LECTURE 9: THETA FUNCTIONS

9.1. **Theta functions in one variable.** Theta function is analytic function of $z \in \mathbb{C}$ is defined as

$$\vartheta(z,\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau + 2\pi i n z} \tag{1}$$

and parameter $\tau \in \mathbb{H}$ takes values in the upper-half plane, i.e., $\operatorname{Im}\tau > 0$. The series converges absolutely and uniformly on compact sets. Indeed for $\operatorname{Im}\tau < c$ and $\operatorname{Im}\tau > \epsilon$ we have

$$|e^{\pi i n^2\tau + 2\pi i n z}| < e^{-\pi\epsilon n^2 + 2\pi c n} < e^{-\pi\epsilon n(n - 2c/\epsilon)}$$

hence starting from $n_0 > 2c/\epsilon$ the series begin to rapidly converge.

Theta function is almost periodic with respect to the lattice $\Lambda = m' + m\tau,\ m', m \in \mathbb{Z}$. Indeed,

$$\begin{aligned}
\vartheta(z+1,\tau) &= \vartheta(z,\tau),\\
\vartheta(z+\tau,\tau) &= \sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau + 2\pi i n(z+\tau)}\\
&= \sum_{n\in\mathbb{Z}} e^{\pi i (n+1)^2\tau - \pi i\tau + 2\pi i n z} = e^{-\pi i\tau - 2\pi i z}\vartheta(z,\tau)
\end{aligned}$$

and in general

$$\vartheta(z+m',\tau) = \vartheta(z,\tau), \quad \vartheta(z+m\tau,\tau) = e^{-\pi i m^2\tau - 2\pi i m z}\vartheta(z,\tau), \quad m', m \in \mathbb{Z}.$$

Theta functions with characteristics are defined as follows

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z,\tau) = \sum_{n\in\mathbb{Z}} \exp\big(\pi i (n+a)^2\tau + 2\pi i (n+a)(z+b)\big) = \\ e^{\pi i a^2\tau + 2\pi i a(z+b)}\vartheta(z + a\tau + b,\tau), \tag{2}$$

Especially important are theta-functions with half-integer characteristics (Jacobi theta functions)

$$\begin{aligned}
\theta_1(z,\tau) &= -\vartheta\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(z,\tau),\\
\theta_2(z,\tau) &= \vartheta\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}(z,\tau),\\
\theta_3(z,\tau) &= \vartheta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(z,\tau) = \vartheta(z,\tau),\\
\theta_4(z,\tau) &= \vartheta\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}(z,\tau).
\end{aligned}$$

Note that $\theta_1(-z) = -\theta_1(z)$ is odd and $\theta_{2,3,4}(-z) = \theta_{2,3,4}(z)$ are even.

These functions satisfy quadratic relations (see Mumford, vol. I).

9.2. **Zeroes.** Theta functions are multivalued on the torus $T_\tau = C/\Lambda$, but its zeroes are well-defined on the torus, as follows from formulas above. We can immediately show that theta function Eq.(1) has one zero in the torus. The number of zeroes is given by the integral

$$\#\text{ zeroes of }\vartheta = \frac{1}{2\pi i}\int_{4\text{ sides}} \frac{d}{dz}(\log f)dz = 1 \tag{3}$$

See Fig. 1. From definition (15) it immediately follows that theta function with characteristics also has one zero. Its location can be determined as follows. It is not hard to show that

$$\vartheta\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right](z,\tau) = -\vartheta\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right](-z,\tau), \tag{4}$$

hence it vanishes at $z=0$. Therefore from (15) one first infers that $\vartheta(z,\tau)$ vanishes at $\frac{1}{2}\tau + \frac{1}{2}$, and next, $\vartheta\left[\begin{smallmatrix}a\\ b\end{smallmatrix}\right](z,\tau)$ has zeros at

$$\Big(a+\frac{1}{2}\Big)\tau + \Big(b+\frac{1}{2}\Big) \qquad \operatorname{mod} \Lambda.$$

We can define the vector space R_k^τ of analytic functions $f(z)$ of $\mathbb{C}$, quasi-periodic with respect to Λ with weight k as follows

$$f(z+1,\tau) = f(z,\tau), \tag{5}$$

$$f(z+\tau,\tau) = e^{-\pi i k\tau - 2\pi i k z} f(z,\tau). \tag{6}$$

Then one can show that

PROPOSITION **9.1.** *(1) For any $k \geqslant 1$, $\dim R_k^\tau = k$*

(2) R_k^τ admits the bases

$$\vartheta\left[\begin{smallmatrix}j/k\\ 0\end{smallmatrix}\right](kz,k\tau), \quad \text{and} \quad \vartheta\left[\begin{smallmatrix}0\\ j/l\end{smallmatrix}\right](z,\tau/k), \quad j=0,...,k-1 \tag{7}$$

(3) if $k=l^2$, then the 3rd basis is given by $\vartheta\left[\begin{smallmatrix}j/l\\ j'/l\end{smallmatrix}\right](kz,k\tau)$, $j,j'=0,...,l-1$

(4) For $k \geqslant 3$ the basis elements $f_j(z)$ of R_k^τ have no common zero in $\mathbb{C}$ and defined an embedding

$$T_\tau = \mathbb{C}/\Lambda \ni z \to (f_1(z),...,f_k(z)) \in \mathbb{CP}^{k-1}$$

Comments on the proof. (1) and (2) will be proven later for any g.

9.3. **Meromorphic functions on the torus.** On $\mathbb{P}^1$ we can construct meromorphic functions as ratios

$$\prod_j \frac{z-a_j}{z-b_j} \tag{8}$$

On the torus theta functions give us several ways to construct meromorphic functions

- The ratios of the basis functions in Eq. (16)

$$\frac{\vartheta\left[\begin{smallmatrix} j/k \\ 0 \end{smallmatrix}\right](kz,k\tau)}{\vartheta\left[\begin{smallmatrix} j'/k \\ 0 \end{smallmatrix}\right](kz,k\tau)}$$

  This can be used to illustrate the Riemann-Roch theorem. Using the same argument as in the proof of Eq. (3), one can show that a basis function f_j of R_k^τ has exactly k zeroes. Let us pick one basis function, e.g. $f_0(z)=\vartheta\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right](kz,k\tau)$ and consider the divisor of its zeroes D_k. We have $\deg D_k = k$. The only holomorphic differential on the torus is dz and its nowhere vanishing, hence $\dim H^0(T,\Omega(-D_k)) = 0$. Hence $\dim H^0(T,\mathcal{O}(D_k)) = \deg[D_k] = k$ and the basis is given by

$$\frac{\vartheta\left[\begin{smallmatrix} j/k \\ 0 \end{smallmatrix}\right](kz,k\tau)}{\vartheta\left[\begin{smallmatrix} 0/k \\ 0 \end{smallmatrix}\right](kz,k\tau)}, \quad j=0,...,k-1.$$

  (for $j=0$ this is constant function).
- We can take quotients of θ itself

$$\prod_{j=1}^{N} \frac{\vartheta(z-a_j)}{\vartheta(z-b_j)} \tag{9}$$

  This is periodic provided $\sum a_j = \sum b_j$. Hence, for $N=1$ we only get constant function as no meromorphic functions with only one simple pole exist on torus.
- One can take second logarithmic derivative

$$\frac{d^2}{dz^2}\log\theta_1(z,\tau) = -\wp(z) + \mathrm{const}$$

  where $\wp(z)$ is Weierstrass $\wp$-function

$$\wp(z,\tau) = \frac{1}{z^2} + \sum_{n^2+m^2\neq 0}\left(\frac{1}{(z+m+n\tau)^2} - \frac{1}{(m+n\tau)^2}\right).$$

  The constant is chosen such that the Laurent expansion of $\wp$ at zero has no constant term. This function has a pole of order 2 at $z=0$. Taking the divisor $D_\wp = 2P_{z=0}$ of degree 2, we have $\dim H^0(T,\mathcal{O}(D_\wp)) = \deg D_\wp = 2$, hence $\wp(z)$ can be written as a sum of constant function and, e.g., meromorphic function of the type (9) with $\theta_1(z)^2$ in the denominator. Indeed,

$$\frac{d^2}{dz^2}\log\theta_1(z,\tau) = \frac{\theta_3''(0)}{\theta_3(0)} - \frac{\theta_1'(0)^2}{\theta_3(0)^2}\frac{\theta_3(z)^2}{\theta_1(z)^2},$$

see Mumford, p.26, for the proof using quadratic relations.

Next, we can recall that by the Abel theorem the divisor of meromorphic function is mapped to zero in the Jacobian torus. In our case the Jacobian torus is isomorphic to the original torus, so if z_1 and z_2 are zeroes of $\wp(z)$ Abel theorem simply means that

$$\int_0^{z_1} dz + \int_0^{z_2} dz = z_1 + z_2 = 0 \mod \Lambda.$$

The actual position of the zero is quite hard to find (see M. Eichler and D. Zagier, Math. Ann. 258, 399-407 (1982)).

- Finally one can also take the combinations of first logarithmic derivatives

$$\sum_{j=1}^N \lambda_j \frac{d}{dz}\log\vartheta(z-a_j,\tau),$$

with $\sum\lambda_j = 0$. For $N=2$ this is proportional to the third kind Abelian differential $\eta_{a_1a_2}$. The second kind differentials are constructed with the help of $\wp$: $\eta_0^{(2)} \sim \wp dz$.

9.4. **Riemann theta function.** Consider now coordinate vector $z_j \in \mathbb{C}^g$ and the lattice $\Lambda = m_j' + \tau_{jl}m_l,\ m, m' \in \mathbb{Z}^g$, where τ is symmetric complex matrix with positive definite imaginary part $\operatorname{Im}\tau > 0$.

Remark: We reduce to the tori of this form, called principally polarized Abelian tori, because its a classical theorem (Siegel) that non-constant meromorphic functions exist only on such Abelian tori. (Mumford, Griffits-Harris). Abelian tori in general correspond to the lattice $\Lambda = m_j'a_j + \tau_{jl}m_l,\ m, m' \in \mathbb{Z}^g, a_k \in \mathbb{N}, a_1 = 1, a_k|a_{k+1}$

Closely related fact is that for such tori there is projective embedding (Lefshetz embedding theorem).

This is parameterized by an open subset in $\mathbb{C}^{g(g+1)/2}$, called the Siegel upper-half plane. Then

$$\vartheta(z,\tau) = \sum_{n_j\in\mathbb{Z}^g} e^{\pi i n_j\tau_{jl}n_l + 2\pi i n_j z_j}, \tag{10}$$

where we drop the indices on z, τ etc., where notation is obvious.

Proposition **9.2.** *ϑ converges absolutely and uniformly in each set* $\max_j|\operatorname{Im} z_j| < c_1$ *and* $\operatorname{Im}\tau_{jl} \geqslant c_2\delta_{jl}$.

Proof.

$$|e^{\pi i n_j \tau_{jl} n_l + 2\pi i n_j z_j}| \leqslant e^{-\pi c_2 \sum n_j^2 + 2\pi c_1 \sum_j |n_j|}. \quad (11)$$

Hence the series are dominated by $\left(\sum_{n\geqslant 0} e^{-\pi c_2 \sum n^2 + 2\pi c_1 |n|}\right)^g$, which we already know converges. □

Theta function is quasi-periodic

$$\vartheta(z_j + m'_j + \tau_{jl} m_l, \tau) = e^{-\pi i m_j \tau_{jl} m_l - 2\pi i m_j z_j} \vartheta(z, \tau), \quad m, m' \in \mathbb{Z}^g. \quad (12)$$

The proof is identical to the one-dimensional case.

By analogy with (5), we can define the vector space R_k^τ of analytic functions $f(z)$ of $\mathbb{C}^g$, quasi-periodic with respect to Λ with weight k as

$$f(z_j + m'_j, \tau) = f(z, \tau), \quad (13)$$

$$f(z_j + m'_j + \tau_{jl} m_l, \tau) = e^{-\pi i k m_j \tau_{jl} m_l - 2\pi i k m_j z_j} f(z, \tau). \quad (14)$$

Theta functions with characteristics are defined as follows

$$\vartheta\begin{bmatrix} a_j \\ b_j \end{bmatrix}(z, \tau) = \sum_{n_j \in \mathbb{Z}^g} \exp\left(\pi i (n_j + a_j)\tau_{jl}(n_l + a_l) + 2\pi i (n_j + a_j)(z_j + b_j)\right) = $$

$$e^{\pi i a_j \tau_{jl} a_l + 2\pi i a_j (z_j + b_j)} \vartheta(z_j + \tau_{jl} a_l + b_j, \tau), \quad (15)$$

Then one can show that

PROPOSITION **9.3.** *(1) For any $k \geqslant 1$, $\dim R_k^\tau = k^g$*

(2) R_k^τ admits the bases

$$f_{a_j}(z) = \vartheta\begin{bmatrix} a_j/k \\ 0 \end{bmatrix}(kz, k\tau), \quad \text{and} \quad g_{b_j}(z) = \vartheta\begin{bmatrix} 0 \\ b_j/l \end{bmatrix}(z, \tau/k), \quad 0 \leqslant a_j, b_j < k \quad (16)$$

(3) if $k = l^2$, then the 3rd basis is given by $\vartheta\begin{bmatrix} a_j/l \\ b_{j'}/l \end{bmatrix}(kz, k\tau)$, $0 \leqslant j, j' < l$

(4) For $k \geqslant 3$ the basis elements $f_{a_j}(z)$ of R_k^τ have no common zero in $\mathbb{C}^g$ and defined an embedding

$$\mathbb{C}^g/\Lambda \ni z \to (f_1(z), ..., f_{k^g}(z)) \in \mathbb{CP}^{k^g - 1}$$

Proof. By Eq. (13), we can expand the function f in Fourier series

$$f(z) = \sum_{n_j} c_{n_j} e^{2\pi i n_j z_j}.$$

Applying now (14),

$$\sum_n c_n e^{2\pi i n(z + \tau m)} = e^{-\pi i k m \tau m - 2\pi i k z m} \sum_n c_n e^{2\pi i n z},$$

$$c_n = \chi_n e^{\pi i \frac{1}{k} n \tau n}, \implies \chi_n = \chi_{n+km}. \quad (17)$$

Hence χ is constant on cosets of $\mathbb{Z}^g/k\mathbb{Z}^g$. Taking

$$\chi_m^a = \begin{cases} 1, & \text{for } m = a + k\mathbb{Z}^g \\ 0, & \text{othewise,} \end{cases}$$

we obtain

$$f(z) = \sum_m \chi_m^a e^{\pi i \frac{1}{k} m\tau m + 2\pi i m z} = \sum_n e^{\pi i \frac{1}{k}(a+kn)\tau(a+kn) + 2\pi i (a+kn) z} = f_{a_j}(z). \tag{18}$$

In order to get g_{b_j}, take $\chi_n^b = e^{2\pi i \frac{1}{k} n_j b_j}$. $\square$