LECTURE 12: THETA DIVISOR

12.1. **Zeroes of the Riemann theta function.** Obviously $\vartheta(e) \not\equiv 0$ for $e \in \mathbb{C}^g$ because it is given by a Fourier expansion with non-zero coefficients.

Let now τ_{jl} be a period matrix of the Riemann surface X of genus g. The function theory on X can be studied using the Jacobean embedding, via theta function

$$f(P) = \vartheta\left(\int_{P_0}^{P} \omega_j - e_j, \tau\right) \tag{1}$$

as a function of a point $P \in X$, for an arbitrary vector $e_j \in \mathbb{C}^g$.

This function is locally single-valued, but globally multi-valued on X. It is invariant around a-cycles. Around b-cycles it transforms as

$$\begin{aligned}\vartheta\left(-e_j + \int_{P_0}^{P} \omega_j + \int_{b_k} \omega_j, \tau\right) &= \vartheta\left(-e_j + \int_{P_0}^{P} \omega_j + \tau_{kj}, \tau\right)\\ &= e^{-\pi i \tau_{kk} - 2\pi i (\int_{P_0}^{P} \omega_k - e_k)} \vartheta\left(-e_j + \int_{P_0}^{P} \omega_j, \tau\right).\end{aligned} \tag{2}$$

It follows that its zeroes are well-defined on X.

The set of zeroes of theta function is called *theta divisor*. The goal of this lecture is to describe the theta divisor in terms of divisors on X.

Theorem **12.1. (Riemann vanishing theorem)**.

1. *Theta function either vanishes identically $f(P) \equiv 0$ on X or has g zeroes (counting multiplicities) $Q_1, ..., Q_g$.*
2. *In the latter case there exists a vector $\Delta_j \in \mathbb{C}^g$, such that*

$$\sum_{l=1}^{g} \int_{P_0}^{Q_l} \omega_j = e_j - \Delta_j \mod \Lambda. \tag{3}$$

Proof. (1) Consider the canonical dissection X_0 and assume all zeroes are separate and $Q_i \in X_0$, $P_0 \in X_0$. Let δ_j be small disks around Q_j. Consider differential df/f and apply Stokes theorem

$$\begin{aligned}0 = \int_{X_0 - \cup \delta_j} d\frac{df}{f} = \int_{\partial(X_0 - \cup\delta_j)} \frac{df}{f} &= -\sum_j \int_{\partial\delta_j} \frac{df}{f}\\ &+ \sum_{l=1}^{g} \left(\int_{a_l^-} - \int_{a_l^+} + \int_{b_l^-} - \int_{b_l^+}\right) d\log f,\end{aligned} \tag{4}$$

since df/f is holomorphic in $X_0 - \cup\delta_j$. Since f is invariant under a-cycles, b-integrals cancel out. Since b_l joins a_l^- and a_l^+ and around b_l-cycle we have Eq. (2) $d\log f|_{a_l^-} - d\log f|_{a_l^+} = 2\pi i\omega_l$. Hence

$$\#\text{ of zeroes of } f = \frac{1}{2\pi i}\sum_j \int_{\partial\delta_j} \frac{df}{f} = \sum_{l=1}^{g} \int_{a_l} \omega_l = g. \tag{5}$$

(2) Here the goal is to derive the formula for the vector of Riemann constants Δ_j

$$\Delta_j = \frac{1}{2} + \frac{1}{2}\tau_{jj} - \sum_{l\neq j} \int_{a_l} \omega_l \int_{P_0}^{P} \omega_j \tag{6}$$

and the idea is to apply the previous argument to the one form $g_k df/f$, where $\omega_k = dg_k$ and $g_k(P_0) = 0$ on X_0.

$$0 = \int_{X_0 - \cup\delta_j} d\big(g_k \frac{df}{f}\big) = ... \tag{7}$$

One can check that Δ_j in Eq. (6) is independent of the integration path, but depends on the base point.

$\square$

12.2. **Theta divisor and X_{g-1}.** In Lec. 7.5 we defined the set of positive divisors of degree n as $X_n = X \times ... \times X/Sym_n$.

Lets denote the image of X_n under the Abel map as $W^n = I(X_n)$. Since Abel map does not distinguish linearly equivalent divisors (Abel theorem Thm. 6.1), W^n really acts the set of equivalence classes of divisors X_n of degree n. From the Jacobi inversion theorem (sec. 7.5) we also know that $I(X_g)$ is surjective, i.e. W^g is equal to the whole of $Jac(X)$.

The following theorem establishes that the theta divisor is isomorphic to W^{g-1} and thus to the set of equivalence classes of divisors X_{g-1} of degree $g-1$.

THEOREM **12.2.** *For a vector $e_j \in \mathbb{C}^g$.*

$$\vartheta(e) = 0 \iff e \in W^{g-1} + \Delta,$$

i.e. $\vartheta(e)$ vanishes iff $\exists D \in X_{g-1}, D > 0$, s.t. $e = I(D) + \Delta$.

Proof. First, we prove $\Leftarrow$. Consider a non-special divisor of degree g, $D \in X^g$, $i(D) = 0$. From the proof of Jacobi inversion theorem we know that in a neighborhood of D all divisors D' are also non-special. So we will prove the statement for the neighborhood and extend to all space.

Let $D = P_1 + ... + P_g$ and set $e = I(D) + \Delta$. Then consider the function Eq. (1). If $\vartheta(I(P) - e) \equiv 0$, then for $1 \leqslant j \leqslant g$,

$$0 = \vartheta(I(P_j) - e) = \vartheta(I(P_1, ...\hat{P}_j..., P_g) + \Delta)$$

where $\hat{D}_j$ is deletion of D_j. Thus the statement of the theorem follows.

On the other hand, if $\vartheta(I(P)-e) \not\equiv 0$, then by Riemann vanishing theorem, there is a divisor D_Q of g zeroes Q_l, $l=1,...,g$, such that $e = I(D_Q)+\Delta$. Hence $I(D_Q) = I(D)$. Hence $D_Q = D+(f)$, where f is meromorphic. But by assumption $i(D)=0$, hence $i(D_Q)=0$, because $i(\cdot)$ depends only on divisor class (see sec. 7.2). If $f \not\equiv const$, then $\exists f \not\equiv const$, such that $(f)+D = D_Q > 0$, and $(f) > -D$, hence $i(D)>0$. But $i(D)=0$ by assumption $\Rightarrow f = const, \Rightarrow D = D_Q$ and $e = I(D)+\Delta$. Then it follows again that

$$0 = \vartheta(I(P_j) - I(D_Q) - \Delta) = \vartheta(I(P) - I(D) - \Delta) = \vartheta(I(P_1, ...\hat{P}_j..., P_g) + \Delta)$$

.

So we proved that $\vartheta(I(P_1,...,P_{g-1})+\Delta)$ vanishes identically on a full neighborhood D^{g-1}, hence identically zero on D^{g-1}.

Next, let us prove $\Rightarrow$, i.e. that $W^{g-1}+\Delta$ is the complete set of zeroes of ϑ in $Jac(X)$. Suppose $\vartheta(e)=0$. There exists $s \leqslant g$ such that

$$\vartheta(I(D_1) - I(D_2) - e) \not\equiv 0 \quad \text{for } D_1, D_2 \in X_s$$

while

$$\vartheta(I(D_1') - I(D_2') - e) = 0 \quad \text{for all } D_1', D_2' \in X_r,\ 0 \leqslant r < s.$$

Indeed, by Jacobi inversion we know that $I(X_g)$ is all of $Jac(X)$ and $\theta(z) \not\equiv 0$ for all $z \in Jac(X)$. Hence the bound $s \leqslant g$.

Let us write $D_1 = P_1 + D_1'$ and $D_2 = P_2 + D_2'$ where P_1, P_2 are two points in D_1 and D_2 and $D_1', D_2' \in X_{s-1}$. Consider theta function

$$f(P) = \vartheta\left(\int_{P_2}^{P} \omega + I(D_1') - I(D_2') - e\right), \quad P \in X.$$

It vanishes on D_2, hence $(f) \geqslant D_2$ and by assumption it does not vanish identically. Hence, by Riemann vanishing the zero divisor $D_3 := (f)$ has degree g and satisfies

$$I_{P_2}(D_3) = e - I_{P_2}(D_1') + I_{P_2}(D_2') - \Delta_{P_2}$$

where subscript P_2 indicates that we take the base point P_2. We can write $D_3 = D_2 + \tilde{D}$, where $\deg \tilde{D} = g - s$. Then from equation above we get

$$e = I_{P_2}(D_2+\tilde{D}) + I_{P_2}(D_1') - I_{P_2}(D_2') + \Delta_{P_2} = I_{P_2}(\tilde{D}+D_1') + \Delta_{P_2}. \tag{8}$$

Finally, we note that $\deg(\tilde{D}+D_1') = g-s+s-1 = g-1$. □

One might wonder why in Eq. (8) the base point is P_2 and not P_0. However,

Proposition **12.1.** *For any $D \in X_{g-1}$, $I_{P_0}(D) + \Delta_{P_0} \in Jac(X)$ is independent of P_0.*

We have $I_{P_0}(D) = I_{P_1}(D) + (g-1)I_{P_0}(P_1)$. From Eq. (8) it follows immediately, that $\Delta_{P_0} = \Delta_{P_1} - (g-1)I_{P_0}(P_1)$.

Corollary **12.2.** *$\vartheta(W^r + \Delta) = 0$ for $0 \leqslant r \leqslant g-1$. In particular $\vartheta(\Delta) = 0$.*

Proof. Apply the Thm. 12.2 to the divisor $D = \underbrace{P_0 + \dots + P_0}_{r\,\text{times}} + P_{r+1} + \dots + P_{g-1} \in X_{g-1}$. □

12.3. **Zeroes of $f(P)$ and special divisors.** .

Let us return to the theta function $f(P) = \vartheta(I(P) - e)$ on Riemann surface $P \in X$ in Eq. (1) and determine the vanishing conditions in terms of speciality of divisors.

Proposition **12.3.** *Let a positive divisor of degree g $D \in X_g$ be a Jacobi inversion of $e - \Delta$,*

$$I(D) = e - \Delta.$$

Then

(1) $f(P) \equiv 0$ iff D is special, i.e. $i(D) > 0$,
(2) $f(P) \not\equiv 0$ iff D is non-special, i.e. $i(D) = 0$. Then D is zero divisor of f.

Proof. For (1):
by Thm. 12.2, $\vartheta(I(P) - e) \equiv 0 \Rightarrow \exists D_P \in X_{g-1}$, such that

$$-I(P) + e = I(D_P) + \Delta$$

Using $I(D) = e - \Delta$ we get $-I(P) + I(D) + \Delta = I(D_P) + \Delta$. Hence $I(D) = I(D_P + P)$. By Abel theorem $D \equiv D_P + P$ are linearly equivalent, i.e. $\exists$ meromorphic f, vanishing at an arbitrary point $P \in X$. Hence $l(-D) > 1$, or equivalently $i(D) > 0$.

On the other hand, if $i(D) > 0 \Rightarrow \exists f$ meromorphic, $f \not\equiv const$, vanishing at an arbitrary point on X, e.g., for example at $P \in X$ and having pole at some of the points in D. Then $-I(P) + e = -I(P) + I(D) + \Delta = I(D - P) + \Delta = I(D + (f) - P) + \Delta$ and $D + (f) - P > 0$ and of degree g. Hence by Thm. 12.2, $\vartheta(I(P) - e) = 0$.

Next, for (2), suppose D is non-special. Then $\vartheta(e - I(P)) = \vartheta(I(D) - I(P) + \Delta) \not\equiv 0$ as it is not of the form as in Thm. 12.2. But Riemann vanishing immediately implies for the divisor $(f(P))$ of zeroes of $f(P)$: $I((f(P))) = e - \Delta = I(D)$. Since D is non-special, this yields $D = (f(P))$. □