LECTURE 5: ABELIAN DIFFERENTIALS

5.1. **Differential forms on a Riemann surface.** We will consider a compact Riemann surface X as a complex manifold of dimension one. In particular, it is an orientable manifold, i.e. given a collection of charts with local complex coordinates (U_α, z_α), with holomorphic transition functions at each intersection $U_\alpha \cap U_\beta$, the orientation is preserved $\frac{i}{2} dz_\alpha \wedge dz_\alpha = \left|\frac{dz_\alpha}{dz_\beta}\right|^2 \frac{i}{2} dz_\beta \wedge dz_\beta$.

Since complex dimension is one, we have the following differential forms on X: scalar functions $f(z, \bar{z})$, 1-forms (*differentials*), which locally look like

$$\omega = \omega_z(z, \bar{z})dz + \omega_{\bar{z}}(z, \bar{z})d\bar{z},$$

and 2-forms $v(z, \bar{z})dz \wedge d\bar{z}$. These objects are invariant under changes of coordinates from local chart to chart, so they are defined on all X. We can also decompose ω into $(1, 0)$ and $(0, 1)$ parts as $\omega_z(z, \bar{z})dz$ (and $\omega_{\bar{z}}(z, \bar{z})d\bar{z}$) and this decomposition is invariant on X since the transition functions are holomorphic, i.e., $\omega_z(z_\alpha, \bar{z}_\alpha) = \omega_z(z_\beta, \bar{z}_\beta)\frac{dz_\alpha}{dz_\beta}$.

5.2. **Homology group and closed differentials.** As a topological space the Riemann surfaces are classified by the genus g, i.e. a number of "handles".

Claim: The homology group $H_1(X, \mathbb{Z})$ of the Riemann surface is generated by $2g$ cycles $a_1, ..., a_g, b_1, ..., b_g$.

Definition **5.1.** *This basis is called canonical if the intersection numbers are*

$$a_j \circ b_l = \delta_{jl}, \quad a_j \circ a_l = b_j \circ b_l = 0. \tag{1}$$

The intersection number of two 1-cycles is ± 1, depending on the orientation of the intersection.

Remark: Canonical bases are not unique. Indeed, let us represent the basis by the $2g$-dimensional vector as follows

$$\begin{pmatrix} a \\ b \end{pmatrix}.$$

Then any other canonical basis is related by the integer matrix $A \in GL(2g, \mathbb{Z})$ transformation

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = A \begin{pmatrix} a \\ b \end{pmatrix},$$

with the condition of preserving the intersection numbers Eq. , i.e.

$$J = AJA^T, \quad J = \begin{pmatrix} a \\ b \end{pmatrix} \circ \begin{pmatrix} a & b \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{2}$$

Hence the new basis is canonical if and only if $A \in Sp(g, \mathbb{Z})$ is a symplectic matrix.

Consider now closed differentials ω, $d\omega = 0$. Given a (canonical) basis of 1-cycles, periods of ω are well-defined

$$\int_{a_j} \omega, \quad \int_{b_j} \omega,$$

i.e. independent of the choice of paths representing the cycles, for the homological chocies of paths. This is because for any two homological closed paths a and a', we have $\int_a \omega = \int_{a'} \omega$ for any closed differential ω.

5.3. **Canonical dissection.** We will often work with the *canonical dissection* of the Riemann surface. The idea is to fix a base point P_0 and then contract the canonical basis a, b so that the cycles start and end at P_0, as illustrated on the picture below.

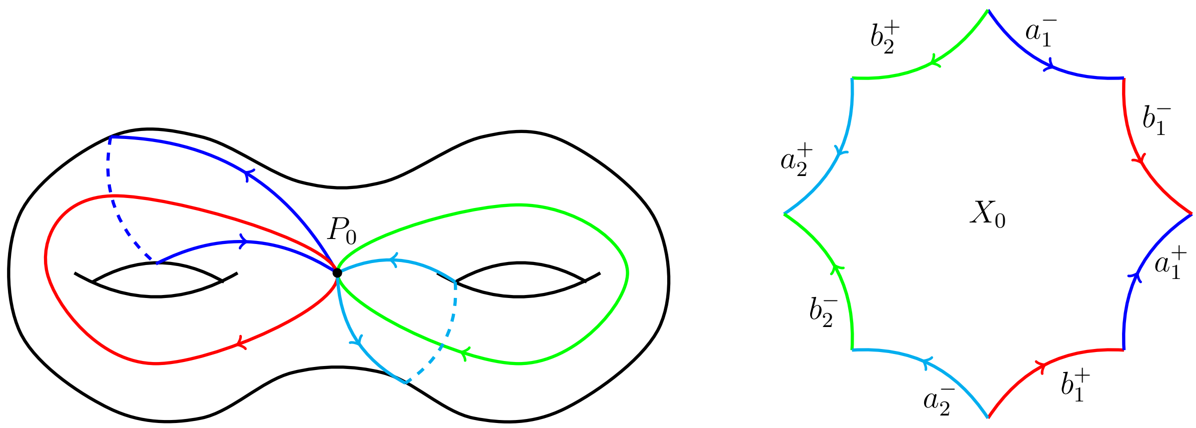

Figure 1. Riemann surface of genus 2 and its canonical dissection.

As a result we end up with the simply-connected 2-cell X_0 with the boundary

$$\partial X_0 = \sum_{j=1}^{g} \left(-a_j^+ - b_j^+ + a_j^- + b_j^- \right),$$

where a_j^+, b_j^+ (resp. a_j^-, b_j^-) are left (resp. right) sides of cuts along cycles a_j, b_j.

5.4. **Riemann's bilinear identity.**

THEOREM **5.1.** *Let X be a genus-g compact Riemann surface, with the canonical basis and corresponding canonical dissection.*

(a) For any two closed differentials ω_1, ω_2 we have

$$\int_X \omega_1 \wedge \omega_2 = \sum_{j=1}^{g} \int_{a_j} \omega_1 \cdot \int_{b_j} \omega_2 - \int_{b_j} \omega_1 \cdot \int_{a_j} \omega_2 \tag{3}$$

(b) For all holomorphic differentials ω, η

$$\sum_{j=1}^{g} \int_{a_j} \omega \cdot \int_{b_j} \eta - \int_{b_j} \omega \cdot \int_{a_j} \eta = 0 \tag{4}$$

(c) and for holomorphic differential $\omega \not\equiv 0$ we have

$$\mathrm{Im} \sum_{j=1}^{g} \int_{a_j} \bar{\omega} \int_{b_j} \omega > 0. \tag{5}$$

Proof. To proof (a), we note that since X_0 is simply-connected, there exists a function f on X_0, s.t. $\omega_1 = df$. Then by Stokes theorem,

$$\begin{aligned}
\int_X \omega_1 \wedge \omega_2 = \int_{X_0} \omega_1 \wedge \omega_2 = \int_{X_0} df \wedge \omega_2 = \int_{X_0} d(f\omega_2) = \int_{\partial X_0} f\omega_2 \\
= \sum_{j=1}^{g} \left(- \int_{a_j^+} - \int_{b_j^+} + \int_{a_j^-} + \int_{b_j^-} \right) f\omega_2 \\
= \sum_{j=1}^{g} \int_{a_j} \big(f \text{ on } a_j^- - f \text{ on } a_j^+\big)\omega_2 + \int_{b_j} \big(f \text{ on } b_j^- - f \text{ on } b_j^+\big)\omega_2.
\end{aligned}$$

Next, we note that df has no discontinuity on a_j or b_j, so f on a_j^+ and a_j^- must differ by a constant, and same for b_j^+, b_j^-. Since the path b_j connects a_j^+ and a_j^- (as can be seen from the Fig. 1), we can write the last expression as

$$\sum_{j=1}^{g} \int_{a_j} \omega_1 \cdot \int_{b_j} \omega_2 - \int_{b_j} \omega_1 \cdot \int_{a_j} \omega_2,$$

establihing (a).

If ω_1, ω_2 are holomorphic, then $\omega_1 \wedge \omega_2 = 0$ and (b) follows.

Next, for holomorphic ω, there exists a holomorphic function f on X_0, s.t. $\omega = df$. We apply (3) for $\omega_1 = \omega$ and $\omega_2 = \bar{\omega}$,

$$\operatorname{Im} \sum_{j=1}^{g} \int_{a_j} \bar{\omega} \int_{b_j} \omega = -\frac{1}{2i} \int_X \bar{\omega} \wedge \omega = -\frac{1}{2i} \int_{X_0} d\bar{f} \wedge df$$
$$= \frac{1}{2i} \int_{X_0} |\partial f|^2 dz \wedge d\bar{z} = \int_{X_0} |\partial f|^2 dx \wedge dy > 0$$

where we used some local complex coordinates $z = x + iy$ and the fact that $dx \wedge dy$ is a everywhere positive 2-form on X_0. □

COROLLARY **5.1.** *From Eq.* (5) *it follows that if all a -periods of a holomorphic differential* ω *vanish, then* $\omega \equiv 0$.

5.5. **Holomorphic differentials and period matrix.** We know from Riemann-Roch theorem (Lecture 4) that the dimension of the vector space $H^0(X, \Omega)$ of holomorphic differentials on X is equal to genus $g = \dim H^0(X, \Omega)$.

From Cor. 5.1 it follows that for any basis ω_j of $H^1(X)$ the matrix of a-periods

$$A_{jl} = \int_{a_j} \omega_l$$

is non-degenerate and invertible. Thus we can normalize the basis of holomorphic differentials as follows

DEFINITION **5.2.** *Given the canonical basis* a_j, b_j *of 1-cycles, the basis of holomorphic differentials normalized as*

$$\int_{a_j} \omega_l = \delta_{jl} \tag{6}$$

is called canonical. The matrix of b-periods of the canonical basis

$$\tau_{jl} = \int_{b_j} \omega_l$$

is called the period matrix of X.

COROLLARY **5.2.** *The period matrix is symmetric* $\tau_{jl} = \tau_{lj}$ *and* $\operatorname{Im} \tau > 0$ *is positive-definite.*

Proof. Symmetry immediately follows by applying (4) to $\omega = \omega_j$ and $\eta = \omega_l$.

Let α_j be a real vector and apply (5) to $\omega = \sum_j \alpha_j \omega_j$. It follows that $\operatorname{Im} \sum_{j,l} \alpha_j \tau_{jl} \alpha_l = \sum_{j,l} \alpha_j (\operatorname{Im} \tau_{jl}) \alpha_l > 0$. □

5.6. **Abel map.** This is the key application of holomorphic differentials on compact Riemann surfaces.

The period matrix of X generates a lattice Λ in $\mathbb{C}^g$

$$\Lambda = \{n_j + i\tau_{jl}m_l,\ n, m \in \mathbb{Z}^g\},$$

generated by the a and b - periods of holomorphic differentials.

Definition **5.3** *The Jacobean variety (equiv., Jacobian) of X is the complex torus*

$$Jac(X) = \mathbb{C}^g/\Lambda.$$

If P_0 is a base point then using holomorphic differentials we obtain the holomorphic map $X \to Jac(X)$ as follows

Definition **5.4** *The Abel map is defined as*

$$\begin{aligned} I:\quad & X \to Jac(X), \\ & P \to \left(\int_{P_0}^{P} \omega_1, ..., \int_{P_0}^{P} \omega_g\right). \end{aligned}$$

This is well-defined because the right hand side is defined modulo period integrals, i.e., modulo Λ, so its a point in $Jac(X)$. The Abel map can be naturally extended to divisors $D = \sum_{P\in X} n_P P,\ n_P \in \mathbb{Z}$, as

$$I(D) = \sum_{P\in X} n_P \int_{P_0}^{P} \omega_j.$$

Note that if $\deg D = 0$, i.e. $D = P_1 + ... + P_N - Q_1 - ... - Q_N$, then the Abel map

$$I(D) = \sum_{m=1}^{N} \int_{Q_m}^{P_m} \omega_j$$

is independent of the base point P_0.

5.7. **Abelian differentials and their properties.**

Definition **5.3.** *A differential η is called meromorphic, or equivalently, Abelian differential, if in a local coordinate z it has the form $h(z)dz$ where $h(z)$ is meromorphic function.*

Zeros and poles of local function $h(z)$ define zeroes and poles of the meromorphic differentials and the notion of order of zero or pole is well-defined, i.e. independent of the choice of local coordinates.

The residue $\mathrm{Res}_{z_0}\eta$ of the Abelian differential at a singular point z_0 is defined as the h_{-1} coefficient in the Laurent expansion around z_0,

$$h(z) = \sum_{n=n_0}^{\infty} h_n(z-z_0)^n.$$

The residue is independent of the choice of local coordinates, since it can be written in the manifestly invariant form

$$\mathrm{Res}_{z_0}\eta = \frac{1}{2\pi i}\int_{\partial B_{z_0}} \eta(z),$$

where B_{z_0} is a disk containing z_0 in the interior, s.t. η is holomorphic on $\bar{B}/\{z_0\}$, e.g. no other singular points in the closure. The following property holds.

LEMMA **5.5.** *Let $z_1, ..., z_m$ be the singular points of the Abelian differential η, then*

$$\sum_{j=1}^{m} \mathrm{Res}_{z_j}\eta = 0.$$

Proof. Let B_j be small disk around z_j containing no other singularities in its closure. Then,

$$\sum_{j=1}^{m} \mathrm{Res}_{z_j}\eta = \frac{1}{2\pi i}\sum_j \int_{\partial B_j} \eta = -\frac{1}{2\pi i}\int_{X-\cup B_j} d\eta = 0, \tag{7}$$

since η is holomorphic on $X - \cup B_j$ and thus closed there. □

5.8. **Differentials of 2nd and 3rd kind.** The following terminology is commonly used:

(a) Holomorphic differentials are called Abelian differentials of the *first kind*,

(b) Meromorphic differentials with poles with vanishing residues are called Abelian differentials of the *second kind*,

(c) Meromorphic differentials with non-zero residues are called Abelian differentials of the *third kind*.

Any meromorphic differential is a combination of differentials of three types.

We have already constructed the canonical basis of differentials of the first kind. Normalized Abelian differentials of the second kind are constructed as follows. The differential of 2nd kind $\eta_P^{(N)}, N \in \mathbb{N}$ has only one singularity of order $N+1$ at $P \in X$, i.e. for a local coordinate $z, z(P) = 0$,

$$\eta_P^{(N)} = \left(\frac{1}{z^{N+1}} + O(1)\right) dz.$$

Remark: This construction depends on the choice of the local coordinate, but the order of the pole is independent of the choice of local coordinate.

Example: Consider $\eta_0^{(N)} = \frac{dz}{z^{N+1}}$ on the sphere.

The basic differential of 3rd kind η_{PQ} has only two singularities at P and Q with opposite residues

$$\mathrm{Res}_P\, \eta_{PQ} = -\mathrm{Res}_Q\, \eta_{PQ} = 1.$$

Example: Consider $\eta_{z_0 z_1} = d\log\frac{z-z_0}{z-z_1}$ on the sphere.

Note that adding holomorphic differentials to $\eta_P^{(N)}$ and η_{PQ} preserves the form of singularities. Taking into account (6), this ambiguity can be used in a straightforward way to normalize the differentials above as follows

$$\int_{a_j} \eta_P^{(N)} = 0, \quad \int_{a_j} \eta_{PQ} = 0,$$

for the a-cycles. Such differentials are called *normalized* Abelian differentials of, resp., 2nd and 3rd kind. We now have to demonstrate their existence and uniqueness.