LECTURE 13: PRIME FORM

13.1. **Prime function.** Eventually we would like to make meromorphic functions out of ratios of theta functions $f(P)$, as we did in the case of the torus. For this we investigate the function

$$E_e(x,y) = \vartheta\left(e - \int_x^y \omega\right), \tag{1}$$

where $e \in \mathbb{C}^g$ is fixed and satisfies $\vartheta(e) = 0$.

Lemma **13.1.** *Let $e \in \mathbb{C}^g$ satisfy $\vartheta(e) = 0$ and $E_e(x,y) \not\equiv 0$. Then there are $2g-2$ points (counting multiplicities) $P_1, ..., P_{g-1}, Q_1, ..., Q_{g-1} \in X$ such that $E_e(x,y) = 0 \iff (x = y)$, or $x = P_j, \forall y \in X$ or $y = Q_j, \forall x \in X$.*

Proof. The proof is an immediate consequence of the previous results. Indeed, from $E_e(x,y) \not\equiv 0$ by Riemann vanishing there are exactly g zeroes $Q_1, .., Q_g$, where $E_e(x, Q_j) = 0$, such that $I_x(D_Q) \equiv e - \Delta_x$. Since $E_e(0,0) = \vartheta(e) = 0$, one of these zeroes is $Q_g = x$ and there exactly $g-1$ left.

On the other hand, $\vartheta(e) = 0 \to \vartheta(-e) = 0$ and $E_e(x,y) = E_{-e}(y,x)$. Applying the same argument as above we obtain g zeroes $P_1, ..., P_g$, such that $I_y(D_P) \equiv -e - \Delta_y$ and one of those is $P_g = y$. $\square$

13.2. **Constructing meromorphic functions.** This can be done with the help of the function $E_e(x,y)$. Consider the divisors of zeroes D_Q and poles D_P, both of degree d and

$$I(D_Q) = I(D_P), \tag{2}$$

by Abel theorem. Then choose $e \in \mathbb{C}^g$, such that

$$\vartheta(e) = 0, \quad E_e(P_j, y) \not\equiv 0, \quad E_e(Q_j, y) \not\equiv 0.$$

Then the function

$$f(y) = \frac{\prod_{j=1}^d E_e(Q_j, y)}{\prod_{j=1}^d E_e(P_j, y)} \tag{3}$$

will have required poles and zeroes. But one has to prove that its a function, i.e. single-valued along a and b period transforms. There is a choice of contours in Eq. (3) which achieves just that (see Mumford, p.159). Choose paths σ_j from P_0 to P_j, resp., τ_j from P_0 to P_j, so that

$$\sum_{j=1}^d \int_{\sigma_j} \omega = \sum_{j=1}^d \int_{\tau_j} \omega. \tag{4}$$

By assumption Eq. (2) this is always true mod Λ, but one could tweak the contours such that its true in $\mathbb{C}^g$. Then in Eq. (3) choose the contours from Q_j , resp., P_j, to y as $-\tau_j$, resp., $-\sigma_j$, followed by the same contour form P_0 to y everywhere. Then along a-cycles $f(y)$ is invariant and along b_k-cycle it pick up the factor

$$\frac{\prod_{j=1}^{d} \exp\left(-\pi i \tau_{kk} - 2\pi i (\int_{Q_j}^{y} \omega_k + e_k)\right)}{\prod_{j=1}^{d} \exp\left(-\pi i \tau_{kk} - 2\pi i (\int_{P_j}^{y} \omega_k + e_k)\right)} = 1,$$

due to the condition (4).

The function $E_e(x, y)$ is called Prime function, it is a close analog of $x - y$ and of $\vartheta_1(x - y)$ on a genus $g > 1$ Riemann surface. There exists even better object, called "Prime form", whose divisor of zeroes is just the diagonal $E(x, y) = 0 \iff x = y$.

13.3. **Vector of Riemann constants and canonical divisor.**

Lemma **13.2.** *Vector of Riemann constants Δ is related to canonical divisor as*

$$2\Delta = -I(K)$$

Proof. Consider a positive divisor $D_1 \in X_{g-1}$, then ϑ function vanishes at

$$e = I(D_1) + \Delta.$$

Then $\vartheta(-e) = 0$ also vanishes by evenness. Hence $\exists D_2 \in X_{g-1}$, such that

$$-e = I(D_2) + \Delta.$$

Adding up, we obtain

$$2\Delta = -I(D_1 + D_2),$$

for $\forall D_1 \in X_g$. Now we need to show that $D = D_1 + D_2 = (\omega)$, where ω is an Abelian differential. This means $i(D) > 0$, or, equivalently by Riemann-Roch, $h^0(D) - 2g + 2 + g - 1 = i(D) > 0$, hence $h^0(D) \geqslant g$.

For proof we refer to Farkas-Kra, p.298.

□

13.4. **Odd theta characteristics.** Consider half-periods of Λ

$$\delta = \delta' + \delta''\tau, \quad \delta', \delta'' \in \{0, \frac{1}{2}\}^g,$$

These are called half-integer characteristics or theta characteristics. Their number is 4^g. Recall the definition of theta function with characteristics (lecture 9, Eq. 15),

$$\vartheta\left[\begin{smallmatrix} \delta' \\ \delta'' \end{smallmatrix}\right](z) = e^{\pi i \delta' \tau \delta' + 2\pi i \delta'(z + \delta'')} \vartheta(z + \delta'' + \delta'\tau). \tag{5}$$

Then we have

$$\vartheta\left[\begin{smallmatrix} \delta' \\ \delta'' \end{smallmatrix}\right](z) = e^{-4\pi i (\delta', \delta'')} \vartheta\left[\begin{smallmatrix} \delta' \\ \delta'' \end{smallmatrix}\right](-z),$$

where $(\delta', \delta'') = \sum_j \delta'_j \delta''_j$. Depending on parity of $4(\delta', \delta'')$ the theta characteristic is called odd or even. It follows that $\vartheta\left[\begin{smallmatrix} \delta' \\ \delta'' \end{smallmatrix}\right](0)$ vanishes, or equivalently $\vartheta(\delta) = 0$ for odd theta characteristics.

Proposition **13.3.** *There is a divisor $D_\delta \in X_{g-1}$ corresponding to each odd theta characteristic*

$$\delta = I(D_\delta) + \Delta, \tag{6}$$

such that $2D_\delta \equiv K$.

Proof. From Thm. 12.2 (lecture 12), from $\vartheta(\delta) = 0$ it follows that $\delta = I(D_\delta) + \Delta,\ D_\delta \in X_{g-1}$. Then

$$\delta = I(D_\delta) + \Delta \Rightarrow 2\delta = I(2D_\delta) + 2\Delta \Rightarrow I(2D_\delta) = -2\Delta = I(K)$$

Since $2\delta \in \Lambda$ and by Lemma 13.2. Then by Abel Thm., $2D_\delta \equiv K$. □

It follows that there exists holomorphic differential ω_δ such that $(\omega_\delta) = 2D_\delta$. Let us now construct it in terms of theta functions.

Theta characteristic δ is called non-singular, if there exists non-vanishing partial derivative of ϑ at δ, $\frac{\partial \vartheta}{\partial z_j}(\delta) \neq 0$, at least for some j's. Existence of non-singular odd theta characteristic follows from Lefschetz embedding theorem (lecture 9, and Mumford p. 128). A version of this theorem states that $z \in \mathbb{C}^g/\Lambda \to (..., \vartheta[\delta', \delta''](z), ...) \in P^{2^g-1},\ \delta', \delta'' \in \frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g$ is an embedding. Hence $d\vartheta[\delta', \delta''](z) \not\equiv 0$ partial derivatives do not vanish simultaneously at any point. Let us look at $\vartheta[\delta', \delta''](0)$. If δ is even, then $\vartheta[\delta](0)$ is even, so $d\vartheta(0) \equiv 0$ for all even characteristics. But there should be some $d\vartheta(0) \neq 0$, and by necessity it corresponds to some odd δ.

Lemma **13.4.** *Let δ be non-singular odd theta characteristic, $D_\delta \in X_{g-1}$ is the corresponding divisor Prop. 13.3, Then the holomorphic differential ω_δ, s.t. $(\omega_\delta) = 2D_\delta$ is given by*

$$\omega_\delta = \sum_{j=1}^{g} \frac{\partial \vartheta}{\partial z_j}(\delta)\omega_j,$$

where ω_j is the canonical basis of holomorphic differentials.

Proof. For any positive divisor $D = P_1 + ... + P_{g-1} \in X_g$ we have $\vartheta(I(D) + \Delta) \equiv 0$. Differentiating this with respect to P_j we get

$$\omega_D = \sum_j \frac{\partial \vartheta}{\partial z_j}(I(D) + \Delta)\omega_j(P_k) = 0,$$

for all P_k. Specifying this for non-singular odd theta characteristic, $\omega_\delta = \sum_{j=1}^{g} \frac{\partial \vartheta}{\partial z_j}(\delta)\omega_j$ vanishes at all $P_k \in D_\delta$, hence $(\omega_\delta) \geqslant D_\delta$.

Let us now demonstrate that D_δ is uniquely determined by Eq. (6), i.e. $i(D_\delta) = 1$. Suppose $i(D_\delta) > 1$, i.e. $\exists f \in H^0(D_\delta)$, $f \not\equiv const$. The divisor of $f(P) - f(P_0)$ for an arbitrary P_0 is $P_0 + D_{P_0} - D_\delta$ with some $D_{P_0} \in X_{g-2}$, $D_{P_0} > 0$. Then

$$\omega_\delta = \sum_{j=1}^{g} \frac{\partial\vartheta}{\partial z_j}(\delta)\omega_j$$

vanishes at D_δ and also at $P_0 + D_{P_0}$, because $\sum_{j=1}^{g} \frac{\partial\vartheta}{\partial z_j}(\delta)\omega_j = \sum_{j=1}^{g} \frac{\partial\vartheta}{\partial z_j}(I(D) + \Delta)\omega_j = \sum_{j=1}^{g} \frac{\partial\vartheta}{\partial z_j}(I(D + (f)) + \Delta)\omega_j = \sum_{j=1}^{g} \frac{\partial\vartheta}{\partial z_j}(I(P_0 + D_{P_0}) + \Delta)\omega_j$. Thus $\omega_\delta(P_0) = 0$, but P_0 is arbitrary. Hence $\omega_\delta(P_0) \equiv 0$, but this is impossible by the assumption of non-singularity of δ. Thus D_δ is unique and $i(D_\delta) = 1$. We have shown that $(\omega_\delta) > D_\delta$ and proved that the space of holomorphic differentials ω vanishing at D_δ is one dimensional, hence ω_δ is given by the formula above, up to multiplication by a constant. $\square$

In particular, all zeroes of ω_δ are double zeroes.

13.5. **Prime form.** We would like to construct a holomorphic function $E(x, y)$ vanishing to first order on the diagonal, and otherwise non-vanishing, i.e. improve the function $E_e(x, y)$ of sec. 13.1. On the sphere there is $x - y$, but it is not a function, as it blows up at infinity. Mumford (p.3.207) suggests to consider instead a $(-\frac{1}{2}, -\frac{1}{2})$-differential

$$E(x, y) = \frac{x - y}{\sqrt{dx}\sqrt{dy}}, \tag{7}$$

where $\sqrt{dx}$ is a formally defined half-differential, or "spinor", which transforms as $x' = 1/x$, $\sqrt{dx'} = \frac{\sqrt{-1}}{x} dx$. Then

$$E(x', y') = \frac{x' - y'}{\sqrt{dx'}\sqrt{dy'}} = \frac{1/x - 1/y}{-\sqrt{dx}/x \cdot \sqrt{dy}/y} = E(x, y),$$

so its well-defined on $\mathbb{P}^1 \times \mathbb{P}^1$.

On a general Riemann surface the generalization of (7) is called the Prime form. In the numerator we can put the Prime function Eq. (1). It vanishes on the diagonal as required, but it has extra zeroes. To compensate for that we divide by something to cancel out those zeroes.

If holomorphic differential has only even zeroes, as e.g. ω_δ, then its square-root $\sqrt{\omega_\delta}$ is the half-differential (spinor). Consider

$$E(x, y) = \frac{\vartheta\left[\begin{smallmatrix}\delta' \\ \delta''\end{smallmatrix}\right]\left(\int_x^y \omega\right)}{\sqrt{\omega_\delta(x)}\sqrt{\omega_\delta(y)}} \tag{8}$$

This function has the following properties

- It is antisymmetric
$$E(x,y) = -E(y,x).$$
This follows from the property of odd theta characteristic.
- It vanishes only on the diagonal to first order. Indeed
$$\vartheta\left[\begin{smallmatrix}\delta'\\ \delta''\end{smallmatrix}\right]\Big(\int_x^y \omega\Big) \sim \vartheta\Big(\delta - \int_x^y \omega\Big)$$
and the zeroes of the latter are characterized by Lemma 13.1, taking into account Eq. (6), $\delta = I(D_\delta) + \Delta$: its either $(y = x)$, or $y = P_k \in D_\delta$ and the same for x: $(x = y), x = P_k \in D_\delta$.

  Now, $\sqrt{\omega_\delta(x)}$ has zeroes on D_δ, so the extra unwanted zeroes cancel out between numerator and denominator.

  For the near-diagonal behavior one can write
$$E(x,y) = \frac{x-y}{\sqrt{dx}\sqrt{dy}}(1 + \mathcal{O}((x-y)^2)))$$
- Prime form is independent of the choice of the odd non-singular characteristic.

  This follows from the fact, that the periodicity properties of the Prime form along a, b-cycles are independent of δ. Indeed, it is invariant under a-cycles and for b-cycle it transforms as
$$E(x + b_k, y) = e^{-\pi i \tau_{kk} - 2\pi i(\int_y^x \omega_k)} E(x,y). \tag{9}$$
- If $D = \sum n_j P_j$ is a divisor of meromorphic function, then $f \sim \prod_j E(x, P_j)^{n_j}$.

The spinor, or half-differential can be described in terms of line bundles as assignment of holomorphic functions f_α for open covering U_α, such that
1. $f_\alpha \sqrt{dz_\alpha} = f_\beta \sqrt{dz_\beta}$ 2. cocycle condition for triple intersections.
In other words, it is a section of line bundle S, such that $S^2 = K$. Degree of S is $g-1$. There exists 4^g non-isomorphic spin bundles on X of genus g.