LARGE *N* EXPANSION OF BETA- ENSEMBLES ON ARBITRARY CONTOURS

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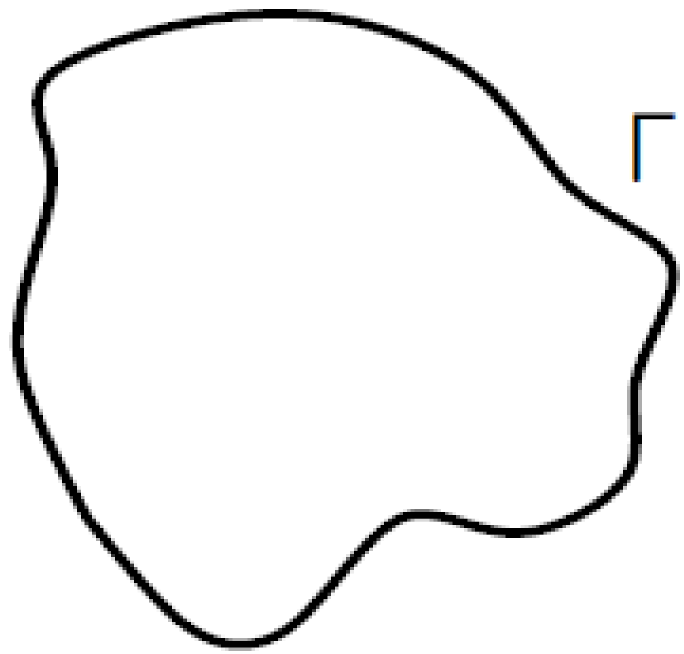
Based on a joint work with P.Wiegmann (in progress)

Beta-ensemble:

2D Coulomb charges confined to a contour Γ



(The Dyson gas on an arbitrary contour)

Partition function:

$$Z_N = \oint_\Gamma \ldots \oint_\Gamma \prod_{i<j} |z_i - z_j|^{2\beta} \prod_{k=1}^{N} e^{\frac{1}{\hbar} W(z_k)} |dz_k|$$

$W(z) = W(z, \bar{z})$ is a (real valued) potential.

Parameters:

- β: "inverse temperature"
- $\hbar$: a "quasiclassical" parameter introduced for large N limit: $\hbar \to 0$ simultaneously with $N \to \infty$

Distribution functions

$$R(a) = \frac{\hbar N}{Z_N} e^{\frac{1}{\hbar}W(a)} \oint_\Gamma |\Delta_{N-1}(z_i)|^{2\beta} \prod_{j=1}^{N-1} |a - z_j|^{2\beta} e^{\frac{1}{\hbar}W(z_j)} |dz_j|$$

$$R(a,b) = \frac{\hbar^2 N(N-1)}{Z_N} |a-b|^{2\beta} e^{\frac{1}{\hbar}(W(a)+W(b))}$$

$$\times \oint_\Gamma |\Delta_{N-2}(z_i)|^{2\beta} \prod_{j=1}^{N-2} |a - z_j|^{2\beta} |b - z_j|^{2\beta} e^{\frac{1}{\hbar}W(z_j)} |dz_j|$$

$\Delta_N(z_i) = \prod_{j<i}^{N} (z_i - z_j)$ is the Vandermonde determinant.

Sum rules:

$$\oint_\Gamma R(z)\,|dz| = N\hbar\,, \qquad \oint_\Gamma R(a,z)\,|dz| = (N-1)\,\hbar\,R(a)$$

Observables:

- Density $$\rho(z) = \hbar \sum_{i=1}^{N} \delta_\Gamma(z, z_i)$$

- Potential

$$\varphi(z) = -\beta\hbar \sum_i \log|z - z_i|^2 = -2\beta \oint_\Gamma \log|z - \xi|\, \rho(\xi)|d\xi|$$

Jump on the contour:

$$\partial_n^+ \varphi(z) - \partial_n^- \varphi(z) = 4\pi\beta\rho(z), \quad z \in \Gamma$$

The Green's functions

The Green's function $G_{\text{int}}(z,\zeta)$ of the Laplace operator in the domain D ($\Gamma = \partial$D):

- $G_{\text{int}}(z,\zeta) = G_{\text{int}}(\zeta,z)$ is harmonic in D in each variable except $z = \zeta$ and

$$G_{\text{int}}(z,\zeta) = \log|z-\zeta| + \dots$$

as $z \to \zeta$;

- $G_{\text{int}}(z,\zeta) = 0$ if z or ζ belongs to the boundary Γ.

In terms of the conformal map w_{int} from D onto the unit disk:

$$G_{\mathrm{int}}(z,\zeta) = \log \left| \frac{w_{\mathrm{int}}(z) - w_{\mathrm{int}}(\zeta)}{1 - w_{\mathrm{int}}(z)\overline{w_{\mathrm{int}}(\zeta)}} \right|$$

Solution to the Dirichlet boundary value problem in D:

$$f_H(z) = \frac{1}{2\pi} \int_{\Gamma} f(\xi) \partial_n G_{\mathrm{int}}(z,\xi) |d\xi|$$

The Green's function $G_{\text{ext}}(z,\zeta)$ of the Laplace operator in the domain $\mathbf{C} \setminus D$:

- $G_{\text{ext}}(z,\zeta) = G_{\text{ext}}(\zeta, z)$ is harmonic in $\mathbf{C} \setminus D$ and bounded at infinity in each variable except $z = \zeta$ and

$$G_{\text{ext}}(z,\zeta) = \log|z-\zeta| + \dots \qquad \text{as } z \to \zeta;$$

- $G_{\text{ext}}(z,\zeta) = 0$ if z or ζ belongs to the boundary Γ.

In terms of the conformal map w_{ext} from $\mathbf{C} \setminus D$ onto the exterior of the unit disk:

$$G_{\text{ext}}(z,\zeta) = \log\left|\frac{w_{\text{ext}}(z) - w_{\text{ext}}(\zeta)}{1 - w_{\text{ext}}(z)\overline{w_{\text{ext}}(\zeta)}}\right|$$

Solution to the Dirichlet boundary value problem in $\mathbf{C} \setminus D$:

$$f^H(z) = -\frac{1}{2\pi}\int_\Gamma f(\xi)\partial_n G_{\text{ext}}(z,\xi)|d\xi|$$

The Neumann jump operator

The Neumann jump operator takes a function f on the contour Γ to the difference between normal derivatives of its harmonic extensions to the interior and to the exterior of Γ:

$$\mathcal{N}_\Gamma f(z) = \partial_n^+ f_H(z) - \partial_n^- f^H(z), \quad z \in \Gamma$$

In terms of the Neumann jump operator

$$\rho(z) = \rho^{(0)}(z) + \frac{1}{4\pi\beta} \mathcal{N}_\Gamma \varphi(z)$$

$$\rho^{(0)}(z) = -\frac{\hbar N}{2\pi} \partial_n G_{\text{ext}}(z, \infty) = \frac{\hbar N}{2\pi} |w'_{\text{ext}}(z)|$$

Correlation functions

Correlation functions of densities are variational derivatives of $\log Z_N$ w.r.t. the potential:

$$\langle \rho(z) \rangle = \hbar^2 \frac{\delta \log Z_N}{\delta W(z)}$$

$$\langle \rho(z_1)\rho(z_2) \rangle_c = \hbar^2 \frac{\delta \langle \rho(z_1) \rangle}{\delta W(z_2)} = \hbar^4 \frac{\delta^2 \log Z_N}{\delta W(z_1)\delta W(z_2)}$$

$$\langle \rho(z_1)\rho(z_2) \rangle_c = \langle \rho(z_1)\rho(z_2) \rangle - \langle \rho(z_1) \rangle \langle \rho(z_2) \rangle$$

is the connected part of the correlation function.

The correlation functions and the distribution functions are related as follows:

$$\langle \rho(a) \rangle = R(a)$$

$$\langle \rho(a)\rho(b) \rangle = R(a,b) + \hbar \langle \rho(a) \rangle \delta_\Gamma(a,b).$$

The loop equation

The obvious identity which follows from invariance of the partition function under reparametrizations of the contour (changes of the integration variables):

$$\sum_j \oint_\Gamma \cdots \oint_\Gamma \partial_{s_j} \left(\epsilon(z_j) \prod_{i<k} |z_i - z_k|^{2\beta} \prod_m e^{\frac{1}{\hbar} W(z_m)} \right) \prod_{l=1}^N |dz_l| = 0$$

$$\Updownarrow$$

$$\left\langle \sum_j \left(\hbar \partial_{s_j} \epsilon(z_j) + \epsilon(z_j) \partial_{s_j} W(z_j) \right) + \hbar\beta \sum_{j \neq k} \epsilon(z_j) \partial_{s_j} \log |z_j - z_k|^2 \right\rangle = 0$$

$$\Updownarrow$$

$$\beta \oint_\Gamma R(z,\xi) \partial_s \log |z - \xi|^2 \, |d\xi| + \partial_s W(z) R(z) - \hbar \partial_s R(z) = 0$$

(this is the first equation of the BBGKY chain)

A better way to write the loop equation

Introduce

$$T(z) = \sum_{j\neq k} \frac{\beta^2\hbar^2}{(z-z_j)(z-z_k)} + \sum_j \frac{\beta\hbar^2}{(z-z_j)^2} + 2\beta\hbar \sum_j \frac{\partial W(z_j)}{z-z_j}$$

(an analog of the holomorphic component of the stress-energy tensor). In terms of the fields φ, ρ it is

$$\boxed{T(z) = (\partial\varphi(z))^2 + (1-\beta)\hbar\,\partial^2\varphi(z) + 2\beta \oint_\Gamma \frac{\partial W(\xi)\rho(\xi)}{z-\xi}\,|d\xi|}$$

The mean value $\langle T(z)\rangle$ is a holomorphic function in the exterior and interior of the contour Γ with a jump

$$\Big[\langle T(z)\rangle\Big]_\Gamma := \langle T(z_+)\rangle - \langle T(z_-)\rangle$$

The loop equation is

$$\boxed{\mathcal{I}m\left(\nu^2(z)\Big[\langle T(z)\rangle\Big]_\Gamma\right) = 0\,, \quad z \in \Gamma}$$

where $\nu(z) = -i dz/|dz|$ is the outward looking unit normal vector to Γ at the point z.

Example: the model on the real line R.

In this case $\nu^2 = -1$ and the jump of $\langle T\rangle$ is purely imaginary, so the loop equation says that the jump vanishes:

$$\Big[\langle T(z)\rangle\Big]_{\mathrm{R}} = \langle T(z+i0)\rangle - \langle T(z-i0)\rangle = 0$$

Since $\langle T(z)\rangle$ is holomorphic everywhere in the upper and lower half-planes and vanishes at infinity, this means that $\langle T(z)\rangle = 0$ everywhere in the plane.

Components of the stress-energy tensor:

$$4T_{nn} = \nu^2 T + \bar{\nu}^2 \bar{T} - 2\Theta$$

$$4T_{ss} = -\nu^2 T - \bar{\nu}^2 \bar{T} - 2\Theta$$

$$4T_{sn} = i\nu^2 T - i\bar{\nu}^2 \bar{T}$$

($\Theta := -\mathrm{tr}\, T$).

The loop equation is equivalent to the boundary condition for the mixed component:

$$\Big[\langle T_{sn}(z) \rangle \Big]_\Gamma = 0 \,, \qquad z \in \Gamma$$

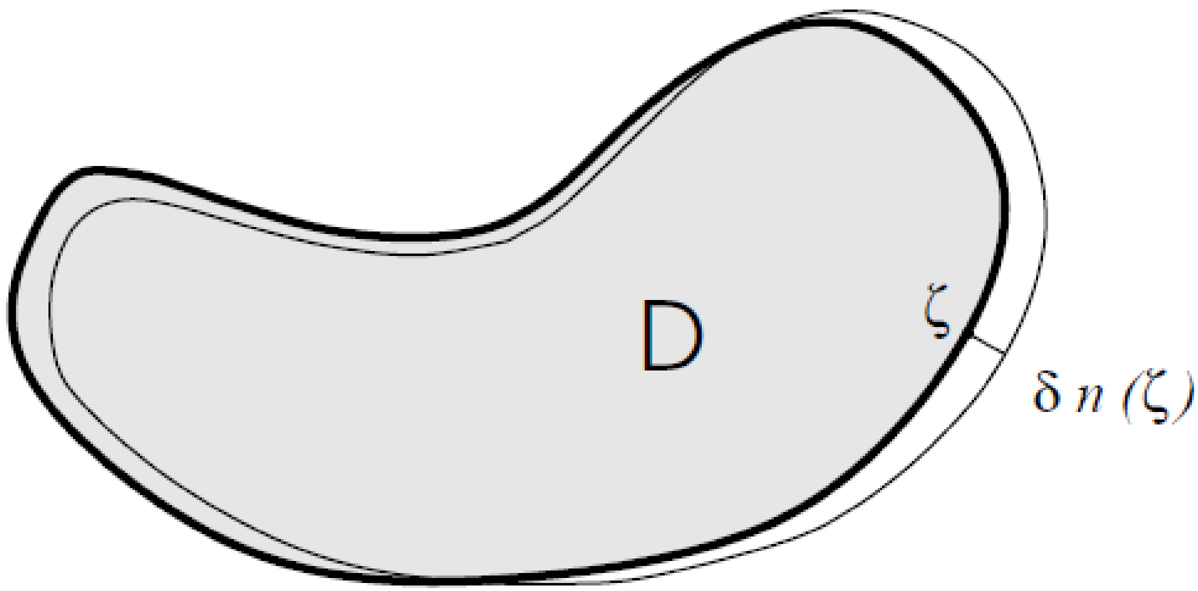
Different forms of the loop equation

$$\langle\rho(z)\rangle\Big(\partial_s\langle\varphi(z)\rangle-\partial_s W(z)\Big)-(\beta-1)\hbar\,\partial_s\langle\rho(z)\rangle$$
$$=\frac{1}{2\pi\beta}\mathcal{I}m\Big(\nu^2(z)\left[\langle(\partial\varphi(z))^2\rangle_{\mathrm{c}}\right]_\Gamma\Big)$$

$$\langle\varphi(z)\rangle=W(z)+\hbar(\beta-1)\log\langle\rho(z)\rangle+\lambda+O(\hbar^3)$$

$$\langle\rho(z)\rangle=\frac{t}{2\pi}|w'_{\mathrm{ext}}|+\frac{1}{4\pi\beta}\mathcal{N}_\Gamma W(z)$$
$$+\frac{(\beta-1)\hbar}{4\pi\beta}\mathcal{N}_\Gamma\log\langle\rho(z)\rangle+O(\hbar^3)$$

Variations of the contour



$\delta n(\xi)$ = normal displacement of the point $\xi \in \Gamma$ under a small deformation

The variation of the free energy:

$$\delta \log Z_N = -\frac{1}{2\pi\beta\hbar^2}\oint_\Gamma \mathcal{R}e\Big(\nu^2(z)\Big[\langle T(z)\rangle\Big]_\Gamma\Big)\,\delta n(z)\,|dz|$$

or $$\delta \log Z_N = -\frac{1}{\pi\beta\hbar^2}\oint_\Gamma \Big[\langle T_{nn}(z)\rangle\Big]_\Gamma \delta n(z)\,|dz|$$

Large N limit

The large N limit: $N \to \infty$, $\hbar \to 0$, $t := N\hbar$ fixed

$$Z_N = N!\, N^{(\beta-1)N} \exp\left(\frac{F_0}{\hbar^2} + \frac{F_{1/2}}{\hbar} + F_1 + O(\hbar)\right)$$

The $\hbar$-expansion of the mean density:

$$\langle \rho(z) \rangle = \rho_0(z) + \hbar \rho_{1/2}(z) + \hbar^2 \rho_1(z) + \dots$$

The leading order

The free energy:

$$F_0 = \beta t^2 \log r + tW^H(\infty) + \frac{1}{8\pi\beta} \oint_\Gamma W \mathcal{N}_\Gamma W \, |dz|$$

where

$$r = \lim_{z\to\infty} \left| \frac{z}{w_{\text{ext}}(z)} \right|$$

is the exterior conformal radius (Robin's constant).

The mean density:

$$\rho_0(z) = \frac{t}{2\pi}\,|w'_{\text{ext}}(z)| + \frac{1}{4\pi\beta}\mathcal{N}_\Gamma W(z)\,, \qquad z \in \Gamma$$

The 2-point function:

$$\langle \rho(z)\rho(\zeta)\rangle_{\text{c}} = \frac{\hbar^2}{4\pi\beta} N(z,\zeta)$$

where

$$N(z,\zeta) = \partial_{n_z}\partial_{n_\zeta} G_{\text{int}}(z,\zeta) + \partial_{n_z}\partial_{n_\zeta} G_{\text{ext}}(z,\zeta)$$

is the kernel of the Neumann jump operator.

The 2-point function of potentials:

$$\frac{\langle \varphi(z)\varphi(\zeta)\rangle_{\mathsf{c}}}{2\beta\hbar^2} \cong \begin{cases} G_{\mathsf{ext}}(z,\zeta) - G_{\mathsf{ext}}(z,\infty) - G_{\mathsf{ext}}(\infty,\zeta) - \log\dfrac{|z-\zeta|}{r}, & z,\zeta \in \mathbf{C}\setminus \mathsf{D} \\ G_{\mathsf{in}}(z,\zeta) - \log\dfrac{|z-\zeta|}{r}, & z,\zeta \in \mathsf{D} \\ -G_{\mathsf{ext}}(\infty,\zeta) - \log\dfrac{|z-\zeta|}{r}, & z \in \mathsf{D}, \zeta \in \mathbf{C}\setminus \mathsf{D} \end{cases}$$

Corollary:

$$\left\langle (\partial\varphi(z))^2 \right\rangle_{\mathrm{c}} \cong \begin{cases} \dfrac{\beta\hbar^2}{6}\{w_{\mathrm{out}}; z\}, & z \in \mathbb{C} \setminus \mathrm{D} \\ \dfrac{\beta\hbar^2}{6}\{w_{\mathrm{in}}; z\}, & z \in \mathrm{D} \end{cases}$$

where

$$\{w; z\} = \frac{w'''(z)}{w'(z)} - \frac{3}{2}\left(\frac{w''(z)}{w'(z)}\right)^2$$

is the Schwarzian derivative.

The next-to-leading order

The free energy:

$$F_{1/2} = (\beta - 1) \oint_{\Gamma} \rho_0 \log \rho_0 \, |dz|$$

The mean density:

$$\rho_{1/2} = \frac{\beta - 1}{4\pi\beta} \mathcal{N}_{\Gamma} \log \rho_0$$

$$\rho_0(z) = \frac{t}{2\pi} |w'_{\rm ext}(z)| + \frac{1}{4\pi\beta} \mathcal{N}_{\Gamma} W(z) \,, \qquad z \in \Gamma$$

Iterative solution of the loop equation

Here we assume that $W(z) = 0$, i.e. we study the partition function

$$Z_N = \oint_\Gamma \dots \oint_\Gamma \prod_{i<j} |z_i - z_j|^{2\beta} \prod_{k=1}^{N} |dz_k|$$

The "stress-energy tensor":

$$T(z) = (\partial\varphi(z))^2 + (1-\beta)\hbar\,\partial^2\varphi(z)$$

Some technical details.

We have the expansions:

$$\langle \rho \rangle = \rho_0 + \hbar \rho_{1/2} + \hbar^2 \rho_1 + \dots$$

$$\langle \varphi \rangle = \varphi_0 + \hbar \varphi_{1/2} + \hbar^2 \varphi_1 + \dots$$

$$\langle T \rangle = T_0 + \hbar T_{1/2} + \hbar^2 T_1 + \dots$$

where

$$T_0 = (\partial \varphi_0)^2$$

$$T_{1/2} = 2\partial \varphi_0 \partial \varphi_{1/2} + (1-\beta)\partial^2 \varphi_0$$

$$T_1 = (\partial \varphi_{1/2})^2 + (1-\beta)\partial^2 \varphi_{1/2} + 2\partial \varphi_0 \partial \varphi_1 + \omega$$

and

$$\omega = \lim_{\hbar \to 0} \hbar^{-2} \left\langle (\partial \varphi)^2 \right\rangle_c$$

The strategy of the iterative solution is as follows: first use the "loop equation"

$$\mathcal{I}m\left(\nu^2(z)[T_j(z)]_\Gamma\right) = 0,$$

to find φ_j on Γ, then extend it harmonically to the complex plane and find F_j from the variation

$$\delta F_j = -\frac{1}{2\pi\beta}\oint_\Gamma \mathcal{R}e\left(\nu^2(z)\,[T_j(z)]_\Gamma\right)\delta n(z)|dz|\,, \quad j = 0, \frac{1}{2}, 1$$

An input for each step of the iterations are the data obtained at the previous steps, so the procedure is well-defined.

The first 2 orders:

$$\boxed{\begin{aligned} F_0 &= \beta t^2 \log r \\ F_{1/2} &= (1-\beta)\,t\log r + \text{const} \end{aligned}}$$

(recall that $t = N\hbar$)

The result for F_1 at $\beta = 1$

$$F_1 = \frac{1}{24\pi} \oint_\Gamma \left(\log |w'_{\text{int}}| \partial_n \log \left| \frac{w'_{\text{int}}}{w^2_{\text{int}}} \right| - \log |w'_{\text{ext}}| \partial_n \log \left| \frac{w'_{\text{ext}}}{w^2_{\text{ext}}} \right| \right) |dz|$$

The Polyakov-Alvarez formula for determinants of Laplace operators in planar domains allows one to see that

$$F_1 = -\frac{1}{2} \log \det(-\Delta_{\text{int}}) - \frac{1}{2} \log \det(-\Delta_{\text{ext}})$$

Using the "gluing formula"

$$\det(-\Delta_{\text{int}}) \det(-\Delta_{\text{ext}}) \det(\mathcal{N}_\Gamma) \cong \text{const}$$

one can write the result as

$$F_1 = \frac{1}{2} \log \det(\mathcal{N}_\Gamma)$$

Effective action

"Path integral" representation of the partition function

$$Z = \int [D\rho] e^{\mathcal{A}[\rho]/\hbar^2}$$

with the effective action

$$\begin{aligned}
\mathcal{A}[\rho] \;=\;& \beta \oint_\Gamma \oint_\Gamma \rho(z) \log |z-\zeta| \, \rho(\zeta) |dz| \, |d\zeta| \\
&+ \oint_\Gamma W(z) \rho(z) |dz| \\
&+ \hbar(\beta-1) \oint_\Gamma \rho(z) \log \rho(z) |dz| \\
&+ \lambda \left(\oint_\Gamma \rho(z) |dz| - t \right)
\end{aligned}$$