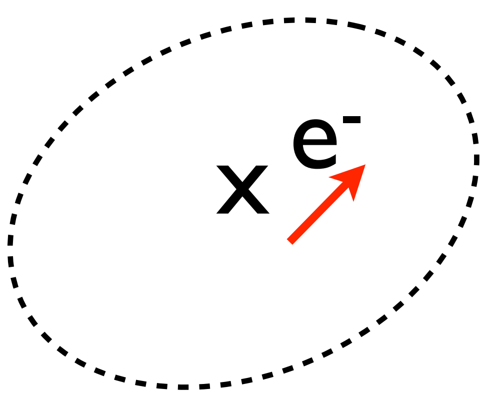


geometric distortion
(preserving inversion symmetry)

shape=metric

creates "curvature"
of metric

- polarizable, B x electric dipole = momentum, origin of "inertial mass"

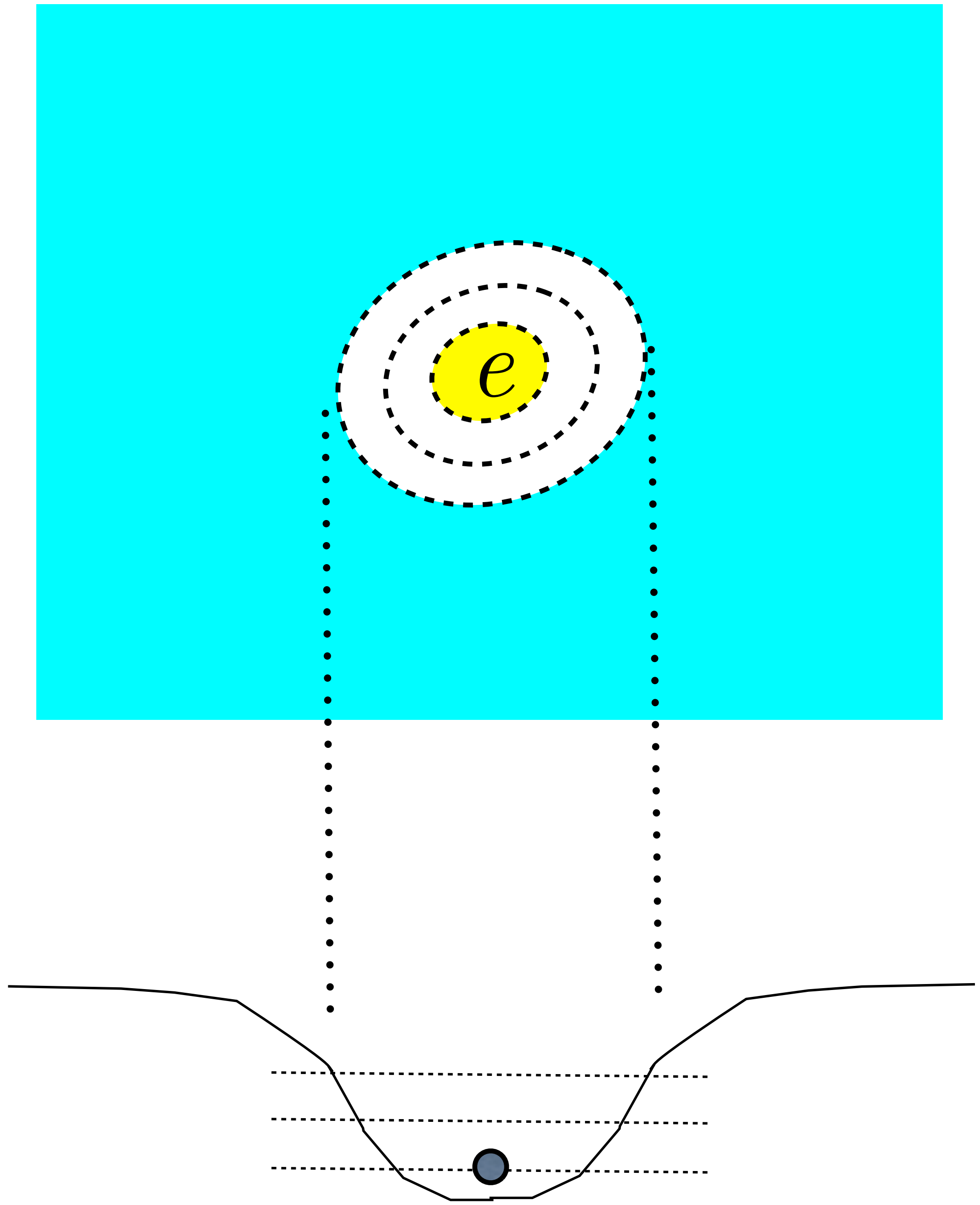


electric polarizability

$1/3$ Laughlin state

If the central orbital is filled, the next two are empty

The composite boson has inversion symmetry about its center

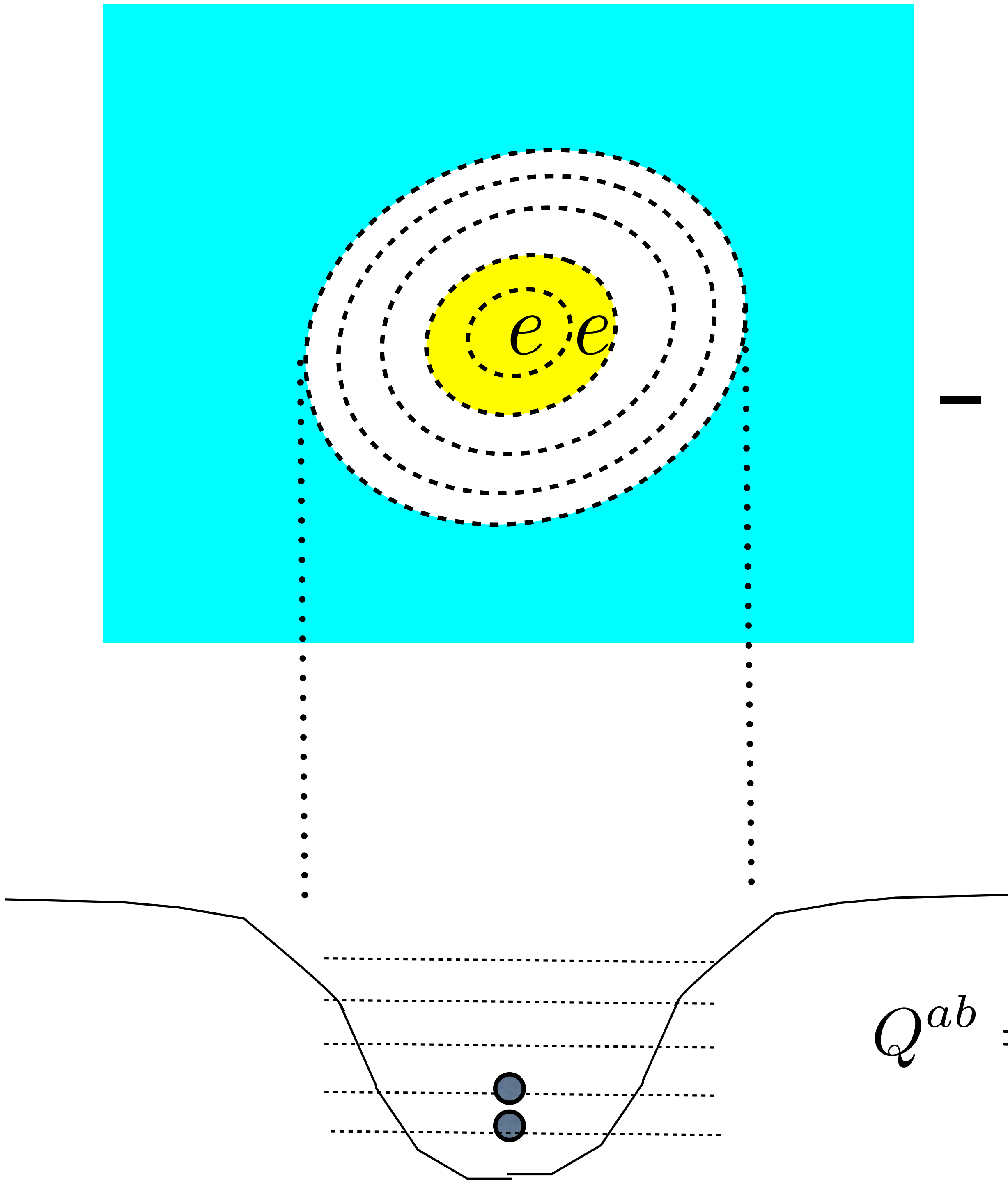


It has a "spin"

$$
\begin{array}{cccccccc}
 & \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & & & & \\
 & \boxed{1} & \boxed{0} & \boxed{0} & \cdots\cdots & & L = \frac{1}{2} & \\
- & \boxed{\frac{1}{3}} & \boxed{\frac{1}{3}} & \boxed{\frac{1}{3}} & \cdots\cdots & - & L = \frac{3}{2} & \\
\hline
 & & & & & & s = -1 &
\end{array}
$$

the electron excludes other particles from a region containing 3 flux quanta, creating a potential well in which it is bound

$2/5$ state



$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$			
1	1	0	0	0	.....
$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	.....

$$L = 2$$
$$- \; L = 5$$
$$s = -3$$

$$L = \frac{g_{ab}}{2\ell_B^2} \sum_i R_i^a R_i^b$$

$$Q^{ab} = \int d^2r \, r^a r^b \delta\rho(r) = s\ell_B^2 g^{ab}$$

second moment of neutral
composite boson
charge distribution

- Furthermore, the local electric charge density of the fluid with $\nu = p/q$ is determined by a combination of the magnetic flux density and the Gaussian curvature of the metric

$$J_e^0(\boldsymbol{x}) = \frac{e}{2\pi q}\left(\frac{peB}{\hbar} - sK_g(\boldsymbol{x})\right)$$

Topologically quantized "guiding center spin"

Gaussian curvature of the metric

- In fact, it is locally determined, if there is an inhomogeneous slowly-varying substrate potential

$$H = \sum_i v_n(\boldsymbol{R}_i) + \sum_{i<j} V_n(\boldsymbol{R}_i - \boldsymbol{R}_j)$$