LECTURE 14: BOSONISATION FORMULAS

14.1. **Divisors, line bundles and holomorphic sections.** We have formulated Riemann-Roch theorem for the dimension of the vector space of meromorphic functions f with divisor $(f) \geqslant -D$

$$\dim H^0(X, \mathcal{O}(D)) = \deg D - g + 1 + i(D),$$

where $i(D)$ is index of speciality. We also gave another point of view on Riemann-Roch formula, in terms of the holomorphic line bundles and their sections.

We can associate to each divisor a class of holomorphic line bundles in the following way. Consider covering $\{U_\alpha\}$ such that each points of D

Given the divisor $D = \sum_j n_j P_j$, we consider a covering $\{U_\alpha\}$ such that each P_j belongs to only one open set. Take for example the meromorphic section ϕ, such that its divisor $(\phi)|_{U_\alpha}$ in each U_α coincides with D_α (D restricted to U_α),

$$(\phi)|_{U_\alpha} = D_\alpha.$$

We can write locally such a section as $(\phi)|_{U_\alpha} = z_\alpha^{n_j}$, where z_α is local coordinate vanishing at $P_j \in U_\alpha$. Then given the basis $f_j, j = 1, ..., h^0(D)$ we can construct the basis of holomorphic sections

$$s_j|_{U_\alpha} = (\phi)|_{U_\alpha} f_j, \quad j = 1, ..., h^0(D) = \dim H^0(X, \mathcal{O}(D)).$$

These are locally holomorphic, when restricted to U_α, because by definition, the divisor $\cup_\alpha (s_j)|_{U_\alpha} = (\phi) + (f_j) = D + (f_j) \geqslant 0$.

If we take another meromorphic section ϕ' with the same divisor $D_\alpha = (\phi)|_{U_\alpha}$, then the bases of sections can be identified by multiplication by $h_\alpha = \phi'_\alpha/\phi_\alpha \in \mathcal{O}^*(U_\alpha)$ and the line bundles corresponding to (ϕ) and (ϕ') are by definition isomorphic. So $L[D]$ will denote the corresponding isomorphic line bundles constructed this way.

On the other hand, two divisors D and D' are linearly equivalent $D \equiv D'$ iff the line bundles $L[D]$ and $L[D']$ are isomorphic.

Indeed, $D \equiv D'$ means there exists meromorphic f on X such that $(f) = D' - D$. Given meromorphic sections ϕ of $L[D]$ we get meromorphic section $\phi' = h\phi$ of $L[D']$ and vice versa. In the opposite direction, if ϕ is meromorphic section of $L[D]$, ϕ' is meromorphic section of $L[D']$ and $L[D]$ and $L[D']$ are isomorphic, then it means that there is $h_\alpha \in O^*(U_\alpha)$ such that $\phi'_\alpha = h_\alpha \phi_\alpha$ and hence the transition functions satisfy $g'_{\alpha\beta} h_\beta = g_{\alpha\beta} h_\alpha$.

Then we can check that $h_\alpha \phi_\alpha / \phi'_\alpha$ is meromorphic on all of X. Indeed

$$h_\alpha \phi_\alpha / \phi'_\alpha = h_\alpha (g_{\alpha\beta} \phi_\beta)/(g'_{\alpha\beta} \phi'_\beta) = h_\beta \phi_\beta / \phi'_\beta.$$

Then divisor of $(h\phi/\phi') = D - D'$, hence $D \equiv D'$.

14.2. **Vandermonde matrix.** Consider N complex numbers $z_1, ..., z_N$. The following matrix is called the Vandermonde matrix

$$V_{ij} = \begin{pmatrix} 1 & z_1 & z_1^2 & ... & z_1^{N-1} \\ 1 & z_2 & z_2^2 & ... & z_2^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & z_N & z_N^2 & ... & z_N^{N-1} \end{pmatrix} \tag{1}$$

There exists a simple formula for its determinant

$$\det V_{ij} = \det z_i^{j-1} = \prod_{i<j}^{N} (z_j - z_i). \tag{2}$$

The proof is purely combinatorial. Here we would like to take a geometric point of view on this formula. Think of z_j as local coordinates on the N copies of P^1. Consider a divisor D of the positive degree $N-1$ (all such divisors are linearly equivalent, because there is a meromorphic function for any degree zero divisor) and the corresponding line bundle $L[D]$. For example we can take $D = (N-1) \cdot P$, where $z_P = 0$ in the local coordinate z. Then $\dim H^0(O(D)) = N$ and we can pick a basis of holomophic sections as $s_j(z) = z^{j-1}, j = 1, ..., N$.

Then the identity Eq. (2) can be proved by comparing zeroes on both sides. As a function of, e.g., z_1 the right hand side has simple zeroes at $z_2, ..., z_N$. Same on the left, due to vanishing of the determinant for equal rows. Then the ratio of the left and right side is a holomorphic function, hence it is constant. More generally, if $\psi_j(z)$ is any basis of holomorphic sections, then

$$\det \psi_i(z_j) = \text{const} \cdot \prod_{i<j}^{N} (z_j - z_i),$$

where constant is normalization dependent.

We are interested in generalization of this formula to higher genus surfaces.

14.3. **Fay's formula.** Here we follow Fay's book *Theta functions on Riemann surfaces* (1973).

PROPOSITION **14.1.** *Let D be a non-special divisor of degree $N + g - 1 \geq g - 1$ with* $\dim H^0(D) = \deg D + 1 - g + i(D) = (g+N-1)+(1-g)+0 = N$. *Suppose $\psi_j, j = 1, ..., N$*

form a basis of holomorphic sections of the line bundle $L[D]$. *Then for* $(z_1, ..., z_N) \in X^N$ *we have*

$$\operatorname{div}_{X^N}\Big(\det \psi_j(z_j)\Big) = \operatorname{div}_{X^N}\left(\vartheta\Big(\sum_1^N z_j - D - \Delta\Big) \cdot \prod_{i<j}^N E(z_i, z_j)\right), \tag{3}$$

where $E(x, y)$ *is the Prime form on* X.

Proof. The formula above is symmetric in z_j's, so it is enough to prove it for, e.g. z_1. The left hand side is a holomorphic section in z_1 of the line bundle $L[D]$. Hence its divisor is positive. We know that its divisor is linearly equivalent to D. In particular, it must have $N + g - 1$ zeroes (counted with multiplicities). Locations of $N - 1$ zeroes are obvious, so we have

$$D_{\text{lhs}} = \operatorname{div}_{X_1}\Big(\det \psi_j(z_j)\Big) = z_2 + z_3 + ... + z_N + P_1 + ... + P_g \equiv D,$$

where $P_1, ..., P_g$ are so far undetermined, but not arbitrary, since they should satisfy the linear equivalence.

Next, the divisor of zeroes on the right is

$$D_{\text{rhs}} = z_2 + ... + z_N + \operatorname{div}_{X_{z_1}} \vartheta\Big(\sum_1^N z_j - D - \Delta\Big).$$

Theta function can be written in our standard notations as

$$\vartheta\Big(\sum_1^N z_j - D - \Delta\Big) = \vartheta\Big(I(z_1) + I(\sum_2^N z_j - D) - \Delta\Big).$$

This could vanish identically on X^N iff $D - \sum_1^N z_j$ is linearly equivalent to a positive divisor of degree $g - 1$ $\tilde{D}_{g-1} \in X_{g-1}$, for all z_j (Thm. 12.2). Let us show that this can't happen. Suppose $\exists f_0$ meromorphic, such that $(f_0) + D - \sum_1^N z_j = \tilde{D}_{g-1} > 0$. Then such $f_0 \in H^0(\mathcal{O}(D))$, because $(f_0) + D \geqslant 0$. Writing $f_0 = \sum c_j f_j$ we arrive at homogeneous system $\sum c_j f_j(z_k) = 0$ for all z_k, which generically has only a trivial solution $c_j = 0$, since we can always make $\det f_j(z_k) \neq 0$. Hence $\dim H^0(\mathcal{O}(D)) > N$, which is a contradiction. Hence, $\vartheta \not\equiv 0$.

Hence, while $\deg D - \sum_1^N z_j = g - 1$, this divisor is in general non-positive, hence by Riemann vanishing the theta function (as a function of z_1) has exactly g zeroes $\tilde{P}_1, ..., \tilde{P}_g$, satisfying

$$I(\tilde{P}_1 + ... + \tilde{P}_g) = \Delta - I(\sum_2^N z_j - D) - \Delta = I(D - \sum_2^N z_j).$$

The takeaway is that divisor of zeroes of the rhs is uniquely determined $D_{\text{rhs}} = z_2 + ... + z_N + \tilde{P}_1 + ... + \tilde{P}_g$. And from equality for Abel maps above it follows that $D_{\text{rhs}} \equiv D_{\text{lhs}}$. But we need to show that $D_{\text{lhs}} = D_{\text{rhs}}$, or equivalently $\tilde{P}_1 + ... + \tilde{P}_g = P_1 + ... + P_g$.

The key point is that the degree g divisor $\tilde{P}_1 + ... + \tilde{P}_g$ is non-special - if it were special then ϑ would be identically zero (lecture 12, Prop. 12.3). Suppose there is a nonconstant meromorphic function f such that $\tilde{P}_1 + ... + \tilde{P}_g + (f) = P_1 + ... + P_g > 0$. Then it follows that $\dim H^0(\sum \tilde{P}) = g + 1 - g + i(\sum \tilde{P}) = 1$, hence f is constant. Hence

$$\tilde{P}_1 + ... + \tilde{P}_g = P_1 + ... + P_g.$$

Hence divisors on the lhs and rhs coincide.

□

Example: Consider the line bundle of degree N on the torus with the holomorphic sections $\psi_j = \vartheta\left[\begin{smallmatrix} \frac{\alpha+j}{N} \\ \beta \end{smallmatrix}\right](Nz, N\tau)$. Then (Fay, 1992, p. 116)

$$\det \psi_i(z_j) = \mu(\tau)\vartheta\left[\begin{smallmatrix} \alpha + \frac{N-1}{2} \\ \beta + \frac{N-1}{2} \end{smallmatrix}\right]\Big(\sum_1^N z_j, \tau\Big)\prod_{i<j}\vartheta\left[\begin{smallmatrix} \frac{1}{2} \\ \frac{1}{2} \end{smallmatrix}\right](z_i - z_j, \tau)$$

$N-1$ zeroes obviously coincide. Both sides are invariant under a-cycles, and under b-cycles (shift $z_1 \to z_1 + \tau$):

$$\vartheta\left[\begin{smallmatrix} \frac{\alpha+j}{N} \\ \beta \end{smallmatrix}\right](Nz_1 + N\tau, N\tau) = e^{-i\pi N\tau - 2\pi i N z_1 - 2\pi i\beta}\vartheta\left[\begin{smallmatrix} \frac{\alpha+j}{N} \\ \beta \end{smallmatrix}\right](Nz_1, N\tau)$$

$$\vartheta\left[\begin{smallmatrix} \alpha + \frac{N-1}{2} \\ \beta + \frac{N-1}{2} \end{smallmatrix}\right]\Big(\sum_1^N z_j + \tau, \tau\Big) = e^{-i\pi N\tau - 2\pi i N z_1 - 2\pi i\beta}\vartheta\left[\begin{smallmatrix} \alpha + \frac{N-1}{2} \\ \beta + \frac{N-1}{2} \end{smallmatrix}\right]\Big(\sum_1^N z_j, \tau\Big)$$

$$\prod_{1<j}\vartheta\left[\begin{smallmatrix} \frac{1}{2} \\ \frac{1}{2} \end{smallmatrix}\right](z_1 - z_j + \tau, \tau) = e^{-i\pi\tau(N-1) - 2\pi i\sum_2^N(z_1 - z_j) - 2\pi i\frac{1}{2}(N-1)}\prod_{1<j}\vartheta\left[\begin{smallmatrix} \frac{1}{2} \\ \frac{1}{2} \end{smallmatrix}\right](z_i - z_j, \tau)$$

The ratio LHS/RHS is a meromorphic function with one zero and one pole, hence constant.

Now we generalized this for special divisors D (Fay, Prop. 2.16).

PROPOSITION **14.2.** *Let D be a divisor of degree $g+n-1$ on X with $\dim H^0(D) = N \geqslant n$. Suppose $\psi_1, ..., \psi_N$ is a basis of the holomorphic sections of the line bundle $L[D]$. Then for any (generic) positive divisor $B = \sum_1^{N-n} b_j$ with $i(D+B) = 0$ and for $(z_1, ..., z_N) \in X^N$ we have*

$$\operatorname{div}_{X^N}\Big(\det\psi_j(z_j)\cdot\prod_{i=1}^N\prod_{j=1}^{N-n}E(z_i, b_j)\Big) = \operatorname{div}_{X^N}\left(\vartheta\Big(\sum_1^N z_j - D - B - \Delta\Big)\cdot\prod_{i<j}^N E(z_i, z_j)\right), \tag{4}$$

Proof. First, let's note that $\dim H^0(D) = (g+n-1) + 1 - g + i(D) = n + i(D) = N$. Hence $i(D) = N - n$.

Again, we consider the zeroes in z_1. On the left hand side, the divisor of zeroes is

$$D_{\mathrm{lhs}} = z_2 + ... + z_N + P_1 + ... + P_{g+n-N} + \sum_{j=1}^{N-n} b_j \equiv D + B.$$

One the rhs:

$$D_{\mathrm{rhs}} = z_2 + ... + z_N + \mathrm{div}_{X_{z_1}} \vartheta\Big(\sum_1^N z_j - D - B - \Delta \Big)$$

Same logic as before tells us that $\vartheta(\sum_1^N z_j - D - B - \Delta) \not\equiv 0$ on X^N, since otherwise $\exists f_0$, such that $(f_0) + D - B - \sum_1^N z_j = \tilde{D}_{g-1} > 0$ and this leads to $\dim H^0(D+B) > N$. But we know that $\dim H^0(D+B) = g+n-1+N-n+1-g+i(D+B) = N+i(D+B) = N$, since by assumption of theorem $i(D+B) = 0$.

Hence there exists a unique divisor of zeroes

$$\mathrm{div}_{X_{z_1}} \vartheta\Big(\sum_1^N z_j - D - B - \Delta \Big) = \tilde{P}_1 + ... + \tilde{P}_g$$

such that $\tilde{P}_1 + ... + \tilde{P}_g \equiv D + B - \sum_2^N z_j$, and

$$D_{\mathrm{rhs}} = z_2 + ... + z_N + \tilde{P}_1 + ... + \tilde{P}_g.$$

We proved that $D_{\mathrm{rhs}} \equiv D_{\mathrm{lhs}}$, but we need to prove $D_{\mathrm{rhs}} = D_{\mathrm{lhs}}$. Same argument as before. Since $\vartheta \not\equiv 0$, $\tilde{P}_1 + ... + \tilde{P}_g$ is non-special and degree g. Since $\tilde{P}_1 + ... + \tilde{P}_g \equiv P_1 + ... + P_{g+n-N} + B$, assume there $\exists f \neq const$ and meromorphic, such that $\tilde{P}_1 + ... + \tilde{P}_g + (f) = P_1 + ... + P_{g+n-N} + B$. Then it follows that $\dim H^0(\sum \tilde{P}) = g+1-g+i(\sum \tilde{P}) = 1$, hence f is constant. Hence

$$\tilde{P}_1 + ... + \tilde{P}_g = P_1 + ... + P_{g+n-N} + B$$

and the divisors coincide.

□

14.4. **Bosonisation formula on higher genus.** Here we discuss some formulas, following from the Fay's master identity.

Theta function identity (Fay, Cor. 2.19)

PROPOSITION **14.3.** *Let $e \in \mathbb{C}^g$ is such that $\vartheta(e) \neq 0$.*

$$\vartheta\Big(\sum_1^n x_i - \sum_1^n y_i - e \Big) \vartheta(e)^{n-1} \prod_{i<j} E(x_i, x_j) E(y_i, y_j) = \det \left(\frac{\vartheta(x_i - y_j - e)}{E(x_i, y_j)} \right) \prod_{i,j} E(x_i, y_j),$$

for all $x_1, ..., x_n, y_1, ..., y_n \in X$.

One has also to require that $\vartheta(e - (x-y)) \not\equiv 0$ for all x, y.

Proof. Take $D = -\Delta + e + \sum_1^n y_i$, $N = n$ and sections

$$\psi_k(x) = \frac{\vartheta(x - y_k - e)}{E(x, y_k)} \prod_1^n E(x, y_i)$$

in Eq. (3). By Riemann vanishin $e - \Delta = P_1 + ... + P_g$ is a

Normaization factor: consider degeneration $x_i \to y_i$ while $y_i \neq y_j$. $\square$