Conformal Scaling Limit
of the Network Model
for the Integer Quantum Hall Transition

Martin R. Zirnbauer (U of Cologne)
@ Rutgers Stat. Mech. Conference
(May 7, 2017)
Integer Quantum Hall Effect

Two-dimensional disordered electron gas at low temperature and in a strong magnetic field. Hall resistance exhibits plateaus: \( R_H = \frac{h}{ne^2} \)

Transition between plateaus is a critical phenomenon (of Anderson-localization type). Could/should be a paradigm, but is not understood in quantitative detail ...
Nonlinear sigma model \cite{Pruisken1983}

\[ L = \frac{\sigma_{xx}}{8} \text{Str} \, \partial_\mu Q \, \partial_\mu Q \\
+ \frac{\sigma_{xy}}{8} \, \epsilon_{\mu\nu} \, \text{Str} \, \partial_\nu Q \, \partial_\mu Q \, \partial_\nu Q \]

\[ \text{weak localization} \]

\[ \Theta \text{-term} \]

Wegner-Efetov SUSY method in target space
\[ Q = \eta \Sigma_3 \eta^{-1}, \quad \Sigma_3 = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} \]

is complex Grassmann manifold $\mathcal{U}/K$

with global symmetry group $\mathcal{U} = \mathcal{U}(r, r | 2r)$

\[ \text{CFT} = ? \]

\[ \text{Pruisken-Khmelnitskii RG flow diagram} \]

\[ \text{(conjectured, 1983)} \]
The Conundrum
Apparent Conflict

- Symmetry group $U = U(r, r|2r)$ must act (by conjugation) on target $X$.

- All known CFTs with continuous symmetries are Wess-Zumino-Witten models or Goddard-Kent-Olive coset theories ($=\text{gauged WZW models}$).

- WZW models (with non-compact target $X$) are ruled out by
  - RG-instability against infinity of relevant $U$-invt perturbations
  - exact results for related critical point ($\text{class C}$)
  
  Mudry, Chamon & Wen, 1995
  Read & Saleur, 2001

- Coset theories $X/H$ with $H \subset U$ greater than center of $U$ are
  ruled out by $U$-symmetry.

- Gauging by center $H = U(1)$ does not suffice to remove RG instability.
WZW model - operator formalism

WZW field \( M : \Sigma \rightarrow X \).

Left and right translations \( M \mapsto g_L M g_R^{-1} \) \((g_L, g_R \in G)\) give rise to

Currents \( J^A = k \text{Str} \left( A \partial M \cdot M^{-1} \right) \) holomorphic

\( \bar{J}^A = k \text{Str} \left( A M^{-1} \bar{\partial} M \right) \) anti-holomorphic.

Current algebra \( \hat{G} \). Operator product expansion:

\[
J^A(z) J^B(w) \sim -\text{Str}(AB) \frac{k}{(z-w)^2} + \frac{J^{[A,B]}(w)}{z-w}
\]

Energy-momentum tensor (Sugawara) \( T = \frac{1/2}{k+c_V} (\bar{J}^e_i J^e_i) \)

is Virasoro: \( T(z) T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2 T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \)

due to \( T(z) J^A(w) \sim \frac{J^A(w)}{(z-w)^2} + \frac{\partial J^A(w)}{z-w} \)
GKO coset CFT (gauged WZW model)

Subgroup $H \subset G$ with anomaly-free action $M \mapsto h_L M h_R^{-1}$

Current subalgebra $\hat{h} \subset \hat{g}$.

Energy-momentum tensor $T^{\hat{g}/\hat{h}} := T^{\hat{g}} - T^{\hat{h}}$ generates Virasoro algebra with central charge $c_{g/h} = c_g - c_h$.

NOTE: gauging by $H$-action is incompatible with action $M \mapsto u M u^{-1}$ by symmetry group $U$.

How to get around this difficulty? Totally new ideas needed??
"Internal" coset construction

Observation. Geometrically, current subalgebras are associated with distributions (i.e. smooth subbundles) of the tangent bundle $TX$.

$U$ acts on $X \equiv X_{r, r|2r}$, hence on $TX$.
To gauge the WZW model without ruining the $U$-symmetry, one needs a $U$-invariant distribution in $TX$.

Fact. Such distributions do exist.

Hint. View the WZW target space as an associated bundle over the nonlinear sigma model target space:

$X_{r, r|2r} = U \times_K (X^+_r \times X^-_r) \hookrightarrow U/K$,

$M = u \begin{pmatrix} M^+ & 0 \\ 0 & M^- \end{pmatrix} u^{-1} = uh^{-1}(h^+M^+h^+ \cdot h^-M^-h^-)h u^{-1}, \quad h \in K$.

 superseding Pruiken's $u \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Gauge by $K$!
Gauged WZW action functional:

\[
S_{k}^{\text{WZW}}[M; A = A^{10} + A^{01}] = \frac{i k}{4 \pi} \int_{\Sigma} \left( \text{Str} \, M^{-1} \partial M \wedge M^{-1} \bar{\partial} M + \frac{1}{3} d^{-1} \text{Str} (M^{-1} d M)^{\wedge 3} \right) \\
+ \frac{i k}{2 \pi} \int_{\Sigma} \text{Str} \left( \partial M \cdot M^{-1} \wedge A^{01} - A^{10} \wedge M^{-1} \bar{\partial} M - A^{10} \wedge M^{-1} A^{01} M + A^{10} \wedge A^{01} \right). 
\]

Let there be two copies of \( M \) for retarded sector \((M_+^+)\) and advanced sector \((M_-)_-\).

Proposed fixed-point action = \( S_{k=4}^{\text{WZW}}[M_+; A_+] + S_{k=4}^{\text{WZW}}[M_-; A_-] \)

\[
+ \frac{i k}{2 \pi} \int_{\Sigma} \text{Str} \left( (u^{-1} \partial u)_{+-} \wedge (u^{-1} \bar{\partial} u)_{-+} - (u^{-1} \partial u)_{++} \wedge M_{-}^{-1} (u^{-1} \bar{\partial} u)_{-+} M_{+} \\
+ (u^{-1} \partial u)_{-+} \wedge (u^{-1} \bar{\partial} u)_{++} - (u^{-1} \partial u)_{++} \wedge M_{+}^{-1} (u^{-1} \bar{\partial} u)_{-+} M_{-} \right). 
\]

Invariance under gauge transformations

\[
M_{s} \mapsto h_{s} M_{s} h_{s}^{-1}, \quad A_{s} \mapsto h_{s} A_{s} h_{s}^{-1} - d h_{s} \cdot h_{s}^{-1}, \quad u \mapsto u \text{ diag}(h_{+}, h_{-})^{-1} 
\]

Interpretation: nonlinear sigma model coupled to maximally gauged WZW models
Network Model

(Chalker & Coddington)
Hilbert space = $\mathbb{C}^{\#\text{links}}$

discrete time evolution $\psi_{t+1} = U \psi_t$

$U = U_r U_s$

$U_r$ diagonal in link basis:
$U_r |l\rangle = |l\rangle e^{i\phi(l)}$

$\phi(l)$ uncorrelated random phases, uniformly distributed

$U_s$ deterministic scattering at nodes:
$U_s |l\rangle = |l_+\rangle a_+ + |l_-\rangle a_-$

$a_\pm = e^{i\pi/4}/\sqrt{2}$
(at criticality)
Gaussian Free Field
Statistics of wave intensities.

\[ \Psi_c = U \Psi_c \] stationary "scattering" state (quasi-energy zero) for incoming wave boundary conditions \[ \Psi_c(c) = 1. \]

\( \rho \) Observable: \[ |\Psi_c(r)|^2 \] for large distances \( |r - c| \)

Prediction from Abelian OPE, crossing symmetry of 4-point fn:

\[ \mathbb{E}(|\Psi_c(r)|^{2q}) \sim |r - c|^{-2\Delta_q}, \quad \Delta_q = X q (1-q). \]

Interpretation: \( \log |\Psi_c(r)|^2 \equiv \phi(r) \)

GTT with background charge \( Q = 1 \), stiffness \( X \).

Numerical simulations give \( X \approx 0.26 (\ldots 0.28) \) \( \sim \) Hypothesis \( X = 1/r \) (level \( k = 4 \))
SUSY Vertex Model
CC network model $\rightarrow$ SUSY vertex model

A variant (due to N. Read) of the Wegner-Efetov supersymmetry method trades the task of taking disorder averages for a statistical mechanics problem of new (collective) variables that admit a continuum limit (at the critical point).

Uses second quantization on a Fock space for bosons and fermions.

**Retarded sector:** $U = e^X$, $\text{Re} X < 0$.

**Bosons:** $\text{Det}^{-1}_{\mathbb{C}^n}(1-U) = \text{Tr}_{S(\mathbb{C}^n)} \phi_b(U)$ where $\phi_b(e^X) = e^{b^t X b}$

**Fermions:** $\text{Det}(1-U) = \text{Str} \phi_f(U)$ where $\phi_f(e^X) = e^{f^t X f}$

**Advanced sector:** $U = e^X$, $\text{Re} X > 0$. $b^t \rightarrow -b$, $b \rightarrow b^t$, $f^t \rightarrow +f$, $f \rightarrow f^t$. 
Random phase average projects Fock space to subspace of $U(1)$ singlets

\[ \mathcal{H} = \bigotimes_{\text{links}} V(l) \]  
state space of SUSY vertex model

where \[ V = \bigoplus \left( S^{n_0^+}(C) \otimes \Lambda^{n_i^+}(C) \otimes S^{n_i^-}(C^*) \otimes \Lambda^{n_i^-}(C^*) \right) \]

\[ n_0^+ + n_i^+ = n_0^- + n_i^- \]

is irreducible highest-weight representation for $U = U(1,1|2)$. $r$ replicas: $U = U(r,r|2r)$

**Unit cell**

interpretation $\sim$ Feynman paths

\[ E\left( |\psi_c^*(r) |^2\right) = \left< \pi_{\text{vac}}(c) \left( B^i B^i \right)^q(r) \right>_{\text{v.h.}} \]

\[ \left< A \right>_{\text{v.h.}} = \text{STr} A \rho(U_s) \]

Global symmetry $U = U(r,r|2r)$
Continuum Limit from
Dirac Approximation
Pure network model \((U = U_s)\) at long wave lengths

\[1 - U_s \approx iD\] (Dirac operator \& discrete holomorphic derivative)

Treat the network model with random phase disorder as a perturbation ...

\(\lambda \) (SUSY) Dirac theory with random gauge potential

and random scalar potential

\(\mathbb{Z}_4\) spectral symmetry

\(\lambda \) 4 Dirac points (at 4th roots of unity)

\[e^{im\pi/2} \cdot 1 - U_s = iD_m \quad (m = 0,1,2,3)\]
Preparatory step:

Assume $A$ with values in $SU(k) \subset U(k)$ ($k = 4$).

Bosonize.

CFT describing continuum limit of disordered critical point is (Bernard & LeClair 2002; Tsvelik et al. 2001):

"$GL(2r|2r)$" WZW model of level $k = 4$.

Target space = supermanifold over Riemannian symmetric space of type A1A

\[
\left( \frac{GL(2r)/U(2r)}{\text{noncompact}} \right) \times U(2r) \quad \text{compact}
\]

What happens when the $U(1)$ gauge field and the random scalar potential are turned on?
Gauged WZW action functional:

\[
S_{k}^{WZW}[M; A=A^{10}+A^{01}] = \frac{ik}{4\pi} \int_{\Sigma} \left( \text{Str} \; M^{-1} \partial M \wedge M^{-1} \bar{\partial} M + \frac{1}{3} e^{-1} \text{Str} \; (M^{-1} e M)^{3} \right) + \frac{ik}{2\pi} \int_{\Sigma} \text{Str} \left( e \partial M \wedge A^{01} - A^{10} \wedge M^{-1} \bar{\partial} M - A^{10} \wedge M^{-1} A^{01} M + A^{10} \wedge A^{01} \right).
\]

Let there be two copies of \( M \) for retarded sector \( (M_{+}) \) and advanced sector \( (M_{-}) \).

Proposed fixed-point action = \( S_{k=+}^{WZW}[M_{+}; A_{+}] + S_{k=+}^{WZW}[M_{-}; A_{-}] \)

\[
+ \frac{ik}{2\pi} \int_{\Sigma} \text{Str} \left( (u^{-1} \partial u)_{++} \wedge (u^{-1} \bar{\partial} u)_{++} - (u^{-1} \partial u)_{+-} \wedge M_{-}^{-1} (u^{-1} \bar{\partial} u)_{-+} M_{+} \\
+ (u^{-1} \partial u)_{--} \wedge (u^{-1} \bar{\partial} u)_{--} - (u^{-1} \partial u)_{-+} \wedge M_{+}^{-1} (u^{-1} \bar{\partial} u)_{+-} M_{-} \right).
\]

Invariance under gauge transformations

\[
M_{s} \rightarrow h_{s} M_{s} h_{s}^{-1}, \; A_{s} \rightarrow h_{s} A_{s} h_{s}^{-1} - dh_{s} \cdot h_{s}^{-1}, \; u \rightarrow u \; \text{diag}(h_{+}, h_{-})^{-1}
\]

Remark: predicts multifractal scaling exponents \( \Delta_{q} = \frac{1}{k} q (1-q) \).
Thank you!