Complex Geometry - Homework 9

1. Problem

(a) Show that

(i) Every meromorphic function on $\hat{\mathbb{C}}$ is rational.

(ii) Any meromorphic function f on $\mathbb{C}$, such that $\lim_{z\to\infty} f(z) = \infty$, is rational.

(b) Let $f \not\equiv 0$ be a meromorphic function on $\hat{\mathbb{C}}$ and $D = N(f) \cup P(f)$ the set of zeros and poles of f in $\hat{\mathbb{C}}$. Show that

$$\sum_{z\in\hat{\mathbb{C}}} \mathbf{ord}_z f = \sum_{z\in D} \mathbf{ord}_z f = 0.$$

(c) Let $z_1, \dots, z_n \in \hat{\mathbb{C}}$ be pairwise distinct points and let $m_1, \dots, m_n \in \mathbb{Z}$ be integers such that $m_1 + \ldots + m_n = 0$. Prove that there exists a meromorphic function f on $\hat{\mathbb{C}}$ with

$$\mathbf{ord}_z f = \begin{cases} m_j & , \text{ if } z = z_j \text{ for some } j \in \{1, \dots, n\}, \\ 0 & , \text{ if } z \notin \{z_1, \dots, z_n\}. \end{cases}$$

2. Problem

Show the following:

(i) Let f be a polynomial and $D = D_r(a) = \{|z-a| < r\}$ a disc, such that $f(z) \neq 0$ for all $z \in \partial D$. One has that

$$\frac{1}{2\pi i}\int_{\partial D} \frac{f'(z)}{f(z)} dz = \sum_{z\in D} \mathbf{ord}_z f$$

is equal to the number of zeros of f in D counted with multiplicity.

(ii) Let $f\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational function and $D = D_r(a) = \{|z-a| < r\}$ a disc such that f has no zeros or poles on ∂D. One has

$$\frac{1}{2\pi i}\int_{\partial D} \frac{f'(z)}{f(z)} dz = \sum_{z\in D} \mathbf{ord}_z f = N_f(0) - N_f(\infty).$$

(iii) Let $f \not\equiv 0$ be a meromorphic function defined in a neighbourhood of $a \in \mathbb{C}$. For sufficiently small $\rho > 0$ one has

$$\mathbf{res}_a(f'/f) = \frac{1}{2\pi i}\int_{|z-a|=\rho} \frac{f'(z)}{f(z)} dz = \mathbf{ord}_a f.$$

(iv) Let $G \subset \mathbb{C}$ be a domain and $f \colon G \to \mathbb{C}$ a holomorphic function. Let $\gamma \colon [0,1] \to G$ a closed path which is homologous to zero and $b \in \mathbb{C}$ such that $f(z) \neq b$ for all $z \in \gamma([0,1])$. Then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-b}dz = n(f \circ \gamma, b) = \sum_{a \in f^{-1}(b)} \nu_f(a) \cdot n(\gamma, a).$$

(v) Let $f \colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational function and $D = D_r(a) = \{|z-a| < r\}$ a disc such that f has no zeros or poles on ∂D. Let $\gamma = \partial D$ the standard parametrization of the boundary of D. One has

$$n(f \circ \gamma, 0) = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \sum_{z \in D} \mathbf{ord}_z f = \sum_{j=1}^{n} n(\gamma, b_j) - \sum_{j=1}^{m} n(\gamma, c_j),$$

where $b_1, \ldots, b_n$ are the zeros and $c_1, \ldots, c_m$ are the poles of f in D counted with multiplicity (i. e. $\nu_f(b_1) = k \Rightarrow b_1 = b_2 = \ldots = b_k$).

3. Problem

Let X be a a compact Riemann surface. Show that:

(a) Two meromorphic functions on X having the same principal part at each of their poles must differ by a constant.

(b) Two meromorphic functions on X having the same zeroes and poles (multiplicities included) agree up to a constant factor.

4. Problem

(a) Let p, q two disjoint points on the Riemann sphere $\mathbb{P}^1$. Find a meromorphic function f on $\mathbb{P}^1$ such that $(f) = p - q$.

(b) Show that two divisors D and D' on $\mathbb{P}^1$ are linearly equivalent if and only if $\deg D = \deg D'$.

(b) Consider the divisor $D = 0 + 1$ on $\mathbb{P}^1$. Calculate $H^0(\mathbb{P}^1, \mathcal{O}(D))$.

5. Problem

Let D be a divisor on the Riemann sphere $\mathbb{P}^1$. Prove

(a) $\dim H^0(\mathbb{P}^1, \mathcal{O}(D)) = \max\{0, 1 + \deg D\}$,

(b) $\dim H^1(\mathbb{P}^1, \mathcal{O}(D)) = \max\{0, -1 - \deg D\}$.

6. Problem

Let X be a complex torus and p a point of X. Show that

$$\dim H^0(X, \mathcal{O}(np)) = \begin{cases} 0, & \text{for } n < 0, \\ 1, & \text{for } n = 0, \\ n, & \text{for } n \geq 1. \end{cases}$$

7. Problem
Let $X = \mathbb{C}/\Gamma$ be a complex torus, where $\Gamma = \mathbb{Z} + \tau\mathbb{Z}$ for $\tau \in \mathbb{H}$. Let $\wp(z, \tau)$ be the associated Weierstrass $\wp$-function.
(a) Calculate the divisors $(\wp)$ and $(\wp')$.
(b) Construct explicitely a meromorphic function on X with an unique pole at a given point p such that the order of this pole is a given integer $n \in \mathbb{N}$. Deduce that $\dim H^0(X, \mathcal{O}(np)) = n$.
(c) Calculate a canonical divisor of X.

8. Problem
Let X be a compact Riemann surface of genus g.
(a) Show that for any point $p \in X$ there exists a meromorphic function on X, which has a pole at p of order $\leq g + 1$ and is holomorphic on $X \setminus \{p\}$. Show that if $g = 1$ the order of the pole cannot be 1.
(b) Show that there exists a branched covering $f : X \to \mathbb{P}^1$ with at most $g + 1$ sheets. Deduce that a compact Riemann surface of genus $g = 0$ is biholomorphic to $\mathbb{P}^1$.
(c) Show that for any $n \geq 2g$ there exists a meromorphic function with a unique pole and this pole has order n.
(d) Let $f : X \to X$ a biholomorphic map different from the identity. Show that f has at most $2g + 2$ fixed points.

9. Problem
Let $X = \mathbb{C}/\Gamma$ be a complex torus, where $\Gamma = \mathbb{Z} + \tau\mathbb{Z}$ for $\tau \in \mathbb{H}$.
(a) Fix a positive integer n, and choose any two sets of n complex numbers $\{a_i : 1 \leq i \leq n\}$ and $\{b_j : 1 \leq j \leq n\}$ such that $\sum_i a_i - \sum_j b_j$ is an integer. Show that the ratio of translated theta functions

$$R(z) = \frac{\prod_{i=1}^{n} \theta\big(z - a_i\big)}{\prod_{j=1}^{n} \theta\big(z - b_j\big)}$$

is a meromorphic Γ-periodic function on $\mathbb{C}$ and so descends to a meromorphic function on X.
(b) Show that any meromorphic function on X has the above form.