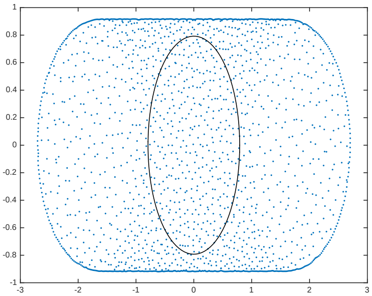
Random perturbations of nonselfadjoint operators, and the Gaussian Analytic Function

Stéphane Nonnenmacher + **Martin Vogel** (Orsay)

Seminar *Mathematical aspects of the Quantum Hall Effect*
Köln, 29 November 2016



Outline

- nonselfadjoint operators: spectral instability, pseudospectrum, quasimodes
- semiclassical nonselfadjoint (pseudo)differential: pseudospectrum vs. classical spectrum
- perturbation by a small random (Gaussian) operator: probabilistic Weyl's law
- can spectral correlations reveal more details of the symbol?
- simplest model in 1D: spectral correlations (k-point functions) lead to the Gaussian Analytic Function point process. Sketch of proof: effective Hamiltonian (Grushin method)
- more general models $\rightsquigarrow$ less elementary processes, still involving the GAF.

Pseudospectrum of nonselfadjoint operators

$P : \mathcal{H} \to \mathcal{H}$ selfadjoint: $\|(P-z)^{-1}\| = \mathrm{dist}(z, \mathrm{Spec}(P))^{-1}$

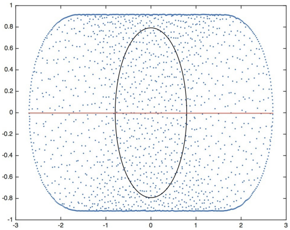
P *not* selfadjoint: $\|(P-z)^{-1}\|$ may be very large far from $\mathrm{Spec}(P)$: pseudospectral effect.

$\rightsquigarrow \mathrm{Spec}_\epsilon(P) \stackrel{\mathrm{def}}{=} \{z \in \mathbb{C}, \ \|(P-z)^{-1}\| \geq \epsilon^{-1}\}$ ϵ-pseudospectrum.

$\Longleftrightarrow$ instability of $\mathrm{Spec}(P)$ w.r.t. perturbations $\Longleftrightarrow$ quasimodes:

$$z \in \mathrm{Spec}_\epsilon(P) \Longleftrightarrow \exists B \in \mathcal{L}(\mathcal{H}), \ \|B\| \leq 1, \ \ z \in \mathrm{Spec}(P + \epsilon B)$$
$$\Longleftrightarrow \exists e_z \in \mathcal{H}, \ \|(P-z)e_z\| \leq \epsilon \|e_z\| \, .$$

<u>Ex:</u> semiclassical (pseudo)differential operator $P_h = \mathrm{Op}_h(p)$, with $p(x,\xi)$ complex-valued. [DENCKER-SJÖSTRAND-ZWORSKI'04]



Red: spectrum of $P_h = -ih\partial_x + e^{2i\pi x}$ on $L^2(S^1)$, $h = 10^{-3}$: $\mathrm{Spec}\, P_h = 2\pi h\mathbb{Z}$.

Blue: spectrum of $P_h^\delta = P_h + \delta Q$, with $\|Q\| \approx 1$, $\delta = 10^{-9}$.

(the spectra are truncated horizontally)

A simple model nonselfadjoint operator

Model [HAGER'06]: $P_h = -ih\partial_x + g(x)$ on $L^2(S^1)$, with $g \in C^\infty(S^1, \mathbb{C})$.
Classical "symbol" $p(x,\xi) = \xi + g(x)$ on T^*S^1. Elliptic $\Longrightarrow$ purely discrete spectrum.

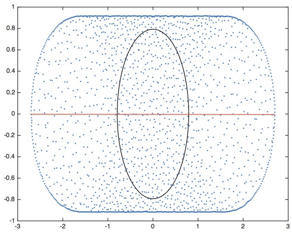
Where is the h^N-pseudospectrum of P_h?

Define the classical spectrum $\Sigma \stackrel{\text{def}}{=} \overline{p(T^*S^1)} = \mathbb{R} + i[\min \operatorname{Im} g, \max \operatorname{Im} g]$.

- $z \in \mathbb{C} \setminus \Sigma$ fixed $\Longrightarrow \|(P_h - z)^{-1}\| \leq C$ uniform when $h \in (0, h_0]$

Hence, if we perturb P_h by a perturbation δQ of size $\delta \sim h^N$, then $\operatorname{Spec}(P_h + \delta Q) \subset \Sigma + o(1)$.

For this model $\operatorname{Spec} P_h = 2\pi h\mathbb{Z} + \bar{g}$ lies on a line.



Main observation: for a generic perturbation δQ, $\operatorname{Spec}(P_h + \delta Q)$ fills the whole of Σ.

The same phenomenon occurs for more general operators.

Ex: 1D Schrödinger operator $P_h = -h^2\partial_x^2 + g(x)$ on S^1 (or $\mathbb{R}$), with a complex-valued potential $g(x)$.

Localized Quasimodes

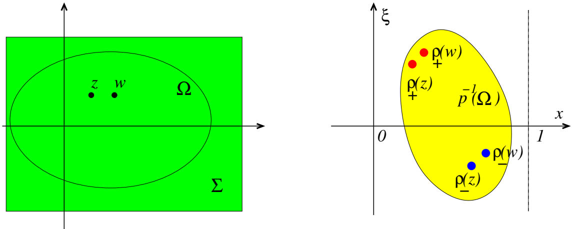
To identify the h^N-pseudospectrum of $P_h = \mathrm{Op}_h(p)$, we construct h^N-quasimodes.

Assumption on $p(x,\xi)$: for any $z \in \Omega \Subset \mathring{\Sigma}$, the "energy shell" $p^{-1}(z) = \{\rho = (x,\xi) \in T^*S^1,\ p(x,\xi) = z\}$ consists in a finite set of points $\rho^j = \rho^j(z) \in T^*S^1$, satisfying $\{\mathrm{Re}\, p, \mathrm{Im}\, p\}(\rho^j) \neq 0$.

Call $\rho = \rho_+$ if $\{\mathrm{Re}\, p, \mathrm{Im}\, p\}(\rho) < 0$ (resp. $\rho = \rho_-$ if $\{\mathrm{Re}\, p, \mathrm{Im}\, p\}(\rho) > 0$).

Then:

- for each $\rho_+(z)$, one can construct a h^∞-quasimode $e_+(z;h)$ of $(P_h - z)$ (that is, $\|(P_h - z)e_+(z;h)\| = \mathcal{O}(h^\infty)$), which is microlocalized on $\rho_+(z)$.
- for each $\rho_-(z)$, one can construct a h^∞-quasimode $e_-(z;h)$ of $(P_h - z)^*$, microlocalized on $\rho_-(z)$.



Localized quasimodes: a "linear normal form"

What do the quasimodes $e_+(z,h)$ *look like?*

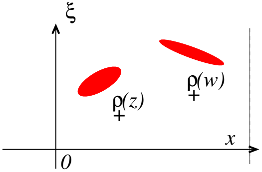
• If we linearize $p(\rho)$ near ρ_+, we are lead (after a symplectic transformation) to a function of the type $a(x,\xi)=\xi-ix$: this is the classical symbol of the annihilation operator $A_h=-ih\partial_x-ix$.

The symbol $a(x,\xi)=\xi-ix$ has classical spectrum $\Sigma=\mathbb{C}$.
For each $z=\Xi-iX\in\mathbb{C}$, the "energy shell" $a^{-1}(z)=\{\rho_+(z)=(X,\Xi)\}$, and satisfies $\{\mathrm{Re}\,a,\mathrm{Im}\,a\}(\rho_+)=-1$.

$\Longrightarrow$ one can construct quasimodes $e_+(z;h)$ of (A_h-z) for all $z\in\mathbb{C}$.

Actually, for all $z=\Xi-iX\in\mathbb{C}$, (A_h-z) admits an *eigenstate*, the coherent state at (X,Ξ), $\eta(x;z,h)=(\pi h)^{-1/4}e^{-(x-X)^2/2h+ix\Xi/h}$.

• For a general $p(x,\xi)$ and $\rho_+\in p^{-1}(z)$, the quasimode $e_+(z,h)$ is approximately a squeezed coherent state centered at the point ρ_+; its shape depends on the linearization $dp(\rho_+)$.



Gaussian random perturbations: probabilitic Weyl's law

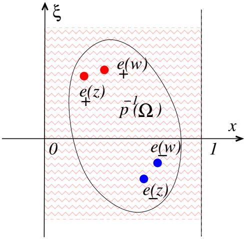
- These quasimodes show that for any $z \in \Omega$, for $h < h_0$, there exists an operator Q, $\|Q\| \sim 1$, such that $z \in \mathrm{Spec}(P_h + \delta Q)$, where $\delta = h^N$.

> *What does the spectrum of $P_h + \delta Q$ look like globally, for a typical perturbation δQ?*

- To construct a typical perturbation Q, consider an orthonormal system (φ_k) microlocally filling a nbhd of $p^{-1}(\Omega)$.

Ex: take $(\varphi_k(x) = e^{2i\pi kx})_{|k|\leq C/h}$

(φ_k is localized on $\{(x, \xi = kh)\}$).



Then define the Gaussian random operator

$$Q = \sum_{k,k'} \alpha_{kk'} \varphi_k \otimes \varphi_{k'}^*, \quad \text{with the } \alpha_{kk'} \text{ i.i.d. } \mathcal{N}_{\mathbb{C}}(0,1) \text{ variables.}$$

Q belongs to the Ginibre ensemble. With high probability $\|Q\|_{HS} \leq \tilde{C}/h$.

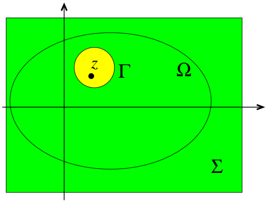
- We then consider the randomly perturbed operator $P_h^\delta = P_h + \delta Q$, with a perturbation strength $\delta = h^N$ $(N \gg 1)$.

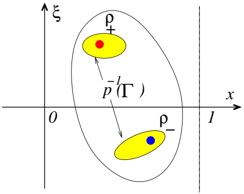
Gaussian random perturbations: probabilitic Weyl's law

Theorem (HAGER'06, HAGER-SJÖSTRAND'07)
With probability $\geq 1 - h^M$, the spectrum of $P_h^\delta = P_h + \delta Q$ satisfies a Weyl's law: for any smooth domain $\Gamma \subset \Omega$,

$$\#(\mathrm{Spec}(P_h^\delta) \cap \Gamma) = \frac{\mathrm{Vol}\left(p^{-1}(\Gamma)\right)}{2\pi h} + o(h^{-1}), \quad \textit{when } h \searrow 0.$$

In particular, w.h.p. the spectrum fills up Ω.





This probabilistic Weyl's law can be expressed in terms of the average spectral density: $D_h(z) = (2\pi h)^{-1} D(z) + o(h^-1)$, with the "classical" density $D(z)\, dz = p^*(dx \wedge d\xi)$.

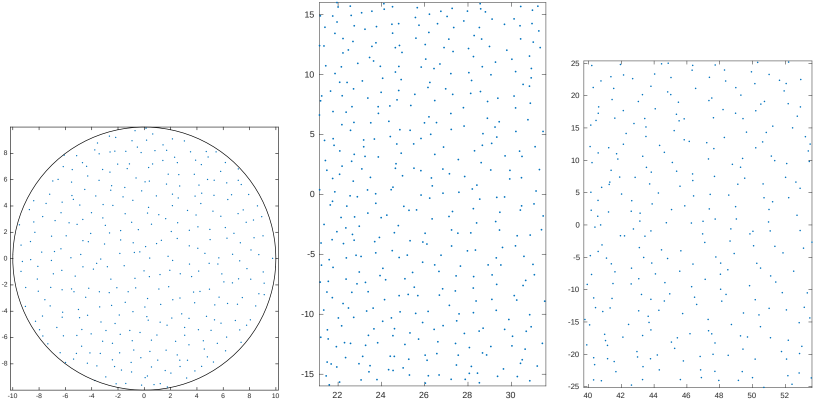
Probabilitic Weyl's law: various settings

$$\#(\mathrm{Spec}(P_h^\delta) \cap \Gamma) = \frac{\mathrm{Vol}\left(p^{-1}(\Gamma)\right)}{2\pi h} + o(h^{-1})$$

This probabilistic Weyl's law has been proved in more and more settings:

- [HAGER'06]: $P_h = -ih\partial_x + g(x)$ on S^1, such that $g^{-1}(z) = \{\rho_+, \rho_-\}$ for any $z \in \mathring{\Sigma}$. Perturbation = Gaussian random operator Q .
- [HAGER-SJÖSTRAND'08]: $P_h = \mathrm{Op}_h(p)$ on $\mathbb{R}^n$.
- [HAGER'06B]: $P = \mathrm{Op}_h(p)$ on $\mathbb{R}^1$, with symmetry $p(x, \xi) = p(x, -\xi)$ (+some assumptions).
  Multiplicative perturbation: random potential $V(x) = \sum_{k \leq C/h} \alpha_k \varphi_k(x)$, with α_k i.i.d. $\mathcal{N}_{\mathbb{C}}(0, 1)$.
  Ex: $P_h = -h^2\partial_x^2 + g(x) + \delta V(x)$, $g(x)$ complex-valued.
- [SJÖSTRAND'08,...]: Same on $\mathbb{R}^n$ or M compact Riemannian mfold.
- [BORDEAUX-MONTRIEUX'10]: P a (nonsemiclassical) differential operator.

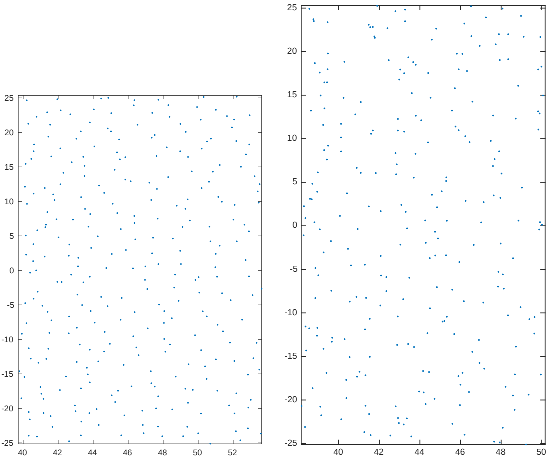
Gaussian random perturbations: experiments



Spectrum (inside some Γ) for various operators on S^1, perturbed by δQ:

$P_1 = -ih\partial_x + e^{2i\pi x}$, $P_2 = -h^2\partial_x^2 + e^{2i\pi x}$, $P_3 = -h^2\partial_x^2 + e^{6i\pi x}$

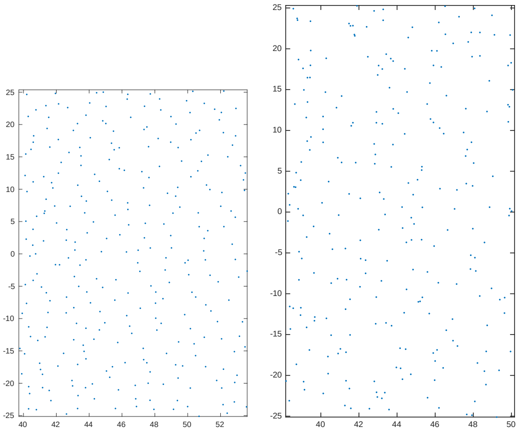
Gaussian random perturbations: experiments



Operator $P_3 = -h^2\partial_x^2 + e^{6i\pi x}$ on S^1, two types of perturbations: random operator δQ (left) vs. random potential δV (right).

Do you see any difference?

Q vs. V perturbation: spectral correlations



Answer: There are differences in the correlations between the eigenvalues.

Q: the eigenvalues seem to "repel" each other on the scale of the mean level spacing, while for V they can present "clusters".

Spectral correlations: k-point functions

The spectrum of P_h^δ defines a random point process on $\mathbb{C}$, represented by the (locally finite) random measure on $\mathbb{C}$

$$\mathcal{Z}_h^\delta = \sum_{z_i \in \mathrm{Spec}\, P_h^\delta} \delta_{z_i} \,.$$

1-point density = average spectral density $D_h(z)$

$$\forall \varphi \in C_c^\infty(\mathbb{C}), \qquad \int_{\mathbb{C}} \varphi(z)\, D_h(z)\, dz = \mathbb{E}[\mathcal{Z}_h^\delta(\varphi)]$$

For any $k \geq 1$, the k-point density of this process is defined (outside the diagonal $\Delta = \{z_i = z_j \text{ for some } i \neq j\}$) as:

$$\forall \varphi \in C_c^\infty(\mathbb{C}^k \setminus \Delta), \qquad \int_{\mathbb{C}^k} \varphi(\vec{z})\, D_h^k(\vec{z})\, d\vec{z} = \mathbb{E}\Big[\sum_{z_1,\ldots z_k \in \mathrm{Spec}\, P_h^\delta} \varphi(z_1, \ldots, z_k) \Big]$$

$$= \mathbb{E}[(\mathcal{Z}_h^\delta)^{\otimes k}(\varphi)] \,.$$

k-point correlation function: normalize the k-point density by the local average densities:

$$\forall (z_1, \ldots, z_k) \in \mathbb{C}^k \setminus \Delta, \qquad K_h^k(z_1, \ldots, z_k) \stackrel{\text{def}}{=} \frac{D_h^k(z_1, \ldots, z_k)}{D_h(z_1) \cdots D_h(z_k)} .$$

2-point function for Hager's model

Given P_h and random perturbation Q, V, can we compute the k-point correlations of $\mathrm{Spec}\, P_h^\delta$?

$\oplus$ [VOGEL'14] computed $K_h^2(z_1, z_2)$ for the operator $P_h^\delta = -ih\partial_x + g(x) + \delta Q$, in the case where $p^{-1}(z) = \{\rho_+(z), \rho_-(z)\}$ for each $z \in \Omega$.

His formula suggests to rescale to the local mean spacing between nearby eigenvalues, namely $D_h(z)^{-1/2} \approx \ell_z h^{1/2}$, $\ell_z = (2\pi/D(z))^{1/2}$.

Theorem (VOGEL'14)

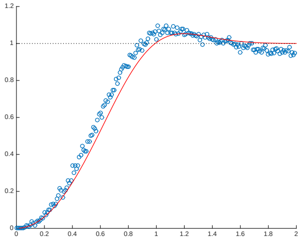
Assume $p^{-1}(z) = \{\rho_+(z), \rho_-(z)\}$ for all $z \in \Omega$.
For any $z_0 \in \Omega$ and any $u_1 \neq u_2 \in \mathbb{C}$, we have a scaling limit

$$K_h^2\big(z_0 + u_1 \ell_{z_0} h^{1/2}, z_0 + u_2 \ell_{z_0} h^{1/2}\big) \xrightarrow{h\to 0} \tilde{K}^2(u_1, u_2)\,,$$

with $$\tilde{K}^2(u_1, u_2) = \kappa\Big(\frac{\pi}{2}|u_1 - u_2|^2\Big), \quad \kappa(t) = \frac{(\sinh^2 t + t^2)\cosh t - 2t\sinh t}{\sinh^3 t}\,.$$

- the limit is uniform for $(u_1, u_2) \in K \Subset \mathbb{C} \times \mathbb{C} \setminus \Delta$
- the scaling limit is universal (dep. of g and z_0 only through $D(z_0)$).
- quadratic repulsion at short rescaled distance: $\kappa(t) = t + \mathcal{O}(t^2)$, $t \to 0$.
- decorrelation at large rescaled distance: $\kappa(t) = 1 + \mathcal{O}(t^2 e^{-2t})$, $t \to \infty$.

$\tilde{K}^2(u_1, u_2)$ and the Gaussian Analytic Function



Red line: $r \mapsto \kappa(r^2)$, where $\tilde{K}^2(u_1, u_2) = k\big(\frac{\pi}{2}|u_1 - u_2|^2\big)$.

Blue circles: numerical data for $P_1 + \delta Q$, using 200 realizations of Q.

$\ominus$ $\tilde{K}^2$ differs from the case of the Ginibre ensemble (= spectrum Q alone): $\tilde{K}^2_{Gin}(u_1, u_2) = 1 - e^{-\pi|u_1 - u_2|^2}$.

$\oplus$ $\tilde{K}^2$ is the 2-point function for the zeros of the Gaussian Analytic Function:

$$G(u) = \sum_{n \geq 0} \beta_j \frac{\pi^{j/2} u^j}{\sqrt{j!}}, \qquad u \in \mathbb{C}, \quad \beta_j \text{ i.i.d. random variables } \mathcal{N}_{\mathbb{C}}(0, 1).$$

[HANNAY'96]

- The zero set of $G(u)$ will be denoted by $\tilde{\mathcal{Z}}_G$, it is a well-studied random point process on $\mathbb{C}$ [NAZAROV-SODIN'10]

GAF: k-point densities from the covariance function

$\tilde{\mathfrak{Z}}_G$ the zero process of the GAF $G(u) = \sum_{n\geq 0} \beta_j \frac{\pi^{j/2} u^j}{\sqrt{j!}}$.

To compute the k-point density $D_G^k(\vec{u})$ of this process, the essential ingredient is the covariance function

$$C(u, \bar{v}) \stackrel{\text{def}}{=} \mathbb{E}[G(u)\overline{G(v)}] = \exp(\pi u \bar{v}).$$

Indeed, the identity between distributions
$\tilde{\mathfrak{Z}}_G(u) \stackrel{\text{def}}{=} \sum_{u_i : G(u_i)=0} \delta(u - z_i) = |G'(u)|^2 \, \delta(G(u))$,
leads to the Kac-Rice-Hammersley formula:

$$D_G^k(\vec{u}) = \mathbb{E}\tilde{\mathfrak{Z}}_G(u_1)\cdots\mathfrak{Z}_G(u_k) = \mathbb{E}|G'(u_1)|^2\delta(G(u_1))\cdots|G'(u_k)|^2\delta(G(u_k))\,.$$

The RHS only depends on the joint distribution of the Gaussian vector $\{G(u_1), \cdots, G(u_k), G'(u_1), \cdots, G'(u_k)\}$, encoded in the
$2k \times 2k$ covariance matrix

$$\begin{pmatrix} \mathbb{E}G(u_i)\overline{G(u_j)} & \mathbb{E}G'(u_i)\overline{G(u_j)} \\ \mathbb{E}G(u_i)\overline{G'(u_j)} & \mathbb{E}G'(u_i)\overline{G'(u_j)} \end{pmatrix} = \begin{pmatrix} C(u_i, \bar{u}_j) & \partial_{u_i} C(u_i, \bar{u}_j) \\ \partial_{\bar{u}_j} C(u_i, \bar{u}_j) & \partial_{u_i}\partial_{\bar{u}_j} C(u_i, \bar{u}_j) \end{pmatrix}.$$

The rescaled spectrum has the statistics of the GAF

Vogel's result for the spectrum of $P_h^\delta = -ih\partial_x + g(x) + \delta Q$ extends to all k-point functions.

Theorem (N-Vogel'16)

Assume $p^{-1}(z) = \{\rho_+(z), \rho_-(z)\}$ for each $z \in \Omega$. Take $\delta = h^N$, $N > 5/2$. For any $k \geq 2$, the k-point correlation function for the eigenvalues of $P_h + \delta Q$ satisfy, near any $z_0 \in \mathring{\Sigma}$, the scaling limit

$$\forall \vec{u} \in \mathbb{C}^k \setminus \Delta, \qquad K_h^k\big(z_0 + \vec{u}\,\ell_{z_0} h^{1/2}\big) \xrightarrow{h\to 0} \tilde{K}^k(\vec{u}),$$

where $\tilde{K}^k(\vec{u})$ is the k-point function of the GAF.

In other words, the rescaled spectral measure centered at z_0,

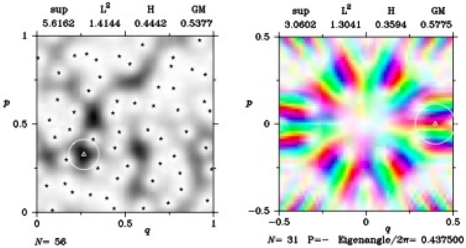
$$\tilde{\mathcal{Z}}_{h,z_0}^\delta = \sum_{z_i \in \mathrm{Spec}\, P_h^\delta} \delta_{u_i = \frac{z_i - z_0}{\ell_{z_0} h^{1/2}}},$$

converges in distribution to the point process $\tilde{\mathcal{Z}}_G$ when $h \searrow 0$.

History: the GAF as a "holomorphic random wave"

The GAF has been used as a model for:

- eigenfunctions (in Bargmann representation) of 1D quantized chaotic maps [LEBOEUF-VOROS'91, BOGOMOLNY-BOHIGAS-LEBOEUF'96, HANNAY'96, PROSEN'96, N-VOROS'98]: $\mathcal{B}_j(z) = \langle \tilde{\eta}(z), \psi_j \rangle$, with $\tilde{\eta}(z)$ holom. coh. st.



Left: random Bargmann function on $\mathbb{T}^2$ (Husimi).

Right: Bargmann eigenfunction of a chaotic quantum map on $\mathbb{T}^2$.

- [BLEHER-SCHIFFMAN-ZELDITCH'00]: M compact Kähler mfold, L positive line bundle: study random holomorphic sections $s(z)$ on $L^{\otimes N}$, in the limit $N \to \infty$ ($N \sim h^{-1}$).
  Covariance $\mathbb{E}s(z)\overline{s(w)}$= $\Pi_N(z, w)$ Bergman kernel, rescale by $N^{-1/2}$
  $\rightsquigarrow$ universal scaling limit $C(u, \bar{v})$ [TIAN, CATLIN, ZELDITCH'98,...]
  $\dim_{\mathbb{C}} M = 1$: at each $z_0 \in M$, the rescaled zero process $\tilde{\mathcal{Z}}_{s,z_0} \overset{N\to\infty}{\to} \tilde{\mathcal{Z}}_G$.
- In the present work, the GAF mimicks an effective spectral determinant.

How to study $\mathcal{Z}_h^\delta$? Use an effective Hamiltonian

Heuristics: the spectrum of P_h^δ near z is governed by the action of P_h^δ, resp. $P_h^{\delta *}$ in the $\sqrt{h}$-neighbourhood of $\rho_\pm(z) \rightsquigarrow$ involves the quasimodes $e_\pm(z)$.

- Idea: construct a Grushin problem: extend $(P_h - z)$ by "filling" its approximate kernel and cokernel $\rightsquigarrow$ invertible operator

$$\mathcal{P}(z) \stackrel{\text{def}}{=} \begin{pmatrix} P_h - z & R_-(z) \\ R_+(z) & 0 \end{pmatrix} : \mathcal{H} \oplus \mathbb{C} \to \mathcal{H} \oplus \mathbb{C}.$$

Auxiliary operators $R_+(z)u = \langle e_+(z), u\rangle$, $R_-(z)u_- = u_- e_-(z)$.

Call $\mathcal{P}(z)^{-1} = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{pmatrix}$.

- Schur's complement formula $\Longrightarrow z \in \operatorname{Spec}(P_h) \Longleftrightarrow E_{-+}(z) = 0$.

$E_{-+}(z)$ is an effective Hamiltonian for P_h.

- $e_\pm(z)$ are h^∞-quasimodes $\Longrightarrow E_{-+}(z) = \mathcal{O}(h^\infty)$, $\forall z \in \Omega$.

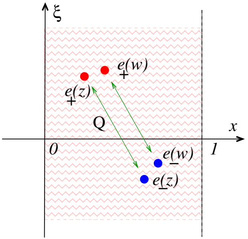
- use same Grushin extension for $P_h^\delta \rightsquigarrow$ randomly perturbed eff. Hamil.

$$E_{-+}^\delta(z) = E_{-+}(z) + \delta F(z) + \mathcal{O}(\delta^2 h^{-1/2}), \quad \text{with} \quad F(z) = -\langle Q e_+(z), e_-(z)\rangle.$$

$\oplus$ w.h.proba., Q couples $e_+(z)$ to $e_-(z)$, so that $F(z) \asymp 1$.

$\Longrightarrow E_{-+}^\delta(z) \approx \delta\, F(z)$ a Gaussian random function of z.

Computing the covariance of $F(z)$



$F(z) = -\langle Qe_+(z), e_-(z)\rangle$ can couple e_+ to e_-, because the phase space spanned by Q contains ρ_+, ρ_-.

We need to study the zeros of the function $\langle Qe_+(z), e_-(z)\rangle + small \rightsquigarrow$ compute its covariance:

$$\mathbb{E}\langle Qe_+(z), e_-(z)\rangle\overline{\langle Qe_+(w), e_-(w)\rangle} = \langle e_+(z), e_+(w)\rangle\langle e_-(w), e_-(z)\rangle + \mathcal{O}(h^\infty)$$

The quasimode $e_+(z)$ is microlocalized in a $\sqrt{h}$-nbhd of $\rho_+(z)$.
For $|z-w| > h^{1/2-\epsilon}$, $\langle e_+(z), e_+(w)\rangle = \mathcal{O}(h^\infty)$. If $z, w = z_0 + \mathcal{O}(\sqrt{h})$,

$$\langle e_+(z_0+\sqrt{h}u), e_+(z_0+\sqrt{h}v)\rangle = \exp\left(\sigma_+ u\bar{v} + \phi_+(u) + \overline{\phi_+(v)} + \mathcal{O}(\sqrt{h})\right),$$

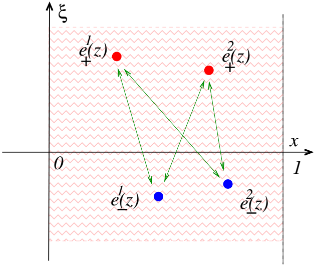
Similar for $\langle e_-(\bullet), e_-(\bullet)\rangle \Longrightarrow$ up to a change of gauge, we get $e^{(\sigma_+ + \sigma_-)u\bar{v}}$.
One shows that $\sigma_+ + \sigma_- = D(z_0)/2$, so rescaling u, v by ℓ_{z_0} we obtain the covariance $e^{\pi u\bar{v}} = C(u, \bar{v})$ in the limit $h \to 0$.

□

1D operators with J quasimodes

Let us now assume that for any $z \in \Omega$, the "energy shell" $p^{-1}(z) = \{\rho_+^1(z), \ldots, \rho_+^J(z), \rho_-^1(z), \ldots, \rho_-^J(z)\}$. Ex: $p(x,\xi) = \xi + e^{2iJ\pi x}$.

$(P_h - z)$ and $(P_h - z)^*$ have J quasimodes $e_\pm^j(z)$ microlocalized on $\rho_\pm^j(z)$. The effective Hamiltonian $E_{-+}^\delta(z)$ is now a $J \times J$ matrix, and $z \in \mathrm{Spec}(P_h^\delta) \iff \det E_{-+}^\delta(z) = 0$.



We get $E^\delta(z)^{ij} = -\delta \langle Q e_+^i(z), e_-^j(z)\rangle + small$
W.h.p., $E_{-+}^\delta(z)$ is dominated by the $J \times J$ matrix $F(z)$ of entries $F_{ij}(z) = \langle Q e_+^i(z), e_-^j(z)\rangle$.

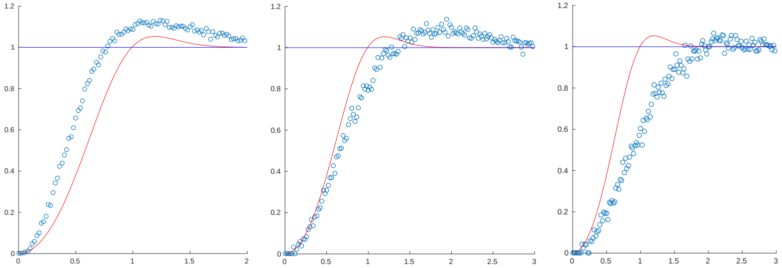
- each $F_{ij}(z)$ is a GAF with rescaled (and re-gauged) covariance $\exp((\sigma_{+i} + \sigma_{-j})u\bar{v})$.
- the $F_{ij}(z)$ are independent of e.o. when $h \to 0$.

Theorem (N-VOGEL'16)

After rescaling by $h^{1/2}$ near z_0, $\tilde{\mathcal{Z}}_{h,z_0}^\delta$ converges to the zero process of the random holomorphic function $G^J(u) = \det\big(G_{ij}(u)\big)$, where $(G_{ij}(u))$ is a matrix of $J \times J$ independent GAFs with variances $e^{(\sigma_{+i}+\sigma_{-j})u\bar{v}}$.

Qu: can we compute the k-correlations of the zeros of $G^J(u)$?

Operators with J quasimodes



Blue circles: numerics for the rescaled 2-point function for operators $P_J = -h^2\partial_x^2 + e^{2i\pi J/2x}$, $J = 2, 6, 10$. In the spectral region we consider, $(P_J - z)$ admits J quasimodes.
Red: for comparison, the function $\tilde{K}^2(u, v)$ as a function of $|u - v|$.

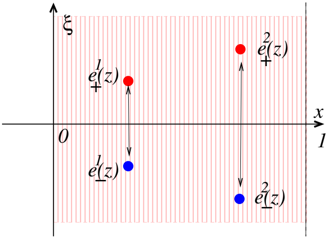
Conjecture

For any $J \geq 1$, the zeros of $G^J(u)$ repel each other quadratically.

(weak form of universality).

Operators with J quasimodes – multiplicative perturbation

Replace the perturbation δQ by the multiplicative perturbation $\delta V(x)$, with $V(x) = \sum_{|k|\leq C/h} \alpha_k \varphi_k(x)$.
The effective Hamiltonian is the $J \times J$ matrix with entries $E^{\delta}_{-+}(z)^{ij} = -\delta \langle V e^i_+(z), e^j_-(z) \rangle + small$.



The coupling of V is *local*:
$\langle V e_+(z), e_-(z) \rangle = \int V\, e_+ \overline{e_-}\, dx$. Hence V can couple $e_+(z)$ with $e_-(z)$ only provided $x_+ = x_-$.
$\Longrightarrow$ consider symmetric symbols $p(x, -\xi) = p(x, \xi)$: then each $\rho^i_+ = (x^i_+, \xi^i_+)$ is associated with $\rho^i_- = (x^i_+, -\xi^i_+)$.

Quasimodes are related by $e^i_-(z) = \overline{e^i_+(z)}$.

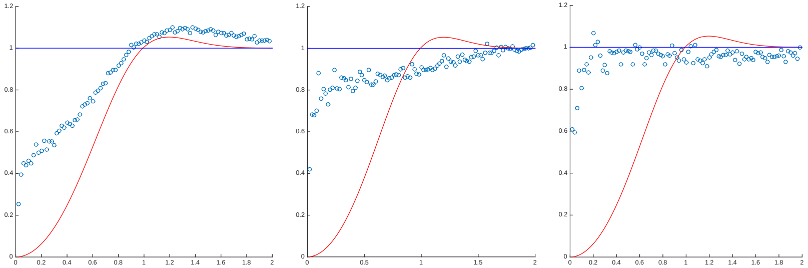
The Gaussian function $F_{ii}(z) = \int V\, e^i_+(z)^2\, dx$ has covariance $\mathbb{E}[F_{ii}(z)\overline{F_{ii}(w)}] = \langle e^i_+(z)^2, e^i_+(w)^2 \rangle$.
After $h^{1/2}$-rescaling, this covariance $\approx e^{2\sigma_i u\bar{v}}$.

The matrix $F_{ij}(z)$ is then approximately diagonal, so that $\det E^{\delta}_{-+}(z) = (-\delta)^J \prod_{j=1}^J F_{ii}(z) + small$.

Theorem (N-Vogel'16)

Around z_0, the $h^{1/2}$-rescaled spectrum of P^{δ}_h converges to the superposition of J independent GAF processes, with respective variances $e^{2\sigma_i u\bar{v}}$.

Operators with J quasimodes –multiplicative perturbation



Blue circles: numerics of the rescaled 2-point correlation function for operators $P_J = -h^2\partial_x^2 + e^{2i\pi J/2x}$, $J = 2, 4, 6$, and perturbation δV.

Observe the absence of quadratic repulsion at the origin, due to the presence of J independent processes, allowing clusters of $\leq J$ eigenvalues.

TO DO: Compute explicitly the k-point densities by superposing the J independent processes.

Conclusion

Small perturbations of nonselfadjoint 1D (pseudo)differential operators lead to interesting spectral phenomena

- ubiquity of Weyl's law
- ubiquity of the Gaussian Analytic Function, and partial universality
- if the law of the perturbation parameters α_{ij} is not Gaussian but sufficiently regular, we expect the same result (CLT)
- spectral correlations are sensitive to the structure (cardinal) of $p^{-1}(z)$, symmetry $\xi \to -\xi$ of the symbol, and type of random perturbation $\Longrightarrow$ probabilistic spectral information on $p(x, \xi)$.
- can we compute the k-point functions for the zeros of $\det(G_{ij}(u))$?
- nonselfadjoint operators in dimension $n > 1$: energy shell is a codimension 2 submanifold $\Longrightarrow$ quasimodes form a subspace of dimension $\sim h^{-n+1} \Longrightarrow E^{\delta}_{-+}(z)$ is a large random matrix. What can be said about spectral correlations?