Bergman kernel, balanced metrics and black holes

Semyon Klevtsov

Rutgers University

M. Douglas, S.K.: arXiv:0811.0367; arXiv:0808.2451

Plan

- Motivation
- Landau levels
- Bergman kernel and Bergman metrics
- Balanced metrics
- Connection to physics and string theory

Motivation

- In string theory space-time dimension is equal to 10. Six extra dimensions are compactified ("hidden" in a compact manifold). According to equations of motion the compactification manifold should allow for Ricci-flat metric. Supersymmetry implies that the manifold should also be Kähler. It's important to find Ricci-flat Kähler metrics. "Donaldson program": approximate everything by Bergman metrics.
- Related to the problem of lowest Landau level for inhomogeneous magnetic field on Kähler manifold in 2 and higher dimensions.

Landau levels

Charged particle on the plane in constant magnetic field B.
Hamiltonian

$$\hat{H} = \frac{1}{2m}(i\hbar\vec{\nabla} + \frac{e}{c}\vec{A})^2$$

Infinite tower of energy levels: $E_n = \frac{\hbar e}{mc}B(n + \frac{1}{2})$

Landau levels

Charged particle on the plane in constant magnetic field B.
Hamiltonian

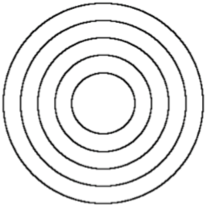
$$\hat{H} = \frac{1}{2m}(i\hbar\vec{\nabla} + \frac{e}{c}\vec{A})^2$$

Infinite tower of energy levels: $E_n = \frac{\hbar e}{mc} B(n + \frac{1}{2})$

Each level is highly degenerate:

Lowest level $n = 0$

Structure:



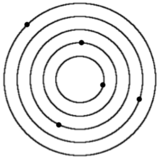
The lowest energy level is special, since wave functions satisfy first order equation

$$\left[\frac{\partial}{\partial \bar{z}} + Bz\right] \psi_k(z) = 0$$

in complex coordinates $z = re^{i\phi}$ on the two-plane. The wave functions are

$$\psi_k(z, \bar{z}) = z^k e^{-B|z|^2}, \quad k = 0, 1 \ldots \infty (\text{or total flux} \int B),$$

Mixed state on the lowest level is described by density matrix



$$\rho(z, z') = \sum_{n=0}^{\infty} c_n \psi_n(z) \psi_n(z')^*.$$

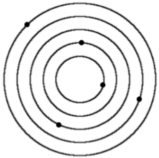
The lowest energy level is special, since wave functions satisfy first order equation

$$\left[\frac{\partial}{\partial \bar{z}} + Bz\right] \psi_k(z) = 0$$

in complex coordinates $z = re^{i\phi}$ on the two-plane. The wave functions are

$$\psi_k(z, \bar{z}) = z^k e^{-B|z|^2}, \quad k = 0, 1 ... \infty (\text{or total flux} \int B),$$

Mixed state on the lowest level is described by density matrix



$$\rho(z, z') = \sum_{n=0}^{\infty} c_n \psi_n(z) \psi_n(z')^*.$$

When the entropy is maximal? $S = -Tr\rho \log \rho$

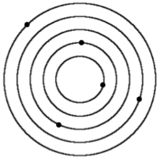
The lowest energy level is special, since wave functions satisfy first order equation

$$\left[\frac{\partial}{\partial \bar{z}} + Bz\right] \psi_k(z) = 0$$

in complex coordinates $z = re^{i\phi}$ on the two-plane. The wave functions are

$$\psi_k(z, \bar{z}) = z^k e^{-B|z|^2}, \quad k = 0, 1 \ldots \infty (\text{or total flux} \int B),$$

Mixed state on the lowest level is described by density matrix



$$\rho(z, z') = \sum_{n=0}^{\infty} c_n \psi_n(z) \psi_n(z')^*.$$

When the entropy is maximal? $S = -Tr\rho \log \rho$

If each microstate is realized with equal probability.

Density matrix

Another definition of maximally entropic state is that probability to find particle at a particular point is independent of a point. In other words density matrix is constant on diagonal

$$\rho(z,z) = \sum_{k=0}^{\infty} \frac{1}{k!}(2B)^k z^k \bar{z}^k e^{-2B|z|^2} = \mathrm{const}$$

True in this case but not in general. We can use this equation as a second (in general much stronger) definition of "maximally entropic" state. So far it was just a standard flat two-plane and constant magnetic field. Same principle, applied to curved space and inhomogeneous magnetic fields, leads to new interesting features.

Question from physical perspective: What is the analog of this picture (rich LLL) in a more general setup, e.g. for compact manifolds, inhomogeneous magnetic fields, any space dimension?

Question from physical perspective: What is the analog of this picture (rich LLL) in a more general setup, e.g. for compact manifolds, inhomogeneous magnetic fields, any space dimension? Sufficient conditions:

- Manifold admits complex coordinates z^a, $\bar{z}^{\bar{a}}$ $(a, \bar{a} = 1, ..., n)$
- ... and Kähler metric $g_{a\bar{a}} = \partial_a \bar{\partial}_{\bar{a}} K$
- Magnetic field is "holomorphic", i.e. only $F_{a\bar{a}}$ components are non-zero.

Question from physical perspective: What is the analog of this picture (rich LLL) in a more general setup, e.g. for compact manifolds, inhomogeneous magnetic fields, any space dimension? Sufficient conditions:

- Manifold admits complex coordinates z^a, $\bar{z}^{\bar{a}}$ $(a, \bar{a} = 1, ..., n)$
- ... and Kähler metric $g_{a\bar{a}} = \partial_a \bar{\partial}_{\bar{a}} K$
- Magnetic field is "holomorphic", i.e. only $F_{a\bar{a}}$ components are non-zero.

Then Hamiltonian (magnetic Schrödinger operator) factorizes:

$$H = g^{a\bar{a}} D_a \bar{D}_{\bar{a}} + g^{a\bar{a}} F_{a\bar{a}}$$

provided $g^{a\bar{a}} F_{a\bar{a}} = constant$, which is equivalent to Maxwell equation $(\nabla F = 0)$. Solution $F_{a\bar{a}} = k g_{a\bar{a}} + u_{a\bar{a}}$
Then it's enough to solve $\bar{D}_{\bar{a}} \psi = 0 \Longrightarrow \psi = s(z) e^K$

Bergman metrics

Consider a Kähler manifold M of complex dimension n, e.g. embedded in some projective space CP^N, $N > n$. Turn on holomorphic line bundle L, or it's k-th power L^k (= magnetic field) and consider its holomorphic sections $s_0(z), \ldots, s_{N_k}(z)$. Denote the metric on L as h, so that inner product is $(s, s') = \int_M hs\bar{s}'$. Then the metric on L^k is h^k.

One can think of $s_\alpha(z)$ as projective coordinates on CP^{N_k}. Therefore a choice of the basis of sections defines an embedding of the manifold M into CP^{N_k}.

A useful choice of metric on M is to pull back the Fubini-Study metric from CP^{N_k}, called Bergman metric

$$\frac{1}{k} g_{FS}|_M = \frac{1}{k}\partial\bar{\partial}\log\left(\sum_{\alpha=0}^{N_k} s_\alpha \bar{s}_\alpha\right) =$$

One can think of s_α as projective coordinates on CP^{N_k}. Therefore a choice of the basis of sections defines an embedding of the manifold M into CP^{N_k}.

A useful choice of metric on the manifold is to pull back the Fubini-Study metric from CP^{N_k}, called Bergman metric

$$\frac{1}{k} g_{FS}|_M = \frac{1}{k}\partial\bar{\partial}\log\left(\sum_{\alpha=0}^{N_k} s_\alpha \bar{s}_\alpha\right) =$$

$$= -\partial\bar{\partial}\log h + \frac{1}{k}\partial\bar{\partial}\log\left(h^k \sum_{\alpha=0}^{N_k} s_\alpha \bar{s}_\alpha\right)$$

We see that it's in the same Kähler class as curvature of the line bundle L

$$F_{a\bar{a}} = k g_{a\bar{a}} = -\partial_a \bar{\partial}_{\bar{a}} \log h^k,$$

Density function (Bergman kernel) has a local expansion for large k

$$\rho_k(z) = h^k \sum_{\alpha=0}^{N_k} s_\alpha(z)\bar{s}_\alpha(\bar{z}) = k^n \left(1 + \frac{1}{2k}R(z) + \dots\right)$$

known as Tian-Yau-Zelditch expansion (Tian' 90, Zelditch' 98).

It's a large-time limit of heat kernel of magnetic Schrödinger operator

$$\rho_k(z) = \lim_{T\to\infty} \langle z|e^{-TH}|z\rangle$$

There are several different representations of the Bergman kernel.

Bergman kernel can be expressed using Feynman-Kac formula

$$\langle z|e^{-TH}|z\rangle = \mathbb{E}_{0,z}^{T,z} \ \exp\left(-i\int_0^T A(B_s)\circ dB_s\right)$$

where B_s is a Brownian bridge from z to z on M. Hard to control at large time. (Ref.: L. Erdös' 94)

Bergman kernel can be expressed using Feynman-Kac formula

$$\langle z|e^{-TH}|z\rangle = \mathbb{E}_{0,z}^{T,z} \;\exp\left(-i\int_0^T A(B_s)\circ dB_s\right)$$

where B_s is a Brownian bridge from z to z on M. Hard to control at large time. (Ref.: L. Erdös' 94)
Path integral representation

$$\langle z|e^{-TH}|z\rangle = \int_{z(0)=z}^{z(T)=z} \prod_{0<t<T} \det g_{a\bar{a}}(z(t))\,\mathcal{D}z(t)\mathcal{D}\bar{z}(t)$$

$$\exp\left(-\frac{1}{\hbar}\int_0^T dt\,(g_{a\bar{a}}\dot{z}^a\dot{\bar{z}}^{\bar{a}} + A_a\dot{z}^a)\right)$$

Some details of calculation

- Quantum hamiltonian should be rewritten in a coordinate covariant fashion and then symmetrized (Weyl-ordering).
  Counterterm $H(\hat{p},\hat{q}) = H_W - \frac{\hbar^2}{4}R$
  DeWitt' 57, Bastianelli, Van Nieuwenhuizen' 06
- Green's function $\left[-\frac{1}{T}\frac{d^2}{d\tau^2} + k\frac{d}{d\tau}\right]\Delta(\tau,\sigma) = \delta(\tau-\sigma)$
- "Mid-point" rule (as for Stratonovich integral).
- Ghosts:

$$\prod_{0<t<T} \det g_{a\bar{a}}(z(t)) = \int \mathcal{D}b(t)\mathcal{D}c(t)e^{-\frac{1}{\hbar}\int dt\; g_{a\bar{a}}b^a c^{\bar{a}}}$$

(Berezin integral)

Some details of calculation

Kähler normal cordinates $K_{a_1 a_2 \ldots \bar{b}}(z_0) = 0$

$$K(z) = K(z_0) + g_{a\bar{a}}(z_0) z^a \bar{z}^{\bar{a}} + \frac{1}{4} R_{ab\bar{a}\bar{b}} z^a \bar{z}^{\bar{a}} (z_0) z^b \bar{z}^{\bar{b}} + \ldots$$

Some details of calculation

Kähler normal cordinates $K_{a_1 a_2 \ldots \bar{b}}(z_0) = 0$

$$K(z) = K(z_0) + g_{a\bar{a}}(z_0) z^a \bar{z}^{\bar{a}} + \frac{1}{4} R_{ab\bar{a}\bar{b}} z^a \bar{z}^{\bar{a}}(z_0) z^b \bar{z}^{\bar{b}} + \ldots$$

Answer in second order perturbation theory

$$\rho = k^n \left[1 + \frac{\hbar}{2k} R + \frac{\hbar^2}{k^2} \left(\frac{1}{3} \Delta R + \frac{1}{24} |\mathrm{Riem}|^2 - \frac{1}{6} |\mathrm{Ric}|^2 + \frac{1}{8} R^2 \right) + \ldots \right]$$

Some details of calculation

Kähler normal cordinates $K_{a_1 a_2 \ldots \bar{b}}(z_0) = 0$

$$K(z) = K(z_0) + g_{a\bar{a}}(z_0) z^a \bar{z}^{\bar{a}} + \frac{1}{4} R_{ab\bar{a}\bar{b}} z^a \bar{z}^{\bar{a}} (z_0) z^b \bar{z}^{\bar{b}} + \ldots$$

Answer in second order perturbation theory

$$\rho = k^n \left[1 + \frac{\hbar}{2k} R + \frac{\hbar^2}{k^2} \left(\frac{1}{3} \Delta R + \frac{1}{24} |\mathrm{Riem}|^2 - \frac{1}{6} |\mathrm{Ric}|^2 + \frac{1}{8} R^2 \right) + \ldots \right]$$

Math result, using peak sections method, by Lu' 99

$$\rho = k^n \left[1 + \frac{1}{2k} R + \frac{1}{k^2} \left(\frac{1}{3} \Delta R + \frac{1}{24} |\mathrm{Riem}|^2 - \frac{1}{6} |\mathrm{Ric}|^2 + \frac{1}{8} R^2 \right) + \ldots \right]$$

Generalizations

- More general magnetic field $F_{a\bar{a}} = kg_{a\bar{a}} + u_{a\bar{a}}$

$$\rho = \det F \ \mathrm{Tr}\Big[1 + \hbar \mathrm{R} F^{-1} +$$

$$+ \hbar F_4 \cdot (F^{-1} \otimes F^{-1} - (F \otimes F + F^2 \otimes g)^{-1}) + \ldots\Big]$$

for large k coincides with Wang' 05 formula for $L^k \otimes \mathcal{E}$.

- Supersymmetric quantum mechanics

$$S_f = \int \bar{\psi}_i^{\bar{a}} D\psi_{i\bar{a}} + F_{a\bar{a}} \bar{\psi}_i^{\bar{a}} \psi_i^a, \qquad i = 1, 2$$

$$\rho = k^n \left(1 + \frac{|N-1|}{2k} R + \ldots\right)$$

Principle of maximal entropy

In quantum mechanics, the Bergman kernel corresponds to density matrix of a particle in strong magnetic field on Kähler manifold, projected to the lowest Landau level.
When does the mixed state on LLL has maximal entropy? If each pure state enters the density matrix with the same weight. If the density matrix is constant everywhere on M.

$$\rho_k(z) = h^k \sum_{\alpha=0}^{N_k} s_\alpha(z)\bar{s}_\alpha(\bar{z}) = k^n \left(1 + \frac{1}{2k} R(z) + \ldots\right) =$$

$$= \text{constant} \approx k^n + \ldots$$

Solve for h^k

$$h^k = \left(\sum_{\alpha=0}^{N_k} s_\alpha(z)\bar{s}_\alpha(\bar{z})\right)^{-1} \cdot \frac{N_k}{\mathrm{Vol} M}$$

Balanced metric

Bergman metric, satisfying the strong maximal entropy principle $\rho =$ constant, is the balanced metric.

$$g = \frac{1}{k}\partial\bar{\partial}\log\left(\sum_{\alpha=0}^{N_k} s_\alpha \bar{s}_\alpha\right)$$

for orthonormal basis of sections with respect to inner-product

$$(s_\alpha, s_\beta) = \int_M h^k s_\alpha \bar{s}_\beta = \frac{N_k}{\mathrm{Vol} M}\int_M \frac{s_\alpha \bar{s}_\beta}{\sum_{\gamma=0}^{N_k} s_\gamma \bar{s}_\gamma} = \delta_{\alpha\beta}$$

This can be more explicitly reformulated in terms of Donaldson's T-map.
Important property: balanced metric converges to constant scalar curvature metric (proved by Donaldson, using from TYZ expansion)

Donaldson's T-map

Donaldson' 05

Sections $s_\alpha(z)$ define an embedding M into $\mathbb{CP}^{N_k}$.
Start with a positive definite Hermitean matrix $G_{\alpha\beta}$ (hermitean metric on vector space of sections) and form it's inverse $G^{\alpha\beta}$
Then T-map is the integral

$$T(G)_{\alpha\beta} = \frac{N_k}{\mathrm{Vol}M} \int_M \frac{s_\alpha \bar{s}_\beta}{\sum_{\gamma,\delta} G^{\gamma\delta} s_\gamma \bar{s}_\gamma} d\mu_G$$

Iterations of T-map $T^{\circ n}$ as $n \to \infty$ converge to a "balanced embedding" $G_\infty^{\alpha\beta}$. Then the balanced metric on the manifold M is given by

$$g = \frac{1}{k} \partial\bar{\partial} \log G_\infty^{\alpha\beta} s_\alpha \bar{s}_\beta$$

Physical meaning of balanced metric

Usually in gravity there is no preferred metric. It's choice depends on the choice of observer. Unless we postulate an observable which singles out one choice metric, say measurements done by a point-like observer who moves on geodesics, there is no way to say which metric is right. Same is true when speaking about quantum corrections to gravity.

Physical meaning of balanced metric

Usually in gravity there is no preferred metric. It's choice depends on the choice of observer. Unless we postulate an observable which singles out one choice metric, say measurements done by a point-like observer who moves on geodesics, there is no way to say which metric is right. Same is true when speaking about quantum corrections to gravity.

Suppose we have a quantum observer in a mixed state LLL. Then from the physical requirement, that observer is in a state with maximal entropy (in the sense explained before) we derive the corresponding metric (" effective metric").

Probe metric

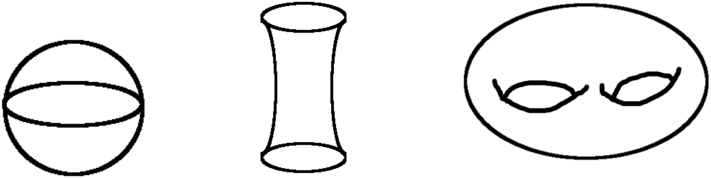
In which physical system the concept of "maximally entropic" metric can be realized?

Probe metric

In which physical system the concept of "maximally entropic" metric can be realized?

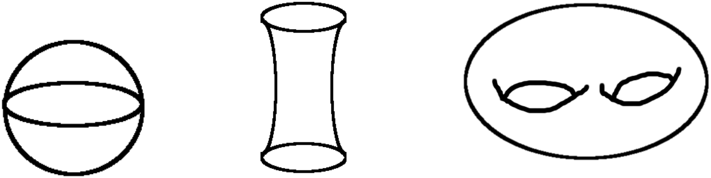
Consider $\mathcal{N} = 2$ black hole solutions in type II string theory, compactified on Calabi-Yau manifold. They have interesting feature called "attractor mechanism": in the near horizon limit the moduli of Calabi-Yau manifold are fixed by equations of motion. The near-horizon geometry is:

$$S^2 \quad \times \quad AdS_2 \quad \times \quad \text{Calabi-Yau}$$



Conjecture: the metric is also fixed if black hole has maximal entropy.

$$S^2 \quad \times \quad AdS_2 \quad \times \quad \text{Calabi-Yau}$$



Black hole solution is characterized by its electric and magnetic charges (q_0, q_A, p^A), which correspond to the charges of D0, D2 and D4-branes wrapped on Calabi-Yau. The problme of entropy counting maps into the problem of counting BPS states of D-branes. Consider configuration with no electric charge. One can show from equations of motion, that D4 induces magnetic field on Calabi-Yau, which is proportional to it's Kähler form (i.e. metric)

$$F_{a\bar{b}} = k g_{a\bar{b}}$$

It's natural to assume that black hole must have maximal entropy, therefore we conclude that that the metric in such a background is the "balanced metric", for finite magnetic field flux.

The probe brane, that "sees" this metric has to be the D2-brane, wrapped around S^2.

One of the consequences of this conjecture is that Einstein equations should receive corrections in accordance with Bergman kernel expansion.

$$R + \alpha' R^2 + \ldots + \alpha' R^4 \ldots = 0$$

Conclusion

We rederived Tian-Yau-Zelditch expansion of Bergman kernel from path integral. We also considered it's generalizations for arbitrary holomorphic line bundle and supersymmetric analogs. Physically Bergman kernel corresponds to density matrix on LLL on Kähler manifold. Balanced metrics correspond to mixed states with constant density. We also formulated the conjecture, that for $\mathcal{N} = 2$ black hole solution moduli space metric is the balanced metric for finite magnetic flux.