Conformal Scaling Limit of the Network Model for the Integer Quantum Hall Transition

Martin R. Zirnbauer (U of Cologne)

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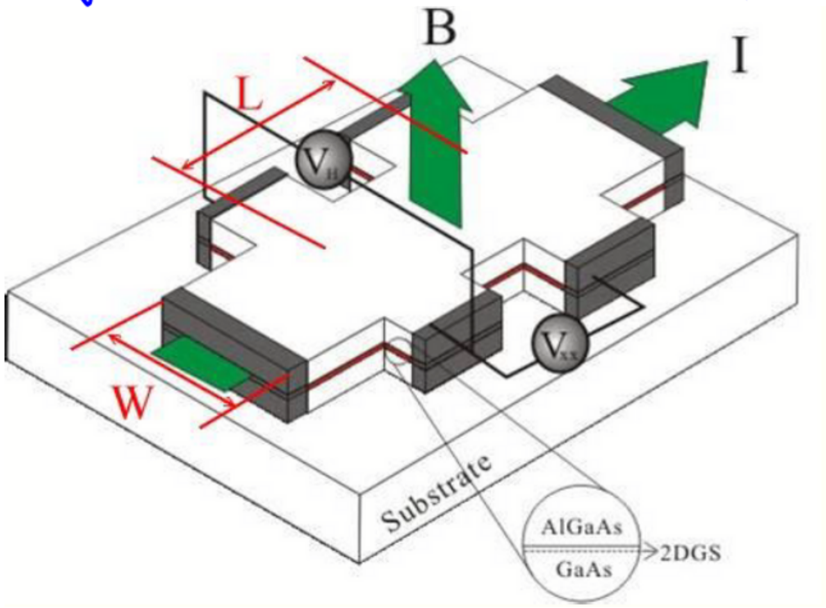
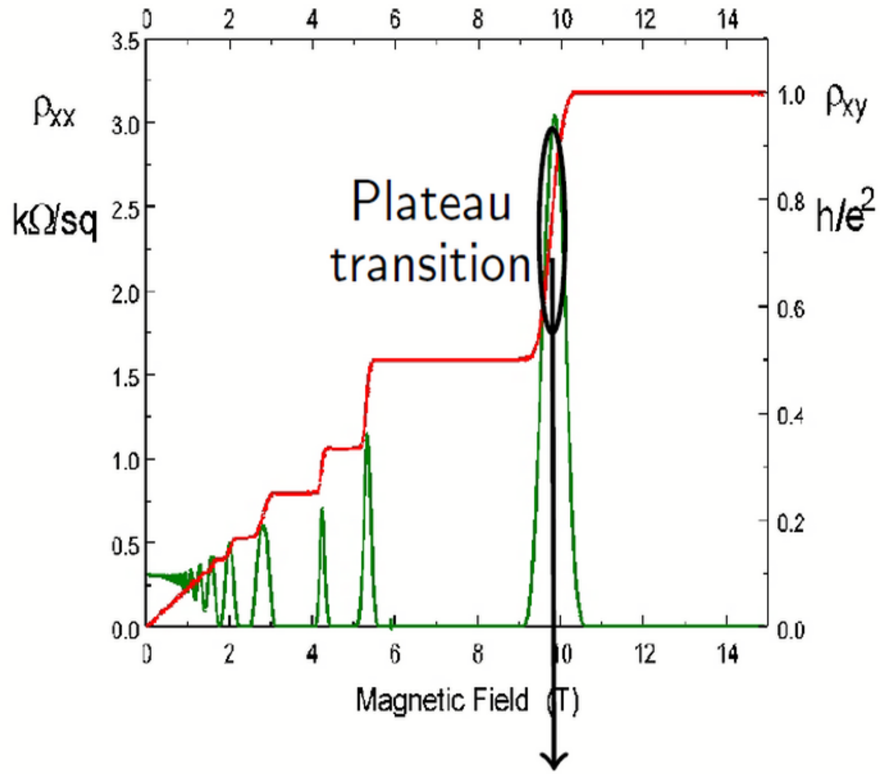
Integer Quantum Hall Effect





Two-dimensional disordered electron gas at low temperature and in a strong magnetic field.

Hall resistance exhibits plateaus: $R_H = \frac{h}{ne^2}$

Transition between plateaus is a critical phenomenon (of Anderson-localization type).

Could/should be a paradigm, but is not understood in quantitative detail ...

Nonlinear sigma model (Pruisken et al., 1983)

weak localization

$$\mathcal{L} = \frac{\sigma_{xx}}{8} \, \mathrm{STr} \, \partial_\mu Q \, \partial_\mu Q + \frac{\sigma_{xy}}{8} \epsilon_{\mu\nu} \, \mathrm{STr} \, Q \, \partial_\mu Q \, \partial_\nu Q$$

θ-term

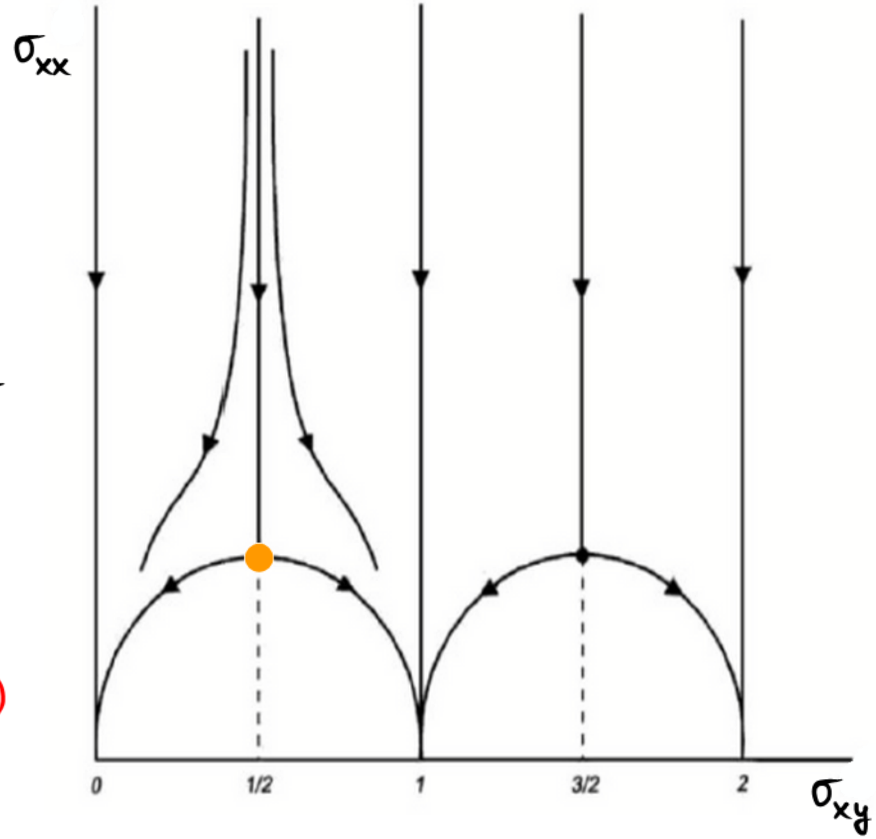
Wegner–Efetov SUSY method ⇝ target space

$$Q = u \, \Sigma_3 \, u^{-1}, \qquad \Sigma_3 = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}$$

is complex Grassmann manifold $\mathcal{U}/K$

with global symmetry group $\mathcal{U} = U(r, r|2r)$

CFT = ?



Pruisken–Khmelnitskii RG flow diagram (conjectured, 1983)

The Conundrum

Apparent Conflict

- Symmetry group $\mathcal{U} = U(r, r|2r)$ must act (by conjugation) on target X.

- All known CFTs with continuous symmetries are Wess-Zumino-Witten models or Goddard-Kent-Olive coset theories (= gauged WZW models).

- WZW models (with non-compact target X) are ruled out by
  - RG-instability against infinity of relevant $\mathcal{U}$-invt perturbations
  - exact results for related critical point (class C)

  Mudry, Chamon & Wen, 1995
  Read & Saleur, 2001

- Coset theories X/H with $H \subset \mathcal{U}$ greater than center of $\mathcal{U}$ are ruled out by $\mathcal{U}$-symmetry.

- Gauging by center $H = U(1)$ does not suffice to remove RG instability.

WZW model – operator formalism

WZW field $M : \Sigma \to X$.

Left and right translations $M \mapsto g_L M g_R^{-1}$ $(g_L, g_R \in G)$ give rise to

Currents $J^A = k\, \mathrm{STr}(A\, \partial M \cdot M^{-1})$ holomorphic

$\bar{J}^A = k\, \mathrm{STr}(A\, M^{-1} \bar{\partial} M)$ anti-holom.

Current algebra $\hat{g}$. Operator product expansion:

$$J^A(z)\; J^B(w) \sim -\mathrm{STr}(AB)\, \frac{k}{(z-w)^2} + \frac{J^{[A,B]}(w)}{z-w}$$

Energy-momentum tensor (Sugawara) $T = \frac{1/2}{k + c_V} (J^{e_i} J^{e_i})$

is Virasoro: $T(z)\, T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}$

due to $T(z)\, J^A(w) \sim \frac{J^A(w)}{(z-w)^2} + \frac{\partial J^A(w)}{z-w}$

GKO coset CFT (gauged WZW model)

Subgroup $H \subset G$ with anomaly-free action $M \mapsto h_L M h_R^{-1}$

Current subalgebra $\hat{\mathfrak{h}} \subset \hat{\mathfrak{g}}$.

Energy-momentum tensor $T^{\hat{\mathfrak{g}}/\hat{\mathfrak{h}}} := T^{\hat{\mathfrak{g}}} - T^{\hat{\mathfrak{h}}}$ generates Virasoro algebra with central charge $c_{\mathfrak{g}/\mathfrak{h}} = c_{\mathfrak{g}} - c_{\mathfrak{h}}$.

NOTE: gauging by H-action is incompatible with action $M \mapsto u M u^{-1}$ by symmetry group $\mathcal{U}$.

How to get around this difficulty?
Totally new ideas needed ??

"Internal" coset construction

Observation. Geometrically, current subalgebras are associated with distributions (i.e. smooth subbundles) of the tangent bundle TX.

$\mathcal{U}$ acts on $X \equiv X_{r,r|2r}$, hence on TX.
To gauge the WZW model without ruining the $\mathcal{U}$-symmetry, one needs a $\mathcal{U}$-invariant distribution in TX.

Fact. Such distributions do exist.

Hint. View the WZW target space as an associated bundle over the nonlinear sigma model target space:

$$X_{r,r|2r} = \mathcal{U} \times_K (X^+_{r|r} \times X^-_{r|r}) \mapsto \mathcal{U}/K ,$$

$$M = u \begin{pmatrix} M_+ & 0 \\ 0 & M_- \end{pmatrix} u^{-1} = u k^{-1} \begin{pmatrix} k_+ M_+ k_+^{-1} & 0 \\ 0 & k_- M_- k_-^{-1} \end{pmatrix} k u^{-1}, \quad k \in K .$$

superseding Pruisken's $u \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} u^{-1}$

Gauge by K!

Gauged WZW action functional:

notation: $d = \partial + \bar\partial$, $\partial = dz \wedge \frac{\partial}{\partial z}$

$$S_k^{WZW}[M; A = A^{10} + A^{01}] = \frac{ik}{4\pi} \int_\Sigma \Big(\mathrm{STr}\, M^{-1}\partial M \wedge M^{-1}\bar\partial M + \tfrac{1}{3} d^{-1} \mathrm{STr}\,(M^{-1}dM)^{\wedge 3} \Big)$$

$$+ \frac{ik}{2\pi} \int_\Sigma \mathrm{STr}\, \Big(\partial M \cdot M^{-1} \wedge A^{01} - A^{10} \wedge M^{-1}\bar\partial M - A^{10} \wedge M^{-1} A^{01} M + A^{10} \wedge A^{01} \Big).$$

Let there be two copies of M for retarded sector (M_+) and advanced sector (M_-).

Riemannian symmetric superspace of type A|A

Proposed fixed-point action $= S_{k=4}^{WZW}[M_+; A_+] + S_{k=4}^{WZW}[M_-; A_-]$

$$+ \frac{ik}{2\pi} \int_\Sigma \mathrm{STr} \Big((u^{-1}\partial u)_{+-} \wedge (u^{-1}\bar\partial u)_{-+} - (u^{-1}\partial u)_{+-} \wedge M_-^{-1} (u^{-1}\bar\partial u)_{-+} M_+$$

$$+ (u^{-1}\partial u)_{-+} \wedge (u^{-1}\bar\partial u)_{+-} - (u^{-1}\partial u)_{-+} \wedge M_+^{-1} (u^{-1}\bar\partial u)_{+-} M_- \Big).$$

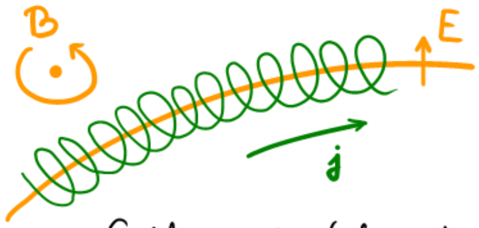
Invariance under gauge transformations

$$M_s \mapsto h_s M_s h_s^{-1}, \quad A_s \mapsto h_s A_s h_s^{-1} - dh_s \cdot h_s^{-1}, \quad u \mapsto u\, \mathrm{diag}(h_+, h_-)^{-1}$$

Interpretation: nonlinear sigma model coupled to maximally gauged WZW models

Network Model

(Chalker & Coddington)



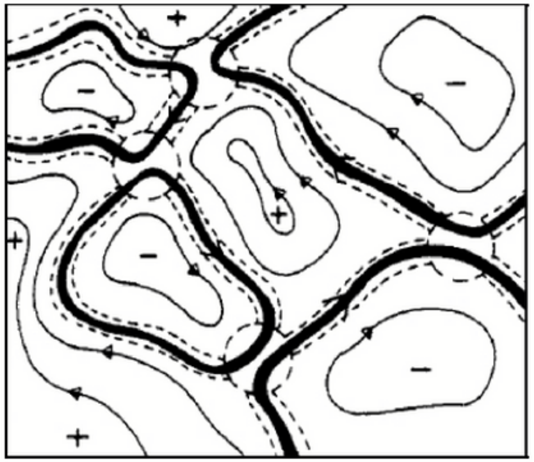
Guiding center (of rapid cyclotron motion)

drifts slowly along equipotential contours

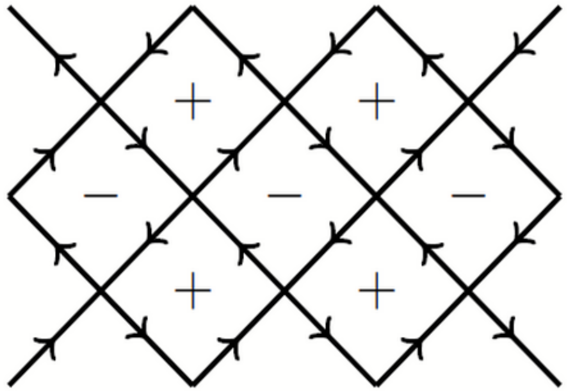
Hilbert space $= \mathbb{C}^{\#\text{links}}$

discrete time evolution $\psi_{t+1} = U \psi_t$

$U = U_r U_s$

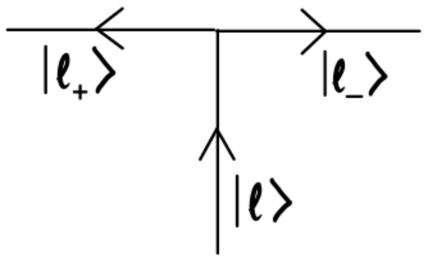


$\longrightarrow$



U_s deterministic scattering at nodes:

$U_s |\ell\rangle = |\ell_+\rangle a_+ + |\ell_-\rangle a_-$



$a_\pm = e^{i\pi/4} / \sqrt{2}$
(at criticality)

U_r diagonal in link basis:

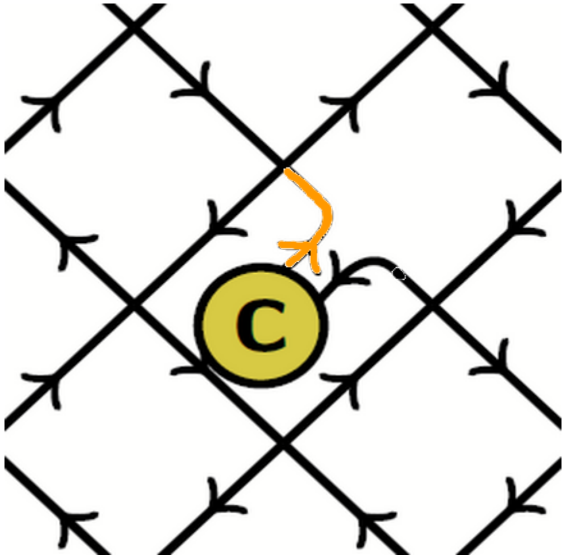
$U_r |\ell\rangle = |\ell\rangle e^{i\phi(\ell)}$

$\phi(\ell)$ uncorrelated random phases, uniformly distributed

Gaussian Free Field

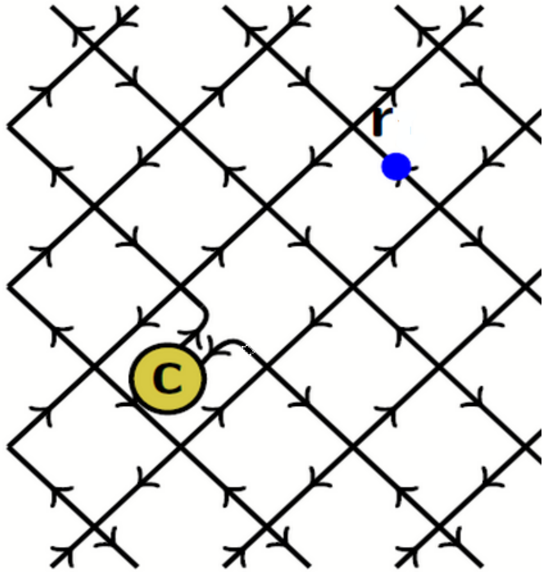
Statistics of wave intensities.

Bondesan, Wieczorek & Z
PRL (2015), NPB-FS (2017) 45 pp



$\psi_c = U\psi_c$ stationary "scattering" state (quasi-energy zero)
for incoming-wave boundary conditions $\psi_c(c) = 1$.

↝ Observable: $|\psi_c(r)|^2$ for large distances $|r-c|$

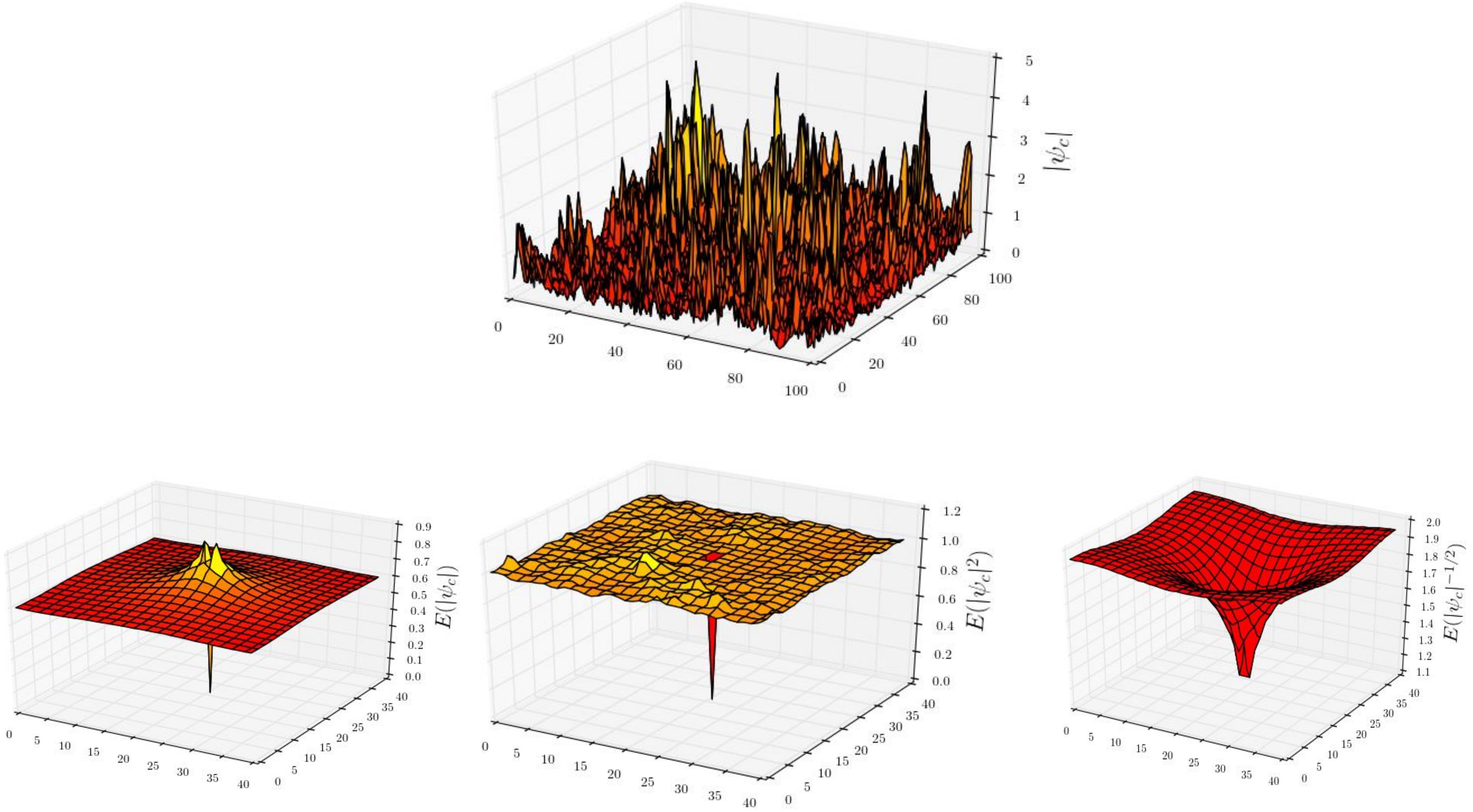


Prediction from Abelian OPE,
crossing symmetry of 4-point fctn:

$$\mathbb{E}\left(|\psi_c(r)|^{2q}\right) \simeq |r-c|^{-2\Delta_q}, \quad \Delta_q = X q(1-q).$$

Interpretation: $\log|\psi_c(r)|^2 \equiv \varphi(r)$
GFF with background charge $Q=1$, stiffness X.

Numerical simulations give $X \cong 0.26\ (\ldots 0.28)$ ↝ Hypothesis $X = 1/k$ (level $k=4$)



$|\psi_c|$
0 1 2 3 4 5
0 20 40 60 80 100
0 20 40 60 80 100
$E(|\psi_c|)$
0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9
0 5 10 15 20 25 30 35 40
0 5 10 15 20 25 30 35 40
$E(|\psi_c|^2)$
0.0 0.2 0.4 0.6 0.8 1.0 1.2
0 5 10 15 20 25 30 35 40
0 5 10 15 20 25 30 35 40
$E(|\psi_c|^{-1/2})$
1.1 1.2 1.3 1.4 1.5 1.6 1.7 1.8 1.9 2.0
0 5 10 15 20 25 30 35 40
0 5 10 15 20 25 30 35 40

SUSY Vertex Model

CC network model $\longrightarrow$ SUSY vertex model

A variant (due to N. Read) of the Wegner–Efetov supersymmetry method trades the task of taking disorder averages for a statistical mechanics problem of new (collective) variables that admit a continuum limit (at the critical point).

Uses second quantization on a Fock space for bosons and fermions.

Retarded sector: $U = e^X$, $\mathrm{Re}\, X < 0$.

bosons : $\mathrm{Det}^{-1}_{\mathbb{C}^N}(1-U) = \mathrm{Tr}_{S(\mathbb{C}^N)}\, \rho_B(U)$

fermions : $\mathrm{Det}(1-U) = \mathrm{STr}\, \rho_F(U)$

where $\rho_B(e^X) = e^{b^\dagger X b}$, $\rho_F(e^X) = e^{f^\dagger X f}$

Advanced sector: $U = e^X$, $\mathrm{Re}\, X > 0$.

$b^\dagger \to -b, \quad b \to b^\dagger,$

$f^\dagger \to +f, \quad f \to f^\dagger.$

Random phase average projects Fock space to subspace of $U(1)$ singlets

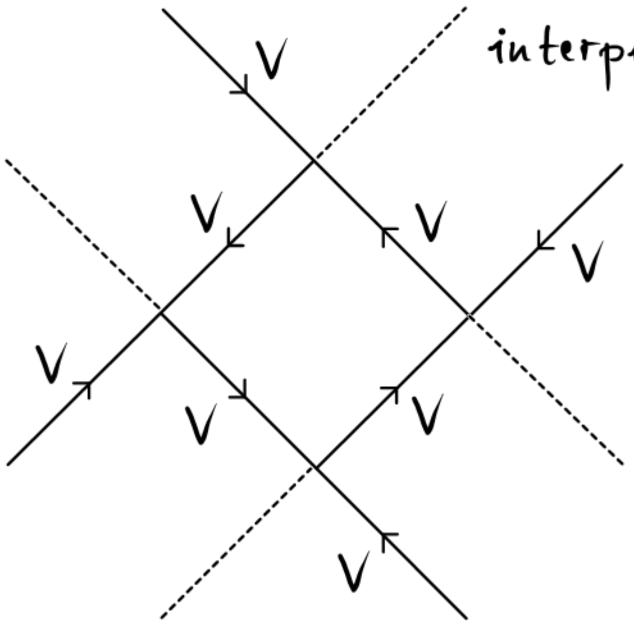
⇝ $\mathcal{H} = \bigotimes_{\text{links}} V(\ell)$ state space of SUSY vertex model

where $$V = \bigoplus_{n_b^+ + n_f^+ = n_b^- + n_f^-} \left(S^{n_b^+}(\mathbb{C}) \otimes \Lambda^{n_f^+}(\mathbb{C}) \otimes S^{n_b^-}(\mathbb{C}^*) \otimes \Lambda^{n_f^-}(\mathbb{C}^*) \right)$$

is irreducible highest-weight representation for $\mathcal{U} = U(1,1|2)$.

r replicas: $\mathcal{U} = U(r,r|2r)$

unit cell



interpretation ⇝ Feynman paths

$$\mathbb{E}\left(|\psi_c(r)|^{2q}\right) = \left\langle \pi_{vac}(c)\, (B^\dagger B)^q(r) \right\rangle_{V.M.}$$

$$\langle A \rangle_{V.M.} = \mathrm{STr}\, A\, \rho(\mathcal{U}_s)$$

Global symmetry $\mathcal{U} = U(r,r|2r)$

Continuum Limit from Dirac Approximation

Pure network model ($U \equiv U_s$) at long wave lengths

$1 - U_s \approx i D$ (Dirac operator ↝ discrete holomorphic derivative)

Treat the network model with random phase disorder as a perturbation ...

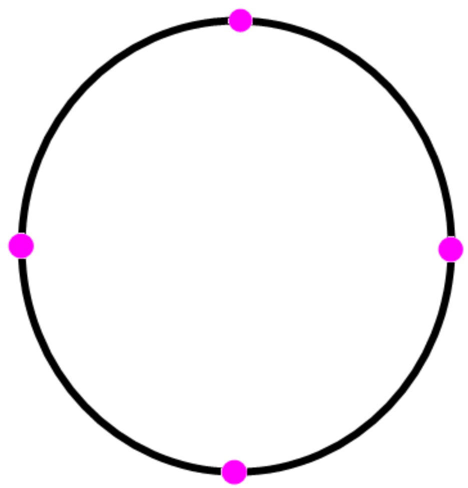
↝ (SUSY) Dirac theory with random gauge potential

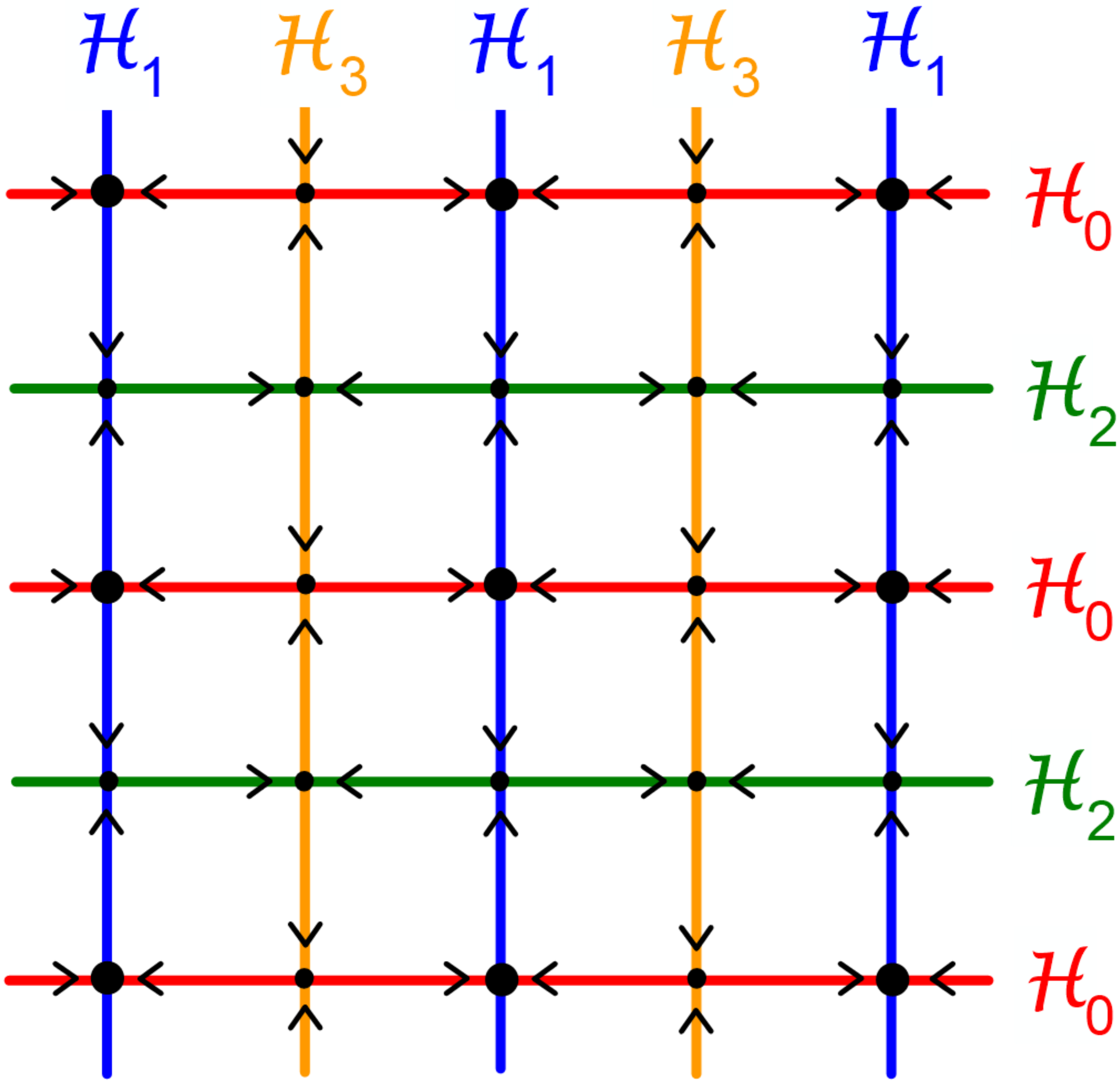
and random scalar potential

$\mathbb{Z}_4$ spectral symmetry

↝ 4 Dirac points (at 4th roots of unity)

$e^{im\pi/2} \cdot 1 - U_s = i D_m$ $(m = 0, 1, 2, 3)$





$\mathcal{H}_1$
$\mathcal{H}_3$
$\mathcal{H}_1$
$\mathcal{H}_3$
$\mathcal{H}_1$
$\mathcal{H}_0$
$\mathcal{H}_2$
$\mathcal{H}_0$
$\mathcal{H}_2$
$\mathcal{H}_0$

Preparatory step:

Assume A with values in $su(k) \subset u(k)$ $(k = 4)$.

Bosonize.

CFT describing continuum limit of disordered critical point is

(Bernard & LeClair 2002; Tsvelik et al. 2001):

"$GL(2r|2r)$" WZW model of level $k = 4$.

Target space = supermanifold over Riemannian symmetric space of type A|A

$$\left(GL(2r)/U(2r) \right) \times U(2r)$$

noncompact × compact

bb ff

What happens when the $U(1)$ gauge field and the random scalar potential are turned on?

Gauged WZW action functional:

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Remark: predicts multifractal scaling exponents $\Delta_q = \frac{1}{k} q(1-q)$.

Thank you !