LECTURE 6: ABEL THEOREM

6.1. **Harmonic differentials.** We would like to prove the existence of 2nd and 3rd kind differentials. Here we sketch the argument, the full construction is based on the decomposition theorem of differentials on Riemann surface, is covered e.g. in [Jost,FarkasKra].

The Hodge $*$-operator is defined on the differential forms as follows

$$\omega = \omega_z(z,\bar z)dz + \omega_{\bar z}(z,\bar z)d\bar z \quad \to \quad *\omega = -i\omega_z(z,\bar z)dz + i\omega_{\bar z}(z,\bar z)d\bar z.$$

Clearly $*^2 = 1$ and type $(1,0)$ (resp. $(0,1)$) forms form eigenspaces of $*$ with eigenvalues $-i$ (resp. i).

Let X be a Riemann surface and consider the Hilbert space $L_2(X)$ of square-integrable differentials with the scalar product

$$(\omega_1, \omega_2) = \int_X \omega_1 \wedge *\bar\omega_2. \tag{1}$$

We have: $(\omega_1,\omega_2) = \overline{(\omega_2,\omega_1)}$ and $(*\omega_1, *\omega_2) = (\omega_1,\omega_2)$

The C^1 (continuous first derivatives, i.e., smooth) 1-form ω is closed (resp., co-closed) if $d\omega = 0$ (resp., $d*\omega = 0$). The C^1 (smooth) 1-form ω is exact (resp., co-exact) if $\omega = df$ (resp., $\omega = *df$).

Next we introduce subspaces B and B^* of exact and co-exact differentials

$$B = \overline{\{df | f \in C^\infty(X)\}},$$
$$B^* = \overline{\{*df | f \in C^\infty(X)\}},$$

where $C^\infty(X)$ are smooth functions on X and bar denotes closure in $L_2(X)$. If X is non-compact then we shall take functions with compact support.

B and B^* are orthogonal, because $(df, *dg) = \int_X df \wedge dg = 0$. Consider orthogonal complements $B^\perp$, $B^{*\perp}$ under the norm (1). We have

Lemma **6.1.** *Let $\alpha \in L^2(X)$ be of class C^1. Then $\alpha \in B^{*\perp} \Leftrightarrow d\alpha = 0$ and $\alpha \in B^\perp \Leftrightarrow d*\alpha = 0$*

Proof. Lets do it for $\alpha \in B^\perp$,

$$(\alpha, df) = \overline{(df,\alpha)} = \overline{\int_X d\bar f \wedge *\alpha} = -\overline{\int_X \bar f \wedge d*\alpha} = 0,$$

for any $f \in C^\infty(X)$. Hence $d*\alpha = 0$. □

Definition **6.2** *A differential α is harmonic if it is smooth and both closed and co-closed.*

It follows immediately from definition that locally, harmonic differentials have the form

$$\alpha = fdz + \bar{g}d\bar{z},$$

where f, g are holomorphic, and also $\alpha = dh$, where where h is harmonic function ($\partial\bar{\partial}h = 0$). The proof is straightforward and is left as an exercise. Then it follows that

$$\alpha + i * \alpha \tag{2}$$

is a holomorphic differential.

Consider now the space $H = B^{\perp} \cap B^{*\perp}$, which is intersection of orthogonal complements. All harmonic differentials by definition are in H. The stronger statement is that H consists only of harmonic differentials (for technical proof based on Weyl's lemma we refer to Jost [Theorem 5.2.1]).

Hence the statement is

Corollary **6.3.** *Every square-integrable differential ω on X is represented by an orthogonal sum*

$$\omega = df + *dg + \alpha$$

of exact, co-exact and harmonic forms.

This discussion can be continued further to prove that $\dim H = 2g$, $\dim H^1(X, \mathbb{C}) = 2g$ and the dimension of space of holomorphic differentials is g, see [FarkasKra].

6.2. **Existence and uniqueness of differentials of 2nd and 3rd kind.**

Theorem **6.4.** *Given points P, Q on a compact Riemann surface X and a canonical basis of cycles there exists unique normalized Abelian differentials $\eta_P^{(N)}, N \in \mathbb{N}$ of 2nd kind and η_{PQ} of 3rd kind.*

Proof. Uniqueness of 2nd and 3rd kind differentials follows from simple considerations: the difference of two normalized differentials is a holomorphic differential with vanishing a-cycles. Hence it vanishes identically due to Cor. 5.1 **??**.

The existence can be verified by the following explicit construction. Consider nested neighbourhoods $P \in U_0 \subset U_1 \subset X$ and a $C^\infty(X)$ interpolating function $\rho : \rho = 1$ on U_0 and $\rho = 0$ on $X \backslash U_1$. Let z be a local coordinate on U_1 centered at P and consider the differential on $X \backslash \{P\}$

$$\psi = d\left(-\frac{\rho}{Nz^N}\right) = \left(-\frac{\partial\rho}{Nz^N} + \frac{\rho}{z^{N+1}}\right)dz - \left(\frac{\bar{\partial}\rho}{Nz^N}\right)d\bar{z}.$$

The $(0,1)$ part of ψ is smooth on X and following Cor. 6.3 it can be decomposed as

$$\psi - i * \psi = df + *dg + \alpha \tag{3}$$

into exact, co-exact and harmonic parts. Consider now differential $\gamma = \psi - df$.

The key claim is that γ is harmonic on $X\backslash P$ and $\gamma - \frac{dz}{z^{N+1}}$ is harmonic on U_0. Indeed,

$$\gamma = d\left(-\frac{\rho}{Nz^N} - f\right),$$

so it is closed on $X\backslash P$, and from Eq. (3) it follows that $\gamma = i * \psi + *dg + \alpha$, so it is co-closed. Hence $\gamma \in H(X\backslash P)$.

Next, observe that $\psi - \frac{dz}{z^{N+1}} \equiv 0$ on U_0 by construction. Hence,

$$\gamma - \frac{dz}{z^{N+1}} = -df = *dg + \alpha \quad \text{on } U_0,$$

so $\gamma \in H(U_0)$.

Then the direct corollary of (2) is that the differential $\eta = \frac{1}{2}(\gamma + i * \gamma)$ is holomorphic on $X\backslash P$ and $\eta - \frac{dz}{z^{N+1}}$ is holomorphic on U_0. Hence η has exactly the pole of the order $N+1$ at P and holomorphic otherwise.

In order to prove the existence of the 3rd kind differential the above construction shall be applied to

$$\psi_{z_1 z_2} = d\left(\rho \log \frac{z - z_1}{z - z_2}\right),$$

for $z_1, z_2 \in U_0$. For arbitrary two points P, Q we can do a telescopic sum of $\psi_{z_1 z_2}$. □

6.3. **Divisors of meromorphic functions and Abelian differentials.**

Definition **6.5** *Divisors on a Riemann surfaces are given by formal finite sums of points*

$$D = \sum_{k=1}^{N} n_k P_k, \quad n_k \in \mathbb{Z},\ P_k \in X,$$

and the sum

$$\deg D = \sum_{k=1}^{N} = n_k$$

is called the degree of D.

Set of divisors $Div(X)$ form an Abelian group with well-defined group operation (summation) and inverse element ($D \to -D$).

Definition **6.6** *Divisor is called positive, if all $n_k \geqslant 0$.*

Divisor (f) of meromorphic function f is a sum of its zeroes $P_1, .., P_N$ and poles $Q_1, ..., Q_M$ with multiplicities

$$(f) = n_1 P_1 + ... + n_N P_N - n'_1 Q_1 - ... - n'_M Q_M.$$

Recall that zeroes and poles of an Abelian is well defined

Definition **6.7** *The divisor of Abelian differential ω is the sum of its zeroes and poles*

$$(\omega) = \sum_{P \in X} \mathrm{ord}(P) P.$$

Here $\mathrm{ord}(P) = n_0$ is the order of zero or singularity in local Laurent expansion $h(z) = \sum_{n=n_0}^{\infty} h_n (z - z_P)^n$ in local coordinate at the point P and locally $\omega = h(z)dz$.

Definition **6.8** *A divisor is called principal if it is a divisor of a meromorphic function.*

Lemma **6.9.** *Divisor of a meromorphic function has degree zero* $\deg(f) = 0$.

Proof. Given a meromorphic function f, consider the Abelian differential df/f. Its residues are equal to the multiplicities of the zeroes and poles of f. By Lemma 5.5. **??**, the sum of residues is zero. □

6.4. **Abel theorem.** The Abel theorem describes what happens to the principal divisors under the Abel map, defined in the previous lecture.

Theorem **6.1.** *The divisor is principal if and only if* $I(D) \equiv 0$.

Proof. Let f be a meromorphic function. As we have already shown in Lemma 6.9, $\deg(f) = 0$. Hence we can write for its divisor

$$(f) = P_1 + \ldots + P_N - Q_1 - \ldots - Q_N,$$

where some of the points could coincide. Consider the meromorphic differential

$$\eta = d \log f$$

Since f is s scalar function, the periods of η can only be integer multiples of $2\pi i$,

$$\int_{a_j} \eta = 2\pi i n_j, \quad \int_{b_j} \eta = 2\pi i m_j, \quad n_j, m_j \in \mathbb{Z}.$$

We need to compute $I(D)$. Following the definition $I(D)$, in sec. 5.6**??**, we have

$$I((f)) = \sum_{k=1}^{N} \int_{Q_k}^{P_k} \omega_j.$$

In order to compute this we need a slightly different version of the Riemann bilinear identity in a more general form. Consider again the setup of Thm. **??**5.1.

Lemma **6.10.** *Let ω_1, ω_2 be two closed differentials, a, b is the canonical basis of 1-cycles and X_0 corresponding canonical dissection. Then*

$$\int_{\partial X_0} \left(\omega_2(P) \int_{P_0}^{P} \omega_1 \right) = \sum_{j=1}^{g} \int_{a_j} \omega_1 \cdot \int_{b_j} \omega_2 - \int_{b_j} \omega_1 \cdot \int_{a_j} \omega_2$$

Proof. By definition, the boundary of X_0 is $\partial X_0 = \sum_{j=1}^{g}\left(-a_j^+ - b_j^+ + a_j^- + b_j^-\right)$, hence

$$\begin{aligned}\int_{\partial X_0}\left(\omega_2(P)\int_{P_0}^{P}\omega_1\right) &= \sum_{j=1}^{g}\left(-\int_{a_j^+} - \int_{b_j^+} + \int_{a_j^-} + \int_{b_j^-}\right)\omega_2(P)\int_{P_0}^{P}\omega_1 \\ &= \sum_{j=1}^{g}\int_{a_j}\left(\omega_2(P_{a_j^-})\int_{P_0}^{P_{a_j^-}}\omega_1 - \omega_2(P_{a_j^+})\int_{P_0}^{P_{a_j^+}}\omega_1\right) \\ &\quad + \sum_{j=1}^{g}\int_{b_j}\left(\omega_2(P_{b_j^-})\int_{P_0}^{P_{b_j^-}}\omega_1 - \omega_2(P_{b_j^+})\int_{P_0}^{P_{b_j^+}}\omega_1\right) \quad (4)\end{aligned}$$

Let $P_{a_j^-}$ and $P_{a_j^+}$ be the points, resp., on a_j^- and a_j^+ which coincide on X, and $P_{b_j^-}$ and $P_{b_j^+}$ are the points, resp., on b_j^- and b_j^+ which coincide on X.

First we note that

$$\omega_2(P_{a_j^-}) = \omega_2(P_{a_j^+}), \quad \omega_2(P_{b_j^-}) = \omega_2(P_{b_j^+}).$$

Next, consulting Fig. 1. one can see that

$$\int_{P_0}^{P_{a_j^-}}\omega_1 - \int_{P_0}^{P_{a_j^+}}\omega_1 = \int_{P_{a_j^+}}^{P_{a_j^-}}\omega_1 = -\int_{b_j}\omega_1$$

and

$$\int_{P_0}^{P_{b_j^-}}\omega_1 - \int_{P_0}^{P_{b_j^+}}\omega_1 = \int_{P_{b_j^+}}^{P_{b_j^-}}\omega_1 = \int_{a_j}\omega_1$$

Plugging this back to (4) we obtain the result

$$\int_{\partial X_0}\left(\omega_2(P)\int_{P_0}^{P}\omega_1\right) = \sum_{j=1}^{g}\int_{a_j}\omega_1\cdot\int_{b_j}\omega_2 - \int_{b_j}\omega_1\cdot\int_{a_j}\omega_2.$$

□

Now we return to the proof of Thm. 6.1. We need to apply Lemma 6.10 to $\omega_1 = \omega_l$, where ω_l is the canonical basis of holomorphic differentials, and to $\omega_2 = \eta$. Note that $\eta = d\log f$ is closed. On the on hand we have

$$\begin{aligned}\frac{1}{2\pi i}\int_{\partial X_0}\left(\eta(P)\int_{P_0}^{P}\omega_l\right) &= \sum_{P\in X}\operatorname{Res}\,\eta(P)\int_{P_0}^{P}\omega_l = \sum_{k=1}^{N}\int_{Q_k}^{P_k}\omega_l \\ &= \frac{1}{2\pi i}\sum_{j=1}^{g}(2\pi i m_j\delta_{jl} - 2\pi i\tau_{jl}n_j) = m_l - \tau_{lj}n_j \in \Lambda,\end{aligned}$$

hence

$$I((f)) \equiv 0.$$

In the opposite direction, consider a divisor $D = P_1 + ... + P_N - Q_1 - ... - Q_N$ of degree zero, such that

$$I(D) \equiv 0. \tag{5}$$

The main idea is to construct the meromorphic function f with the divisor $(f) = D$.

Let us consider the normalized abelian differentials of the third kind $\eta_{P_k Q_k}$ and let

$$\tilde{\eta} = \sum_{k=1}^{N} \eta_{P_k Q_k}. \tag{6}$$

Then a-periods of η vanish by definition. Consider b-periods of $\eta_{P_k Q_k}$ and apply Lemma 6.10 for $\omega_1 = \omega_l$ (canonical basis of holomorphic differentials) and $\omega_2 = \eta_{P_k Q_k}$. Then

$$\int_{b_l} \eta_{P_k Q_k} = \int_{\partial X_0} \left(\eta_{P_k Q_k}(P) \int_{P_0}^{P} \omega_l \right) = 2\pi i \left(\int_{P_0}^{P_k} \omega_l - \int_{P_0}^{Q_k} \omega_l \right) = 2\pi i \int_{Q_k}^{P_k} \omega_l.$$

Hence,

$$\int_{b_l} \tilde{\eta} = \sum_{k=1}^{N} \int_{b_l} \eta_{P_k Q_k} = 2\pi i \sum_{k=1}^{N} \int_{Q_k}^{P_k} \omega_l = 2\pi I(D) = 2\pi i (n_l + \tau_{lj} m_j) \in \Lambda,$$

for some $n_l, m_l \in \mathbb{Z}$, according to the assumption (5). Then the function

$$f(P) = \exp\left(\int_{P_0}^{P} \tilde{\eta} - 2\pi i \sum_{j=1}^{g} m_j \int_{P_0}^{P} \omega_j \right),$$

where η is given by Eq. (6) is single-valued on X and is a meromorphic function with the divisor D.

□