# Partial Bergman kernels and the quantum Hall effect

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Mathematical aspects of QHE/QM2 Seminar

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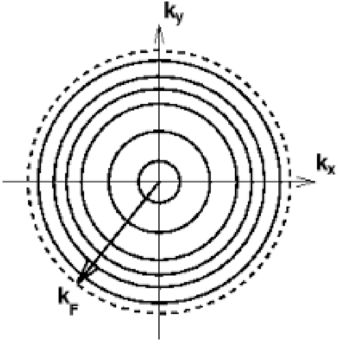
# Motivating problems from the QHE

Motivating question from the QHE (quantum Hall effect):

Let $M$ be a surface and let $D \subset M$ be a bounded domain in $M$. How do you fill a domain $D$ with quantum states in the LLL (lowest Landau level) with minimal spill-over into $M \backslash D$? Below, $M = \mathbb{C}$ and $D$ is the disc of radius $R^2$. The quantum states with Planck constant $\frac{1}{k}$ are $\psi_{k,j}(z) = \frac{(\sqrt{k}z)^j}{\sqrt{j!}} e^{-k|z|^2}$ with $j \leq R$.



# Fill up a domain with quantum states?

Roughly speaking, if the states are denoted $\psi_j$, then we want $\psi_j$ to be in the LLL (i.e. zero modes for the Landau Hamiltonian) and we want the *density profile*

$$\frac{1}{N}\sum_{j=1}^{N}|\psi_j(z)|^2 \simeq \mathbf{1}_D$$

to be approximately equal to the characteristic function $\mathbf{1}_D$ of $D$.

There are a number of inequivalent ways in which this question could be formulated mathematically.

# Mathematical interpretations

- In this talk, approach the filling problem in terms of the spectral theory of semi-classical Toeplitz operators and partial Bergman kernels. This was also motivated by work of J. Ross-M.Singer. It works in the general context of compact Kähler manifolds.
- A related approach is to consider the Toeplitz operator $\Pi_k \mathbf{1}_D \Pi_k$ where $\Pi_k$ is the Bergman kernel. It is almost a projection operator. The eigensections with eigenvalues close to 1 are localized in $D$. Closely related to work of Slepian, Pollack, Landau in the real domain; some results of Berman in the complex domain.
- One could try to construct a weight $e^{-\varphi}$ so that the full Bergman kernel for the weight is concentrated in a 'droplet' $D$ = support of the equiibrium measure assocaited to the weight (Berman, Wiegmann-Zabrodin, Makarov et al).

# How the physicists say it (for the disk)

Jainendra.K. Jain, *Composite Fermions* (p. 33): *The Landau level degeneracy can be obtained by considering a region of radius* $R$ *centered at the origin, and asking how many states lie inside it. For the lowest Landau level, the eigenstate* $|0, m\rangle$ *has its weight located at the circle of radius* $r = \sqrt{2m}\ell$*. Thus the largest value of* $m$ *for which the state falls inside our circular region is given by* $m = R^2/2\ell^2$*, which is also the total number of eigenstates in the lowest Landau level that fall inside the disk.*

# How the physicists say it

Xiao-Gang. Wen, *Quantum Field Theory of Many-Body Systems* (pages 285-287, paraphrasing):

*The density profile of the IQH wave functions: Let* $\ell_B^2 = \frac{\hbar}{|B|}$ *where* $|B|$ *is the magnetic field strength (*$B = \omega$ *in geometer's notation). Then the zeroth LLL state of angular momentum* $m$ *is*

$$\psi_m = \sqrt{N_m} z^m e^{-|z|^2/4\ell_B^2}.$$

*Then* $|\psi_m(z)|^2$ *concentrates in the mth ring centered at* $|z| = r_m = \sqrt{2m}\,\ell_B$. *The ring has area* $= 2\pi\ell_B^2$ *so the m states fill out an area of* $\pi r_m^2 = 2\pi m\ell_B^2$. *Then the density profile where the first m levels are filled has a graph similar to the characteristic function of* $[0, m\ell_B^2]$.

# What is this graph?

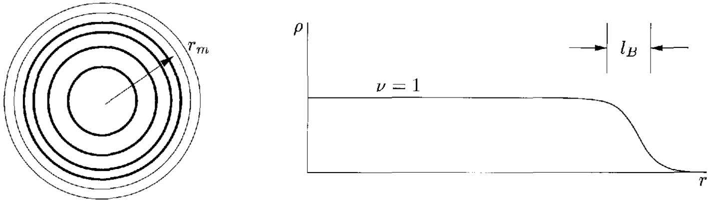



FIG. 7.11. The density profile of the $\nu = 1$ droplet, where the first $m$ levels (represented by the thick lines) are filled.

## Circular symmetry

The above descriptions are for the case where the domain $D$ is the disk of radius $R$. What if $D$ is a more complicated domain in $\mathbb{C}$ or in some other surface?

Wen writes: *The above many-body (Laughlin) wave function describes a circular droplet with uniform density because every electron occupies the same area $2\pi\ell_B^2$. This property is important. A generic wave function may not have a uniform density in the limit $N \to \infty$ and...does not have a sensible thermodynamic limit.*

Here, Wen is talking about the associated many-body wave function, which will not be considered in this talk. I quote it to illustrate what physicists say in cases where the domain is not circular.

In the first part of this talk, we will discuss the case of circular symmetry. In the second part we consider the general case with no symmetry.

# Mathematical definitions

We now define (i) LLL $= H^0(M, L^k)$, (iii) $\Pi_{h^k}$ and the partial density of states on a general Kähler manifold.

The lowest Landau level consists of the space $H^0(M, L^k)$ of holomorphic sections of the kth power of a (positive Hermitian) holomorphic line bundle $L \to M$ where $(M, \omega)$ is a Kähler manifold. For instance, $(M, \omega)$ could be any Riemann surface with any Riemannian metric.

# Bergman kernels on positive line bundles

This talk concerns the space $H^0(M, L^k)$ of holomorphic sections of the kth power of a positive Hermitian holomorphic line bundle $L \to M$ over a Kähler manifold $(M, \omega)$. The Hermitian metric is denoted by $h$ and in a local frame $e_L$ it is denoted by $|e_L(z)|_h^2 = e^{-\varphi}$. Positive Hermitian means that $i\partial\bar{\partial} \log h = \omega$ is a Kähler form. The key object in this talk is the orthogonal

projection,

$$\Pi_{h^k} : L^2(M, L^k) \to H^0(M, L^k)$$

with respect to the inner product

$$\langle s_1, s_2 \rangle := \int_M (s_1(z), s_2(z))_{h^k} \frac{\omega^m}{m!}.$$

The Schwartz kernel of $\Pi_{h^k}(z, w)$ relative to the volume form $\frac{\omega^m}{m!}$ is known as the semi-classcial Bergman kernel or Szego kernel.

# The motivating problem from the QHE

Suppose $\Omega \subset M$ is a domain in a kahler manifold $M^m$ of dimension $m$. We would like to fill it up with quantum states from $H^0(M, L^k)$, with no 'spill-over' into $M \backslash \Omega$. If the states are $\{s_j^k\}_{j=1}^{d_{k,\Omega}}$ then heuristically we want

$$k^{-m} \sum_{j=1}^{d_{k,\Omega}} |s_j^k(z)|_{h^k} \simeq C_m k^m \mathbf{1}_\Omega(z).$$

Here, $\mathbf{1}_\Omega$ is the characteristic function of $\Omega$. Also, $d_{k,\Omega}$ is the dimension of the relevant subspace.

Of course, this is not literally possible. How close can we come? What does the minimal 'spill-over look like".

# Partial Bergman kernels

Given Planck constants $\hbar_k = \frac{1}{k}$ and subspaces $\mathcal{S}_k \subset H^0(M, L^k)$,the

partial Bergman kernel is the projection operator

$$\Pi_{h^k, \mathcal{S}_k} : H^0(M, L^k) \to \mathcal{S}_k.$$

If $\{s_{k,j}\}_{j=1}^{d_k}$ is an orthonormal basis of $\mathcal{S}_k$ then

$$\Pi_{h^k, \mathcal{S}_k} = \sum_{j=1}^{d_k} s_{k,j}(z) \overline{s_{k,j}(w)}.$$

# Spectral partial Bergman kernels

The subspaces we consider are spectral subspaces

$$\mathcal{S}_k := \mathcal{H}_{k:[E_1,E_2]} := \bigoplus_{\mu_{k,j} \in H^{-1}([E_1,E_2])} V_{\mu_{k,j}} \tag{1}$$

of a quantum Hamiltonian $\hat{H}_k$ Here, $\mu_{k,j}$ are the eigenvalues of $\hat{H}_k$ and

$$V_k(\mu_{k,j}) := \{s \in H^0(M, L^k) : \hat{H}_k s = \mu_{k,j} s\}. \tag{2}$$

Given $\Omega$ we find a function $H : M \to \mathbb{R}$ so that

$$\Omega = \{z : E_1 \leq H(z) \leq E_2\}.$$

# Partial Density of States

Thus, the partial Bergman kernels are the spectral projections

$$\Pi_{h^k,[E_1,E_2]} : H^0(M, L^k) \to \mathcal{H}_{k:[E_1,E_2]}. \tag{3}$$

A simple result is that as $k \to \infty$,

$$k^{-m}\Pi_{h^k,[E_1,E_2]}(z,z) \to \mathbf{1}_{H^{-1}[E_1,E_2]}$$

and thus "fills the domain $H^{-1}[E_1, E_2]$ with lowest Landau levels". Thus, the partial density of states $k^{-m}\mathbf{1}_{[E_1,E_2]}(\hat{H}_k)(z,z)$ essentially equals 1 in $H^{-1}([E_1 E_2])$ and zero in the complement, giving a reasonable notion of "filling" the 'lowest Landau level' in the region $H^{-1}([E_1, E_2])$.

# Toeplitz quantization of a Hamiltonian

Let us assume $\Omega = \{z \in M : 0 \leq H(z) \leq E\}$ for some Hamiltonian $H : M \to \mathbb{R}$.

The quantization of $H$ is the self-adjoint zeroth order Toeplitz operator

$$\hat{H}_k := \Pi_{h^k}(\frac{1}{i2\pi k}\nabla_\xi + H)\Pi_{h^k} : H^0(M, L^k) \to H^0(M, L^k). \qquad (4)$$

Here, $\Pi_{h^k} : L^2(M, L^k) \to H^0(M, L^k)$ is the orthogonal (Bergman) projection, $\xi = \xi_H$ is the Hamilton vector field of $H$, $\nabla_\xi$ is the Chern covariant deriative on sections, and $H$ acts by multiplication.

# Recap on the spectral subspaces

Thus our partial Bergman kernels are the projections

$$\Pi_{h^k,[E_1,E_2]} : H^0(M, L^k) \to \mathcal{H}_{k:[E_1,E_2]}. \tag{5}$$

onto the subspaces

$$\mathcal{S}_k := \mathcal{H}_{k:[E_1,E_2]} := \bigoplus_{\mu_{k,j} \in H^{-1}([E_1,E_2])} V_{\mu_{k,j}} \tag{6}$$

where $\mu_{k,j}$ are the eigenvalues of $\hat{H}_k$ and

$$V_k(\mu_{k,j}) := \{s \in H^0(M, L^k) : \hat{H}_k s = \mu_{k,j} s\}. \tag{7}$$

Note: We could choose $H = \mathbf{1}_\Omega$. The discontinuity introduces new technical problems.

# Circular symmetric case

When the Hamiltonian flow of $H$ is $2\pi$-periodic, i.e. an $S^1$ action, the eigenvalues $\mu_{k,j} = \frac{j}{k}$ are integer multiples of the Planck constant $\frac{1}{k}$ lying in the range $P_0 := H(M) \subset \mathbb{R}$.

The eigenspaces $V_k(j) \subset H^0(M, L^k)$ of the $S^1$ action have dimensions of order $k^{m-1}$ and the 'equivariant Bergman kernels' $\Pi_{k,j} : H^0(M, L^k) \to V_k(j)$ resemble transverse Gaussian beams along the energy level $H^{-1}(\frac{j}{k})$, and have complete asymptotic expansions that can be summed over $j$ to give asymptotics of the density of states $\Pi_{h^k,[E_1,E_2]}(z,z)$ of partial Bergman kernels as $k \to \infty$. In the allowed region $H^{-1}([E_1, E_2])$,

$$k^{-m}\Pi_{h^k,[E_1,E_2]}(z,z) \simeq k^{-m}\Pi_{h^k}(z,z) \simeq 1$$

while in the complementary forbidden region the asymptotics are rapidly decaying.

# General case: Allowed and forbidden regions

THEOREM

*Let $\omega$ be a $C^\infty$ metric and let $H \in C^\infty(\mathbb{R})$. Let $P = [E_1, E_2] \subset H(M)$. Then the density of states of the partial Bergman kernel is given by the asymptotic formulas:*

$$k^{-m}\Pi_{|kP}(z,z) \sim \begin{cases} (1 + c_1 k^{-1} + c_2 k^{-2} + \cdots), & z \in \mathcal{A} = H^{-1}(P) \\ O(k^{-\infty}), \;\; z \notin H^{-1}(P) \end{cases}$$

*The asymptotics in the forbidden region $\mathcal{F} = M \backslash H^{-1}(P)$ are uniform on compact sets of $\mathcal{F}$.*

# Toric Kähler manifolds

Partial Bergman kernels first arose from toric Kähler manifolds $M^m$. Then $H^0(M, L^k)$ is spanned by monomials $z^\alpha$ where $\alpha \in kP \cap \mathbb{Z}^m$ is a lattice point in the kth dilate of the polytope $P$ corresponding to $M$, i.e. the image

$$\mu : M \to P$$

under the moment map. Subspaces may be defined by choosing sub-polytopes $P' \subset P$ which are 'Delzant'. The corresponding $z^\alpha$'s vanish to high order on the divisor at infinity.

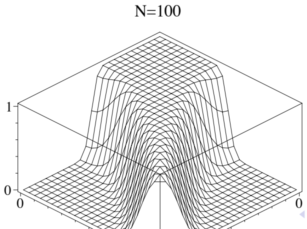
- Shiffman -Zelditch (2004) In the *allowed region* $\mathcal{A} := \mu^{-1}(P')$, the PBK asymptotics are the same as for the full Bergman kernel. In the *forbidden region* $\mathcal{F} := M \backslash \mu^{-1}(P')$, they are exponentially decaying. The decay rate is an explicit Agmon type function $b_{P'}$.
- (2014) Pokorny-Singer: Generalized the allowed asymptotics of (Sh-Z) to any toric Kahler manifold and toric divisor. Main novelty: distributional expansion of PBK on $\partial\mathcal{A}$.

# Density of states for a toric sub-polytope PBK


Figure : Gaussian decay from allowed to forbidden

# Allowed vs Forbidden

Allowed: the flat top; Forbidden the flat bottom.



# Interface results

The most interesting behavior occurs in $k^{-\frac{1}{2}}$-tubes around interfaces $H = E_1, H = E_2$. It suffices to work at one interface $\{H = E\}$ at a time. The tube of radius $k^{-\frac{1}{2}}$ around $\{H = E\}$ is the flowout of this energy surface under the gradient flow

$$F^t(z) := \exp t \frac{\nabla_\omega H(z)}{2\pi}$$

. Consider the small time flow $F^{\beta/\sqrt{k}}$ for $\beta \in [-\epsilon, \epsilon]$. Thus it suffices to study the partial density of states at points $F^{\beta/\sqrt{k}}(z_0)$ with $z_0 \in H^{-1}(E)$. The interface result is the same as for holomorphic $S^1$ actions:

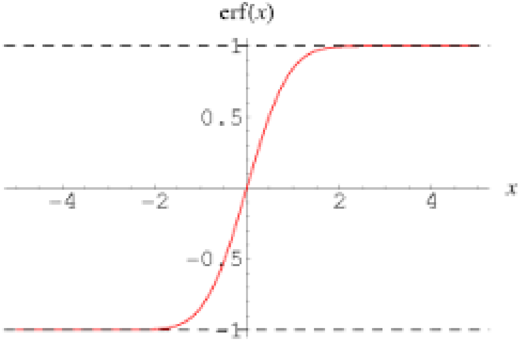
**Theorem**

*For $z_0 \in H^{-1}(E)$ and $\beta \in [-\epsilon, \epsilon]$,*

$$k^{-m} \sum_{\mu_{j,k} > E} \Pi_{k,j}(F^{\beta/\sqrt{k}} z_0) = \int_{-\infty}^{-\beta\sqrt{|\xi_H(z_0)|}} e^{-t^2/2} \frac{dt}{\sqrt{2\pi}} + O(k^{-1/2}).$$

# Incomplete Gaussian

The ‘limit shape’ of the interface is an complete Gaussian:




There is an *allowed region* where the PBK is almost 1 and the *forbidden region* where it is almost zero. The transition region has width $O(\frac{1}{\sqrt{k}})$. This picture is in most standard texts on QHE.

# Ideas of proof

Given a function $f \in \mathcal{S}(\mathbb{R})$ (Schwartz space) one defines

$$f(\hat{H}_k) = \int_{\mathbb{R}} \hat{f}(\tau) e^{i\tau \hat{H}_k} d\tau = \int_{\mathbb{R}} \hat{f}(t) U_k(t) dt, \tag{8}$$

to be the operator on $H^0(M, L^k)$ with the same eigensections as $\hat{H}_k$ and with eigenvalues $f(\mu_{k,j})$. Here,

$$U_k(t) = \exp itk\hat{H}_k. \tag{9}$$

is the unitary group on $H^0(M, L^k)$ generated by $\hat{H}_k$. Thus, if $s_{k,j}$ is an eigensection of $\hat{H}_k$, then

$$f(\hat{H}_k)\hat{s}_{k,j} = f\left(\mu_{k,j}\right) \hat{s}_{k,j} \tag{10}$$

# Partial Bergman kernels

Given an interval $[E_1, E_2] \subset P_0 = H(M)$ the subspace (6) is defined as the range of $f(\hat{H}_k)$ where $f = \mathbf{1}_{[E_1,E_2]}$ and the partial density of states is given by the metric contraction of the kernel,

$$\Pi_{|kP}(z, z) = f(\hat{H}_k)(z, z) = \sum_{j:\mu_{k,j} \in P} \Pi_{\mu_{k,j}}(z, z). \tag{11}$$

For a smooth test function $f$, it is the metric contraction of the Schwartz kernel of $f(\hat{H}_k)$ at $z = w$, is given by

$$f(\hat{H}_k)\hat{\Pi}_{h^k}(\hat{z}, \hat{w})|_{z=w} = \sum_{j:\mu_{k,j} \in P_0} f(\mu_{k,j}) \Pi_{\mu_{k,j}}(z, z). \tag{12}$$

# Interface result for smoothed partial Bergman kernel

**Theorem**

*Let $\omega$ be a $C^\infty$ Kähler metric, and let $H$ be a $C^\infty$ Hamiltonian. Fix $E \in H(M)$, and let $z = F^{\beta/\sqrt{k}} z_0$ for some $z_0 \in H^{-1}(E), \beta \in \mathbb{R}$, and let $f \in C_b(\mathbb{R})$. Then there exists a complete asymptotic expansion,*

$$\sum_{j:\mu_{k,j}\in P_0} f(\sqrt{k}(\mu_{k,j}-E))\Pi_{k,j}(F^{\beta/\sqrt{k}}z_0) \simeq k^m I_m(f,E) + k^{m-\frac{1}{2}} I_{m-\frac{1}{2}}(f,E) + \cdot$$

*in descending powers of $k^{\frac{1}{2}}$, with leading coefficient*

$$[I_m(f,E) = \lim_{k\to\infty} k^{-m} \sum_{j:\mu_{k,j}\in P_0} f(\sqrt{k}(\mu_{k,j}-E))\Pi_{k,j}(F^{\beta/\sqrt{k}}z_0)$$

$$= \int_{-\infty}^{\infty} f(x) e^{-\frac{1}{2}\left(\frac{2x\sqrt{\pi}}{|\nabla_H|} - \beta\frac{|\nabla H|}{\sqrt{\pi}}\right)^2} \frac{2dx}{\sqrt{2|\nabla H|(z_0)}}.$$

# Three types of smooth sums over eigenvalues

There are several 'localization scales' that may be studied rigorously:

- Interface localization: $\sum_{j:\mu_{k,j} \in P_0} f(\sqrt{k}(\mu_{k,j} - E))\Pi_{k,j}(z,z)$, with $z = F^{\beta/\sqrt{k}}(z_0)$ with $H(z_0) = E$.
- Energy range localization: $\sum_{j:\mu_{k,j} \in P_0} f(\mu_{k,j} - E)\Pi_{k,j}(z,z)$;
- Energy level localization: $\sum_{j:\mu_{k,j} \in P_0} f(k(\mu_{k,j} - E))\Pi_{k,j}(z,z)$;

The 'sharp versions' use characteristic functions $f = \mathbf{1}_{[E_0,E_1]}$. Tauberian arguments bridge smooth and sharp asymptotics.

# Toeplitz quantization of maps

A key step in the analysis is to construct

$$U_k(t) = \exp ikt\hat{H}_k$$

in a nice dynamical way.

**PROPOSITION**

*$\hat{U}_k(t,x,y)$ is a semi-classical Fourier integral operator. There exists an analytic symbol $\sigma_{k,t}$ so that if $\pi(x) = z$, the unitary group (9) has the form*

$$\begin{aligned} U_k(t,z,z) &= \hat{U}_k(t,x,x) := \hat{\Pi}_k(\hat{g}^{-t})^* \sigma_{k,t} \hat{\Pi}_k(x,x) \\ &= \hat{\Pi}_k e^{2\pi ik \int_0^t H(\exp sX_H(z))ds)} \quad (\exp tX_H^h)^* \sigma_{k,t} \hat{\Pi}_k(x,x). \end{aligned} \tag{13}$$

# Expression for interface asymptotics

The smoothed interface asymptotics thus amount to the asymptotics of the dilated sums,

$$\sum_j f(\sqrt{k}(\mu_{k,j} - E))\Pi_{k,j}(f^{\beta/\sqrt{k}}(z_0))$$

$$= \int_{\mathbb{R}} \hat{f}(t) e^{-iE\sqrt{k}t} \hat{\Pi}_{h^k} \sigma_{kt} (\hat{g}^t)^* \hat{\Pi}_{h^k}(f^{\beta/\sqrt{k}}(z_0)) dt$$

where $z_0 \in \partial \mathcal{A} = H^{-1}(E_1)$ and where $\hat{f} \in L^1(\mathbb{R})$, so that the integral on the right side converges. We employ the Boutet-de-Monvel-Sjostrand parametrix to give an explicit formula for the right side.

# Boutet de Monvel-Sjostrand parametrix

The projections $\Pi_{h^k}$ onto $H^0(M, L^k)$ lift to projections $\hat{\Pi}_{h_k}$ on the principal $S^1$ bundle $\partial D_h^* \subset L^*$ where $D_h^* = \{(z, \lambda) : |\lambda|_{h_z} < 1\}$. This is a strictly pseudo-convex domain in $L^*$. The sum $\Pi = \sum_{k \geq 0} \hat{\Pi}_{h_k}$ is the true Szego kernel

$$\hat{\Pi} : L^2(\partial D_h^*) \to H^2(\partial D_h^*)$$

onto boundary values of holomorphic functions on $D_h^*$

Near the diagonal in $\partial D_h^* \times \partial D_h^*$, the Boutet de Monvel-Sjostrand parametrix is:

$$\hat{\Pi}(x, y) = \int_0^\infty e^{-\sigma \psi(x,y)} \chi(x, y) s(x, y, \sigma) d\sigma + \hat{R}(x, y). \qquad (14)$$

Here, $\chi(x, y)$ is a smooth cutoff to the diagonal; $s(x, y, \sigma)$ is a semi-classical symbol of order $m = \dim_{\mathbb{C}} M$.

# The phase

When the Kähler metric $\omega$ is real analytic, the phase $\psi$ is constructed from the Kähler potential $\varphi(z)$ of $\omega_0$ by

$$\psi(x, y) = \psi((z, \lambda), (w, \mu)) = 1 - \lambda\bar{\mu}e^{\varphi(z, \bar{w})} \qquad (15)$$

where $\varphi(z, \bar{w})$ is the analytic extension of $\varphi(z) = \varphi(z, \bar{z})$ into the complexification $M \times \bar{M}$ of $M$. Also,

$$s \sim \sum_{n=0}^{\infty} \sigma^{m-n} s_n(x, y) \qquad (16)$$

is an analytic symbol in the sense of Boutet de Monvel. Finally, the remainder term $\hat{R}(x, y)$ is real analytic in a neighborhood of the diagonal. If $\omega$ is only $C^\infty$ then $\psi(z, w)$ is defined by an almost-analytic extension and the remainder $R$ is $C^\infty$.

# Osculating Bargmann Fock representations

At each $z \in M$ there is an osculating Bargmann-Fock or Heisenberg model associated to $(T_z M, J_z, h_z)$. We denote the model Heisenberg Bergman kernel on the tangent space by

$$\Pi_{h_z, J_z}^{T_z M}(u, \theta_1, v, \theta_2) : L^2(T_z M) \to \mathcal{H}(T_z M, J_z, h_z) = \mathcal{H}_J. \qquad (17)$$

In K-coordinates with respect to a K-frame,

$$\Pi_{h_z, J_z}^{T_z M}(u, \theta_1, v, \theta_2) = \pi^{-m} e^{i(\theta_1 - \theta_2)} e^{u \cdot \bar{v} - \frac{1}{2}(|u|^2 + |v|^2)}$$

$$= \pi^{-m} e^{i(\theta_1 - \theta_2)} e^{i \Im u \cdot \bar{v} - \frac{1}{2}(|u - v|^2}$$

Note that $\Im u \cdot \bar{v} = \omega(u, v)$.

# Linearization approximation

**Proposition**

*In K-coordinates in a K-frame at z,*

$$\hat{\Pi}_{h^k}(\hat{g}^{\frac{t}{\sqrt{k}}} z, z) \simeq k^m e^{2\pi t H(z)} \Pi^{T_z M}_{J_z, h_z}(t\xi_H, 0, 0, 0)\left(1 + k^{-1} A_t + \cdots\right).$$

**Proposition**

*In K-coordinates in a K-frame at z,*

$$\hat{U}_k(t/\sqrt{k}, z, \theta, z, \theta) \quad = \quad k^m e^{2\pi i t\sqrt{k} H(z))} e^{-|t\xi_H(z)|^2}[1 + O(k^{-\frac{1}{2}})]$$

# Putting it together

## Lemma

$$\sum_j f\sqrt{k}(\mu_{k,j} - E))\Pi_{k,\mu_{k,j}}(F^{\frac{u}{\sqrt{k}}} z_0, F^{\frac{u}{\sqrt{k}}} z_0)$$

$$= \int_{\mathbb{R}} \hat{f}(t) e^{-iE\sqrt{k}t} \hat{U}_k(t/\sqrt{k}, F^{\frac{u}{\sqrt{k}}} z_0, g^{\frac{u}{\sqrt{k}}} F^{\frac{u}{\sqrt{k}}} z_0) dt$$

$$= k^m \int_{-\infty}^{\infty} \hat{f}(t) e^{2\pi i t\sqrt{k} H(z))} e^{-iE\sqrt{k}t} e^{-|t\xi_H(z)|^2} \, dt \, [1 + O(k^{-\frac{1}{2}})] \, ,$$

By the Plancherel formula,

$$k^{-m} \sum_{j=1}^{d_k} f(\sqrt{k}(\mu_{k,j} - E))\Pi_{k,j}(z_k, z_k)$$

$$= \int_{-\infty}^{\infty} f(x) e^{-\frac{1}{2}\left(\frac{2x\sqrt{\pi}}{|\nabla H(z_k)|} - \beta\frac{|\nabla H(z_k)|}{\sqrt{\pi}}\right)^2} \frac{2dx}{\sqrt{2|\nabla H(z_0)|}} + O(k^{-1/2}.$$