Random normal matrices and Kähler metrics

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based on 1309.7333,
1410.6802 (with Ferrari)
1404.0659 (with Zelditch)
and previous work with Ferrari and Zelditch

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Motivation

We will study large N behavior of the integral of the following type

$$\int_{\mathbb{C}^{\otimes N}} |\Delta(z)|^{2\beta} e^{-N\sum_i W(z_i)} \prod_{i=1}^{N} d^2 z_i$$

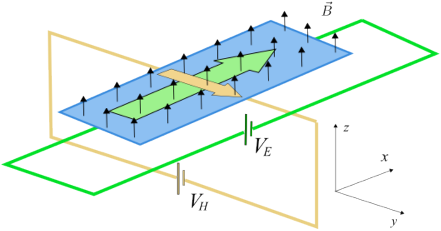
- $W = |z|^2$, $\beta = 1$ (Ginibre)
- $W = V(|z|^2) + \sum_k t_k z^k + \bar{t}_k \bar{z}^k$, $\beta = 1$ (Chau, Yu, Zaboronsky)
- W is arbitrary real (Wiegmann-Zabrodin)

This integral appears in the quantum Hall effect.

Quantum Hall effect

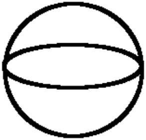
Observed in two-dimensional electron systems subjected to low temperatures and strong magnetic fields. Hall conductance is quantized $\sigma_H = I/V_H = \nu$, where ν is integer for integer QHE, or a fraction for fractional QHE. Involves many ($N \sim 10^6$) electrons on lowest Landau level, described by a collective (Laughlin) state.

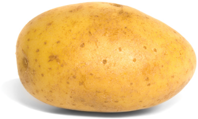
$$\Psi(\{z_i\}) = \prod_{i<j}^{N} (z_i - z_j)^\beta e^{-\frac{B}{2}\sum_i |z_i|^2}, \quad \beta = 1/\nu \in \mathbb{Z}_+$$



Motivation

Quantum Hall effect happens in a planar sample. Here we will ask, what happens when QHE (Laughlin states) is considered on a curved geometry instead of a perfectly flat plane. So far our methods allow us to study what happens in the "bulk", neglecting effects of the boundary. The main question is to calculate the response of the system to the curvature of the sample.



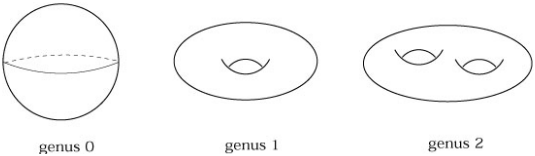


$$R(z) = 2 \qquad\qquad \int_{\text{potato}} R(z) d^2 z = 2$$

The information about how curved is the sample is encoded by a single function, called "scalar curvature" $R(z)$. We ask: how does the density profile of electrons depend on $R(z)$?

Motivation

Same question for higher-genus surfaces. At $\beta = 1$ (integer quantum Hall) we also can find large N scaling limit on any Riemann surface with any metric



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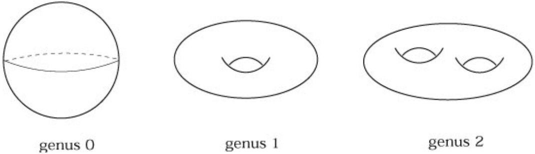
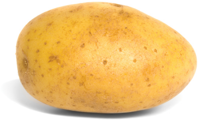
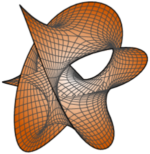




QHE can be defined on higher-dimensional Kähler manifolds



Motivation

In physics one can obtain important information about the system by putting it on a manifold with a Riemannian metric

$$ds^2 = g(z, \bar{z}) dz d\bar{z}$$

Prototypical example (Polyakov, 1981): CFT partition function on a compact Riemann surface (M, g_0) has the following behavior under the transformation of the reference metric g_0 to a new metric $g = e^{2\sigma} g_0$

$$\log \frac{Z^{CFT}(g)}{Z^{CFT}(g_0)} = \frac{c}{12\pi} S_L(g_0, \sigma),$$

where $c \in \mathbb{R}$ is the central charge of the CFT, and the Liouville action is

$$S_L(g_0, \sigma) = \int_M (\partial\sigma\bar{\partial}\sigma + R_0\sigma) d^2 z,$$

and $R_0 = -\frac{1}{\sqrt{g_0}} \partial\bar{\partial} \log \sqrt{g_0}$ is the scalar curvature of g_0.

Lowest Landau level on a Riemann surface

The lowest Landau level wave functions on the plane in constant magnetic field B are given by

$$\psi_k(z,\bar{z}) = z^k e^{-B|z|^2}, \quad k = 0, 1 ... \infty \text{ (or total flux } \int B),$$

Constant magnetic field on a Riemann surface M with the metric g_0: $B_0 = dA = kg_0$. Shrödinger equation for the lowest energy level reduces to

$$(\bar{\partial} + A_{\bar{z}})\psi = 0, \quad \text{where } A_{\bar{z}} = -k\bar{\partial}\log h_0$$

with many solutions $\psi_i^0(z,\bar{z}) = s_i(z)h_0^k(z,\bar{z})$. Mathematically, magnetic field is described by the holomorphic line bundle $L^k \to M$, $s_i(z)$ is the basis of holomorphic sections ($i = 1, .., N_k$). Examples:

- S^2, $s_i(z) = z^{i-1}$, $i = 1, .., k+1$.
- T^2, $s_i(z) = \theta_{\frac{i}{k},0}(kz, k\tau)$, $i = 1, .., k$
- on a surface of genus h there are $N_k = k - h + 1$ sections.

Kähler parameterization of the metric

In two dimensions we are used to parameterizing the metric in conformal class.

$$g|dz|^2 = e^{2\sigma(z,\bar{z})} g_0 |dz|^2$$

Landau levels prefer Kähler parameterization (Kähler class)

$$g|dz|^2 = (g_0 + \partial\bar{\partial}\phi)|dz|^2$$

where the scalar function $\phi(z,\bar{z})$ is called Kähler potential. QHE droplet is incompressible – deformations have the same area, and all metrics in the Kähler class have the same area.
If $\psi_i(z,\bar{z}) = s_i(z) h_0^k(z,\bar{z})$ are normalized LLL wave functions for the magnetic field $B_0 = kg_0$, then (non-normalized) wave functions for the magnetic field $B = k(g_0 + \partial\bar{\partial}\phi)$ are

$$\psi_i(z,\bar{z}) = s_i(z) h_0^k(z,\bar{z}) e^{-k\phi(z,\bar{z})}$$

Laughlin wave function on Riemann surface

The Laughlin wave function of N_k non-interacting fermions (integer QHE) is given by Slater determinant

$$\Psi(z_1, \ldots, z_{N_k}) = \frac{1}{\sqrt{N_k!}} \det \psi_i(z_j)$$

$$= \frac{1}{\sqrt{N_k!}} [\det s_i(z_j)] \cdot \prod_{j=1}^{N_k} h_0^k(z_j, \bar{z}_j) \cdot e^{-k \sum_j \phi(z_j, \bar{z}_j)}$$

where I plugged LLL wavefunctions on curved metric

$$\psi_i(z, \bar{z}) = s_i(z) h_0^k(z, \bar{z}) e^{-k\phi(z, \bar{z})}$$

The Laughlin wave function for the fractional QHE is

$$\Psi_\beta(z_1, \ldots, z_{N_k}) = \frac{1}{\sqrt{N_k!}} (\det \psi_i(z_j))^{\beta}$$

Partition function for integer QHE

Partition function (generating functional)

$$Z^{QHE}(g_0, g) = \int_{M^{\otimes N_k}} |\Psi(z_1, \dots, z_{N_k})|^2 \prod_{i=1}^{N_k} \sqrt{g} d^2 z_i =$$

$$= \frac{1}{N_k!} \int_{M^{\otimes N_k}} |\det s_i(z_j)|^2 e^{-k \sum_i \phi(z_i, \bar{z}_i)} \prod_{i=1}^{N_k} h_0^k(z_i, \bar{z}_i) \sqrt{g} d^2 z_i.$$

Varying $Z^{QHE}(g_0, g)$ with respect to $\delta\phi(z)$ allows to define density correlation functions $\rho(z) = \frac{1}{k} \sum_i \delta(z - z_i)$. On the complex plane, Wiegmann-Zabrodin (2006):

$$Z^{FQHE, S^2}(W) = \frac{1}{N!} \int_{\mathbb{C}^{\otimes N}} |\Delta(z)|^{2\beta} e^{-N \sum_i W(z_i)} \prod_{i=1}^{N} d^2 z_i,$$

Derivation

The integer QHE partition function enjoys determinantal representation

$$Z^{QHE}(g_0, g) = \det_{ij} \int_M \bar{s}_i s_j h_0^k e^{-k\phi} \sqrt{g} d^2 z,$$

studied by Donaldson (2004). Variation of the free energy wrt $\delta\phi(z)$

$$\delta \log Z^{QHE}(g_0, g) = \int_M (-k\rho_k + \Delta\rho_k)\delta\phi \sqrt{g} d^2 z.$$

is controlled by the density of states function

$$\rho_k(z) = \sum_{i=1}^{N_k} \bar{\psi}_i(z)\psi_i(z) = \sum_{i=1}^{N_k} |s_i(z)|^2 h_0^k e^{-k\phi}.$$

In math this is known as the Bergman kernel on the diagonal.

Bergman kernel

Density function (Bergman kernel) has a local expansion for large k

$$\rho_k(z) = \sum_{i=1}^{N_k} \bar{\psi}_i(z)\psi_i(z) = \sum_{i=1}^{N_k} |s_i(z)|^2 h_0^k e^{-k\phi}$$

$$= k^n \left(1 + \frac{1}{2k}R(z) + \frac{1}{3k^2}\Delta R + ...\right)$$

known as Tian-Yau-Zelditch expansion (Zelditch' 98). Total number of states is $N_k = \int_M \rho_k(z) g d^2 z$, in complex dimension $n = 1$ we get

$$N_k = k + \frac{2 - 2h}{2}$$

(Riemann-Roch)

Bergman kernel from path integral

In physics the Bergman kernel expansion can be derived from the path integral for a particle in the magnetic field $B = dA = kg$ (Douglas, S.K., 2008), taking large time limit of

$$\rho_k(z) = \lim_{T\to\infty} \langle z | e^{-TH} | z \rangle =$$

$$= \lim_{T\to\infty} \int_{z(0)=z}^{z(T)=z} e^{-\frac{1}{\hbar}\int_0^T dt\,(g_{a\bar a}\dot z^a \dot{\bar z}^{\bar a} + A_a \dot z^a)} \prod_{0<t<T} \sqrt{g(z(t))}\,\mathcal{D}z(t)\mathcal{D}\bar z(t) =$$

$$= k^n \left[1 + \frac{\hbar}{2k} R + \frac{\hbar^2}{k^2}\left(\frac{1}{3}\Delta R + \frac{1}{24}|\mathrm{Riem}|^2 - \frac{1}{6}|\mathrm{Ric}|^2 + \frac{1}{8}R^2 \right) + \ldots \right].$$

Derivation, cont'd

Using the expansion of ρ_k we can integrate out the free energy order by order in k (in principle to all orders)

$$\delta \log Z^{QHE}(g_0, g) = \int_M (-k\rho_k + \Delta\rho_k)\delta\phi\, g d^2 z$$

$$\log Z^{QHE}(g_0, g) = -2\pi k^2 S_{AY}(g_0,\phi) + \frac{k}{2} S_M(g_0,\phi) + \frac{1}{12\pi} S_L(g_0,\phi) + \mathcal{O}(1/k)$$

where the following functionals appear

$$S_{AY}(g_0,\phi) = \int_M (\frac{1}{2}\phi\partial\bar{\partial}\phi + \phi g_0) d^2 z \qquad \text{Aubin} - \text{Yau}$$

$$S_M(g_0,\phi) = \int_M (-\phi R_0 + g \log \frac{g}{g_0}) d^2 z \qquad \text{Mabuchi}$$

$$S_L(g_0,\sigma) = \int_M (\partial\sigma\bar{\partial}\sigma + R_0\sigma) d^2 z, \qquad \text{Liouville}$$

Derivation, cont'd

All these functionals satisfy one-cocycle condition on the space of metrics: $S(g_0, g_2) = S(g_0, g_1) + S(g_1, g_2)$. Starting from order $1/k$ this becomes easy, since $S(g_0, g) = S(g) - S(g_0)$ (exact one-cocycle):

$$\log Z^{QHE}(g_0, g) = -2\pi k^2 S_{AY}(g_0, \phi) + \frac{k}{2} S_M(g_0, \phi) + \frac{1}{12\pi} S_L(g_0, \phi)$$
$$-\frac{5}{96\pi k}\left(\int_M R^2 g d^2 z - \int_M R_0^2 g_0 d^2 z\right) + \ldots$$

Derivation, cont'd

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Conjecture: all remainder terms (starting order $1/k$ and lower) are exact one-cocycles.

Derivation, cont'd

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$$-\frac{5}{96\pi k}\left(\int_M R^2 g d^2 z - \int_M R_0^2 g_0 d^2 z\right) +$$
$$+\frac{1}{2880\pi k^2}\left(\int_M (29R^3 - 66R\Delta R)\, g d^2 z - \right.$$
$$\left. - \int_M (29R_0^3 - 66R_0\Delta_0 R_0)\, g_0 d^2 z\right) + \mathcal{O}(1/k^3)$$

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Derivation, cont'd

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Fractional QHE (any β)

The large N expansion was derived by Can, Laskin, Wiegmann (2014), who generalized WZ loop equation method to the sphere (more in the next talk by Tankut Can)
We rederive this expansion by a different method, based on free field representation (check arxiv on Monday). We also propose a new path integral parametrix of the remainder series

$$\log Z^{FQHE}(g_0, g) = -\beta 2\pi k^2 S_{AY}(g_0, \phi) + \beta \frac{k}{2} S_M(g_0, \phi) + \frac{3\beta - 1}{24\pi} S_L(g_0, \phi) +$$

$$+ \log \int \left(\int_M e^{i\sqrt{\beta}\sigma(z)} \sqrt{g} d^2 z \right)^{N_k} e^{-\frac{1}{4\pi} S(g,\sigma)} \mathcal{D}_g \sigma -$$

$$- \log \int \left(\int_M e^{i\sqrt{\beta}\sigma(z)} \sqrt{g_0} d^2 z \right)^{N_k} e^{-\frac{1}{4\pi} S(g_0,\sigma)} \mathcal{D}_{g_0} \sigma$$

QHE and random geometry

When Polyakov derived his formula for the conformal field theory $Z^{CFT}(g) = e^{\frac{c}{12\pi} S_L(g_0,\sigma)} Z^{CFT}(g_0)$, he immediately realized that it can be used to define the probability measure on two dimensional metrics

$$d\mu_g^L = e^{\frac{c}{12\pi} S_L(g_0,\sigma)} \mathcal{D}g,$$

thus defining "random geometry", induced by free fields. This gave rise to a beautiful subject of Liouville theory and non-critical string theory.
Following the same logic, the QHE effect induces its own "random geometry":

$$d\mu_g^{QHE} = Z^{QHE}(g, g_0) \mathcal{D}g$$

where $\mathcal{D}g$ is an appropriate measure on metrics.

Random Kähler metrics

With F.Ferrari and S.Zelditch we develop an approach to random metrics, based on Kähler geometry. Consider Bergman metrics

$$g_B = \frac{1}{k}\partial\bar{\partial}\log\sum_{i,j}^{N_k}\bar{s}_i(\bar{z})H_{ij}s_j(z), \quad \text{where } H^\dagger = H, \text{ and positive} - \text{def.}$$

As $N_k \to \infty$, the space of Bergman metrics (space of positive-definite hermitian matrices) approximates the space of all Kähler metrics. The (regularized) sum over the metrics is given by the large N scaling limit

$$d\mu(H) = e^{-\gamma S(H)}(\det H)^{-N_k}[dH], \quad [dH] = \prod_{i,j} d\mathrm{Re}\, H_{ij} d\mathrm{Im}\, H_{ij}$$

where $S(H)$ is regularized action, given by one of the functionals (Aubin-Yau, Mabuchi, Liouville). This partition function is not eigenvalue-type $[dH] = [d\lambda][dU]$. Measure $d\mu(H)$ is probability measure for $\gamma > \gamma_{\mathrm{crit}}$. Convergence of these integrals is related to stability in Kähler geometry (SK, Zelditch 1404.0659)

Thank you