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Bergman kernel from the lowest Landau level

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We use path integral representation for the density matrix, projected on the lowest Landau level, to generalize the expansion of the Bergman kernel on Kähler manifold to the case of arbitrary magnetic field.

1. Setup

Quantum mechanics in curved space is a much studied subject since the pioneering work of DeWitt [1]. One the main results in this area on the interface of geometry and physics is the short time expansion of the heat kernel on a Riemannian manifold using path integral technics (see [2] for a comprehensive review). A less known recent example of a similar geometric expansion, is the expansion of the Bergman kernel on a Kähler manifold, developed by Tian, Yau, Zelditch, Lu and Caitlin [3] and reproduced in [4] from the path integral. Although this result is yet to find more applications in physics, it has a simple interpretation in terms of quantum mechanics of a particle in a magnetic field on the lowest Landau level.

Consider a particle on a compact Kähler manifold with complex coordinates z^a, $\bar z^{\bar a}\,(a,\bar a = 1,\dots,n)$ in a magnetic field. If the field strength has only mixed components $F_{a\bar a}$ nonzero, with $F_{ab} = F_{\bar a\bar b} = 0$, meaning that the underlying line bundle is holomorphic, then the hamiltonian can be rewritten as

$$H = g^{a\bar a}F_{a\bar a} + g^{a\bar a}D_a\bar D_{\bar a}. \qquad (1)$$

Hence it follows that if the first term here is constant, then every wavefunction, satisfying $\bar D_{\bar a}\psi = 0$, will be degenerate and lie in the lowest Landau level. For simplicity here we will assume, that the zero-point energy is subtracted. Therefore the condition of holomorphy of the magnetic field guarantees the degeneracy of the lowest energy level. Equation $g^{a\bar a}F_{a\bar a} = \mathrm{const}$ is known as the hermitian Yang-Mills equation, and in the holomorphic case is equivalent to ordinary Maxwell equation. It has the following generic solution

$$F_{a\bar a} = kg_{a\bar a} + u_{a\bar a}, \qquad (2)$$

where k is some constant and $u_{a\bar a}$ is traceless (it can be nonzero, if the manifold has $b^{1,1} > 1$). In [4] we considered the case $u = 0$, and derived the strong field $1/k$-expansion of the density matrix projected on the lowest level for large but finite k. The purpose of this note is to generalize previous result for nonzero u.

2. TYZ expansion

Consider first the case of field strength, proportional to the Kähler form

$$F_{a\bar a} = kg_{a\bar a} = k\partial_a\bar\partial_{\bar a}K, \qquad (3)$$

where K is the Kähler potential. We are interested in the large time limit of the diagonal of the density matrix, which can be written as the

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following quantum-mechanical path integral

$$\rho(z) = \int_{z(0)=z}^{z(T)=z} \mathcal{D}z\mathcal{D}\bar{z}\mathcal{D}b\mathcal{D}c\, e^{-\frac{1}{\hbar}S}, \tag{4}$$

with the standard action

$$S = \int_{-1}^{0} d\tau \left[\frac{1}{T} g_{a\bar{a}}\dot{z}^a\dot{\bar{z}}^{\bar{a}} + \bar{A}_{\bar{a}}\dot{\bar{z}}^{\bar{a}} + g_{a\bar{a}}b^a c^{\bar{a}} - \frac{\hbar^2}{4}TR\right]$$

The anticommuting ghost fields b, c are introduced to make the integration measure covariant under general coordinate transformations and the time is rescaled as usual [2], so that τ is a dimensionless. The last term here is a famous Weyl-ordering counterterm, discovered in [1]. In general the large time limit of the heat kernel is not very well-defined and is usually nonlocal. In the case under consideration it turns out to be well defined, due to the large degeneracy of the lowest level. Making use of the normal coordinate frame around the point z, adopted to the Kähler case, we can expand the Kähler potential as

$$K = g_{a\bar{a}}z^a\bar{z}^{\bar{a}} + \frac{1}{4}R_{ab\bar{a}\bar{b}}z^a\bar{z}^{\bar{a}}z^b\bar{z}^{\bar{b}} + \ldots, \tag{5}$$

and from this all other geometric quantities. Then, using the free theory propagator Δ, satisfying

$$\left[-\frac{1}{T}\frac{d^2}{d\tau^2} + k\frac{d}{d\tau}\right]\Delta(\tau,\sigma) = \delta(\tau-\sigma), \tag{6}$$

and Dirichlet boundary conditions, we can proceed with perturbation theory expansion. One of the nontrivial checks of the procedure is the fact, that all T-divergent terms cancel, provided the Weyl-ordering counterterm is taken into account. In the infinite time T limit we find

$$\rho = k^n \left(1 + \frac{\hbar}{2k}R + \frac{\hbar^2}{3k^2}\Delta R + \ldots)\right), \tag{7}$$

see [4] for the full answer in order $1/k^2$. In mathematical literature this expansion is known as Tian-Yau-Zelditch expansion of the Bergman kernel, and has many interesting applications [3].

3. Generic magnetic field

For the general magnetic field (2) one can still use the Kähler normal coordinates, with only exception that the magnetic vector potential has a more general expansion

$$\bar{A}_{\bar{a}} = F_{a\bar{a}}z^a + \frac{1}{2}F_{a\bar{a}b\bar{b}}z^a z^b\bar{z}^{\bar{b}} + \ldots$$

As a consequence, instead of (6) the free theory propagator satisfies now a more complicated matrix-valued differential equation

$$\left[-g_{a\bar{b}}\frac{1}{T}\frac{d^2}{d\tau^2} + F_{a\bar{b}}\frac{d}{d\tau}\right]\Delta^{b\bar{b}}(\tau,\sigma) = \delta_a^b\delta(\tau-\sigma),$$

which complete solution can be found in [5]. The calculation then becomes more complicated, since each Feynman integral in parturbation theory becomes a matrix-valued integral. Proceeding along the same lines as before, we computed the expansion up to the first order in $\hbar$

$$\rho(z) = \det F(1 + \hbar R_{a\bar{a}}(F^{-1})^{a\bar{a}} + \hbar F_{a\bar{a}b\bar{b}}\cdot \tag{8}$$

$$\cdot(F^{-1}\otimes F^{-1} - [F\otimes F + F^2\otimes g]^{-1})^{a\bar{a}b\bar{b}} + \ldots)$$

Here $(F^{-1})^{a\bar{a}}$ is the matrix, inverse to $F_{a\bar{a}}$, with indices raised by $g^{a\bar{a}}$, and $\otimes$ is the usual tensor product of matrices. One can easily check, that for $u = 0$ this answer coincides with (7) up to $\hbar$ term. In the large k limit this expansion was obtained by Wang [6] using different methods. The main difference of [6] with the path integral method is that the connection doesn't have to satisfy the hermitian Yang-Mills equation.

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