Geometry and Large N limits in Laughlin states

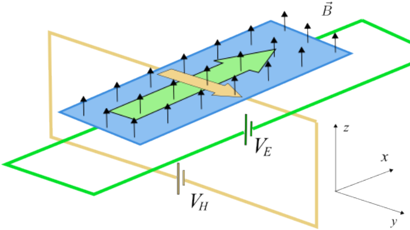
Semyon Klevtsov

University of Cologne

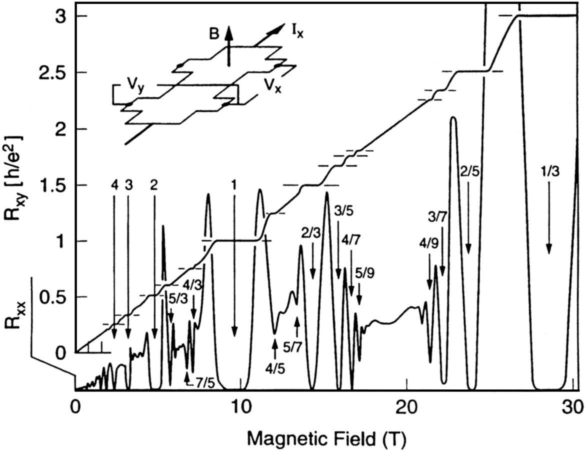
"Interfaces ...", Mittag-Leffler Institute, July 6, 2016

Classical and Quantum Hall effect

The classical Hall effect is the production of a voltage difference (the Hall voltage) across an electrical conductor, transverse to an electric current in the conductor and a magnetic field perpendicular to the current. It was discovered by Edwin Hall in 1879.



Quantum Hall effect (QHE) is observed in certain two-dimensional electron systems (GaAs heterostructures) subjected to low temperatures and strong magnetic fields [von Klitzig'80, Tsui-Stormer'83]. Most recently was reported in graphene.



On the plateaux Hall conductance is quantized $\sigma_H = 1/R_{xy} = \nu$, where $\nu \in \mathbb{Z}$ for integer QHE (labelled $1, 2, 3, 4..$) and $\nu \in \mathbb{Q}$ for fractional QHE (all other labels), to an extremely high precision (order 10^{-9}).

Laughlin state

On the plateaus QHE is described by collective multi-particle wavefunction, called the Laughlin state

$$\Psi(z_1, \ldots, z_N) = \prod_{i<j}^{N} (z_i - z_j)^\beta e^{-\frac{B}{4}\sum_i |z_i|^2}, \quad \beta \in \mathbb{Z}_+$$

[Laughlin'83]

$\beta = 1$: Integer QHE, non-interacting electrons.
$\beta = 3, 5, 7, ..$: Fractional QHE, interacting (via Coulomb forces) electron system. Hall conductance $\sigma_H = 1/\beta$.
Other candidate states were proposed for other plateaus.

Mathematically, the Laughlin state defines a sequence of probability measures on the configuration space $\mathbb{C}^N/S_N$ of N point-particles

$$\mu_N = |\Psi(z_1, \dots, z_N)|^2 \prod_{j=1}^{N} d^2 z_j$$

The total mass of this (unnormalized) measure is called the partition function (this is L^2 norm of Ψ)

$$Z = \int_{\mathbb{C}^N} \prod_{i<j}^{N} |z_i - z_j|^{2\beta} e^{-\frac{B}{2}\sum_i |z_i|^2} \prod_{j=1}^{N} d^2 z_j$$

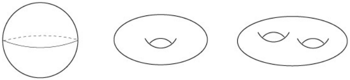
$$= \int_{\mathbb{C}^N} e^{-\frac{B}{2}\sum_i |z_i|^2 + \beta \sum_{i \neq j} \log|z_i - z_j|} \prod_{j=1}^{N} d^2 z_j$$

Central object in Log-gases (other names: Coulomb gas, Dyson gas, random matrix β-ensemble).

Why Hall conductance is quantized? It was understood very soon that Hall conductance is a Chern class (Thouless et.al. 1982). Laughlin state is a section of a vector bundle over certain manifold $V \to Y$. What is Y here? There are two major ways to think about this:
1) Bellissard et.al.: QH states on the lattice, vector bundles over the Brillouin zone, noncommutative geometry Chern classes.
2) Avron, Seiler, Simon, Zograf: QH states on Riemann surfaces, vector bundles on moduli space, Quillen metric.

Laughlin state on Riemann surfaces

We would like to consider Laughlin state on a compact Riemann surface Σ of genus g



genus 0 genus 1 genus 2

Problem: Define Laughlin state(s) Ψ_r on a genus-g Riemann surface Σ with arbitrary geometry: metric g, complex structure J, inhomogeneous magnetic field B, flat connections moduli. Determine the partition function $Z = \sum_r \langle \Psi_r, \Psi_r \rangle^2_{L^2}$ as a function of these geometric parameters

$$Z = Z[g, J, B, \varphi, ...]$$

in the limit of large number of particles.

Why: geometric adiabatic transport

Main idea is *geometric adiabatic transport* [Thouless et.al.; Avron, Seiler, Simon, Zograf, ...]. Laughlin states on a Riemann surface (Σ, g, J) form a vector bundle over the parameter space Y (e.g. moduli space of flat connections $Y = Jac(\Sigma)$ or complex structure moduli $Y = \mathcal{M}_{\mathrm{g}}$). Let d_Y be an exterior derivative along the parameter space. Then adiabatic (Berry) connection and curvature are

$$\mathcal{A} = \langle \Psi, d_Y \Psi \rangle_{L^2}, \quad \mathcal{R} = d_Y \mathcal{A} = -\partial_Y \bar{\partial}_Y \log Z.$$

Transport coefficients:

- For $Y = Jac(\Sigma)$, $\mathcal{R} = \sigma_H \Omega_{\mathrm{flat}}$ (σ_H is Hall conductance) [Thouless et.al.1982, Avron-Seiler 1985, Avron-Seiler-Zograf 1994]
- For adiabatic transport on the moduli space of a torus $\mathcal{M}_1$: $\mathcal{R} = \eta_H \Omega_P$, "anomalous (Hall) viscosity" $\eta_H = 1/4$. IQHE: [Avron-Seiler-Zograf 1995, Levay 1995], FQHE: [Tokatly-Vignale 2007, Read 2009]
- Transport on higher genus: $\mathcal{R} = \big(\eta_H k + \frac{c_H}{12}\chi(M)\big)\Omega_{WP}$, new transport coefficient, dubbed "Hall central charge" [SK-Wiegmann 2015]

Lowest Landau level (LLL)

Consider compact connected Riemann surface (Σ, g, J) and positive holomorphic line bundle (L^k, h^k). The latter corresponds to the magnetic field. The curvature $(1,1)$ form of the hermitian metric $h^k(z,\bar{z})$ is given by $F = -i\partial\bar{\partial}\log h^k$ and $\frac{1}{2\pi}\int_\Sigma F = k$, so k is "total flux" of the magnetic field. Notation: $B = g^{z\bar{z}}F_{z\bar{z}}$. On the plane and for constant magnetic field $B = k$, this corresponds to $h^k = e^{-\frac{B}{2}|z|^2}$.
Shrödinger equation for the one-particle wave functions of lowest energy in the strong magnetic field reduces to

$$\bar{\partial}_{L^k}\psi = 0$$

Solutions are holomorphic sections of L^k,

$$\psi_i = s_i(z), \quad i = 1, \ldots, N_k = \dim H^0(M, L^k)$$

These are wave functions on the lowest Landau level (LLL).
We will also consider tensoring $L^k \otimes K^s$, where K is canonical line bundle and s is called the spin.

Examples

1. Round sphere S^2, constant magnetic field: $h^k = \frac{1}{(1+|z|^2)^k}$ Then $s_j = z^{j-1},\ j = 1, ..k+1$ is a complete basis of holomorphic sections (LLL), with finite L^2 norm

$$\int_{S^2} \bar{s}_i(\bar{z}) s_i(z) h^k \frac{d^2 z}{(1+|z|^2)^2} < \infty$$

2. Flat torus T^2, constant magnetic field $h^k = e^{-\pi i k \frac{(z-\bar{z})^2}{\tau-\bar{\tau}}}$. The basis of sections:

$$s_j = \theta_{\frac{j}{k},0}(kz+\varphi, k\tau), \quad j = 1, \dots, k$$

Here $\varphi = \varphi_2 + \varphi_1 \tau$ is complex coordinate on $Jac(T^2)$: flat connection moduli $dA = 0$

$$A = \frac{\varphi}{\tau - \bar{\tau}} d\bar{z} - \frac{\bar{\varphi}}{\tau - \bar{\tau}} dz, \quad \int_a A = \varphi_1, \quad \int_b A = \varphi_2.$$

3. On Σ of genus g the number of LLL wave functions is

$$N_k = \dim H^0(\Sigma, L^k) = k + 1 - \mathrm{g}$$

for k large.

Definition of Laughlin state (integer QHE)

For integer QHE, $\beta = 1$. Consider N_k points on Σ: $z_1, z_2, \ldots, z_{N_k}$. The (holomorphic part F of the) Laughlin state is completely antisymmetric combination of one-particle wave functions on LLL (holomorphic sections):

$$F(z_1, \ldots, z_{N_k}) = \det[s_i(z_j)]_{i,j=1}^{N_k}$$

Also called Slater determinant. For example, on the sphere S^2, we had $s_j = z^{j-1}$, $j = 1, .., k+1$. We get

$$F^{S^2}(z_1, \ldots, z_{k+1}) = \det z_i^{j-1} = \prod_{i<j}^{k+1} (z_i - z_j)$$

(Vandermonde determinant). Torus:

$$F^{T^2}(z_1, .., z_k) = \det \theta_{\frac{j}{k},0}(kz_i + \varphi, k\tau) = \theta(z_c + \varphi, \tau) \prod_{i<j} \frac{\theta_1(z_i - z_j, \tau)}{\eta(\tau)}$$

where $z_c = \sum_j z_j$ is center-of-mass coordinate. Bosonization formula (higher genus analog of Vandermonde determinant)

Definition of Laughlin state (fractional QHE)

Consider now line bundle $(L^{\beta k}, h^{\beta k})$. But number of points is still $N_k = \dim H^0(\Sigma, L^k)$, i.e. only fraction of LLL states is occupied (thus *fractional* QHE). The (holomorphic part F of the) Laughlin state satisfies

- $F(z_1, ..., z_{N_k})$ is completely anti-symmetric
- Fix all z_j except one, say z_m. Then $F(\cdot, ..., z_m, ..., \cdot)$ is a holomorphic section of $L^{\beta k}$.
- Vanishing condition near diagonal $z_i \sim z_j$ in local complex coordinate system on Σ, $F(z_1, ..., z_{N_k}) \sim \prod (z_i - z_j)^\beta$.

Examples:

- Round sphere: $F(z_1, ..., z_{k+1}) = \prod_{i<j}(z_i - z_j)^{\beta}$.
- Novel feature: for $\mathrm{g} > 0$ Laughlin states are degenerate. Torus

$$F_r(z_1, .., z_k) = \theta_{\frac{r}{\beta},0}(\beta z_c + \varphi, \beta\tau) \prod_{i<j} \left(\frac{\theta_1(z_i - z_j, \tau)}{\eta(\tau)} \right)^{\beta}.$$

Center-of-mass: $z_c = \sum z_i$.

- Higher genus: via Jacobean embedding $\Sigma \to Jac(\Sigma)$: $\vec{z} = \int_{z_0}^{z_i} \vec{\gamma}$

$$\Psi_{\vec{r}}(z_1, .., z_{N_k}) = \theta_{\frac{\vec{r}}{\beta},0}(\beta\vec{z_c} + \vec{\varphi} + \vec{\Delta}, \beta\tau) \prod_{i<j} E(z_i - z_j, \tau),$$

where $\vec{r} = (1, ..., \beta)^{\mathrm{g}}$, $\vec{\gamma}$ is a basis of holomorphic 1-forms, $\vec{\Delta}$ is vector of Riemann constants and $E(z, \tau)$ is prime-form. The number of Laughlin states on Σ of genus g is $n_\beta = \beta^{\mathrm{g}}$: vector bundle of rank n_β over Y.

Arbitrary metric and inhomogeneous magnetic field

The advantage of the language of holomorphic line bundles is that it gives us a clear idea how to put the Laughlin state on Σ with arbitrary metric g and inhomogeneous magnetic field B. Consider some fixed (constant scalar curvature) metric g_0, and constant magnetic field B_0 (and corresponding hermitian metric $h_0^k(z,\bar{z})$). Arbitrary metrics are parameterized by:

- Kähler potential $\phi(z,\bar{z})$: $g_{z\bar{z}} = g_{0\bar{z}z} + \partial_z \bar{\partial}_{\bar{z}}\phi$,
- "magnetic" potential $\psi(z,\bar{z})$: $F = F_0 + \partial\bar{\partial}\psi$, $B = g^{z\bar{z}}F_{z\bar{z}}$

Partition function

For the integer QHE ($\beta = 1$), the partition function on arbitrary Σ is

$$Z = \int_{\Sigma^{N_k}} |\det s_i(z_j)|^2 \prod_{j=1}^{N_k} h_0^k(z_j, \bar{z}_j) e^{-k\psi(z_j, \bar{z}_j)} \sqrt{g}(z_j) d^2 z_j$$

For the fractional QHE ($\beta = 3, 5, 7, ..$)

$$Z = \sum_{r=1}^{n_\beta} \int_{\Sigma^{N_k}} |F_r(z_1, .., z_{N_k})|^2 \prod_{j=1}^{N_k} h_0^{\beta k}(z_j, \bar{z}_j) e^{-k\beta\psi(z_j, \bar{z}_j)} \sqrt{g}(z_j) d^2 z_j.$$

Adiabatic connection on $Y = Jac(\Sigma) \times \mathcal{M}_{\mathrm{g}}$

$$\mathcal{A}_{rs} = \langle F_r, d_Y F_s \rangle_{L^2}$$

Partition function for IQHE

Terminology: we write for the magnetic field $F = dA$, and use components of gauge-connection one form $A_z = i\partial_z \log h^k$. We also write $Ric(g) = d\omega$, where $\omega_z = i\partial \log g_{z\bar{z}}$ is spin-connection.
Result [SK'13; SK, Ma, Marinescu, Wiegmann'15]:

$$\log Z = \frac{1}{2\pi}\int_\Sigma (A_z A_{\bar{z}} + \frac{1-2s}{2}(A_z\omega_{\bar{z}} + A_{\bar{z}}\omega_z) + \left(\frac{(1-2s)^2}{4} - \frac{1}{12}\right)\omega_z\omega_{\bar{z}})$$

$$+\mathcal{F}[B,R]$$

where $\mathcal{F}[B,R]$ is a local functional of magnetic field B and scalar curvature R, which admits large k asymptotic expansion, with first terms given by

$$\mathcal{F} = -\frac{1}{2\pi}\int_\Sigma \left[\frac{1}{2}B\log B + \frac{2-3s}{12}R\log B + \frac{1}{24}(\log B)\Delta_g(\log B)\right]\sqrt{g}d^2z + \mathcal{O}(1/k).$$

Derivation of $\log Z$ in IQHE

For $\beta = 1$ the partition function satisfies determinantal formula:

$$
\begin{aligned}
Z &= \int_{\Sigma^{N_k}} |\det s_i(z_j)|^2 \prod_{j=1}^{N_k} h_0^k(z_j, \bar{z}_j) e^{-k\psi(z_j, \bar{z}_j)} \sqrt{g}(z_j) d^2 z_j \\
&= \det \langle s_i, s_j \rangle_{L^2}
\end{aligned}
$$

Denoting $G_{jl} = \langle s_j, s_l \rangle$, we get

$$
\begin{aligned}
\delta \log Z &= \delta \,\mathrm{Tr}\, \log \langle s_j, s_l \rangle = \\
&= -\frac{1}{2\pi} \sum_{j,l} G^{-1}_{lj} \int_\Sigma \left(\frac{s-1}{2} (\Delta_g \delta\phi) + k\delta\psi \right) \bar{s}_j s_l h^k \sqrt{g}^{1-s} d^2 z \\
&= -\frac{1}{2\pi} \int_\Sigma \left(\frac{s-1}{2} (\Delta_g B_k(z, \bar{z}))\, \delta\phi + k B_k(z, \bar{z})\, \delta\psi \right) \sqrt{g}^{1-s} d^2 z,
\end{aligned}
$$

where $B_k(z, \bar{z})$ is the Bergman kernel on diagonal.

Bergman kernel

B_k is the Bergman kernel on the diagonal. For orthonormal basis of sections $\{s_j\}$:

$$B_k(z,\bar{z}) = \sum_{i=1}^{N_k} ||s_i||^2_{h^k} = B + \frac{1-2s}{4}R + \frac{1}{4}\Delta_g \log B + \frac{2-3s}{24}\Delta_g(B^{-1}R) + \frac{1}{24}\Delta_g(B^{-1}\Delta_g \log B) + \mathcal{O}(1/k^2)\,.$$

[Boutet de Monvel-Sjöstrand, Tian, Zelditch, Catlin, ...]

We need 3 leading terms in the most general form at $n=1$, due to [Ma-Marinescu'06 (Book), Ma-Marinescu'12(Crelle)]

In QM, Bergman kernel is the density of states ψ_i on "completely filled" LLL

$$B_k(z,\bar{z}) = \sum_{i=1}^{N_k} |\psi_i(z)|^2 = \lim_{T\to\infty} \int_{x(0)=z}^{x(T)=z} e^{-\int_0^T (\dot{x}^2 + A\dot{x})dt} \mathcal{D}x(t)$$

[Douglas, SK'09]

Constant magnetic field case

The case of $B = k$ (Kähler potential is equal to magnetic potential $\phi = \psi$) is of special interest

$$\log Z = \int_M (-k + \Delta_\phi) B_k(z)\, \delta\phi\, \sqrt{g} d^2 z$$

The answer is given by geometric functionals ([Donaldson'04] studied this as $Z = \det \mathrm{Hilb_k}$, where Hilb_k parameterizes inner-products on $H^0(M, L^k)$)

$$\log Z = -\frac{k^2}{2\pi} S_{AY}(g_0, \phi) + \frac{k}{4\pi} S_M(g_0, \phi) + \frac{1}{12\pi} S_L(g_0, \phi) + \overbrace{\frac{1}{k} \int R^2 + \dots}^{\text{local densities}}$$

[SK'13]

First three terms of the expansion are geometric functionals: Aubin-Yau(-Futaki-Mabuchi), Mabuchi and Liouville. The remainder terms are integrals of local densities of higher-order curvature invariants.

Geometric functionals

Aubin-Yau(-Futaki-Mabuchi): $\delta S_{AY} = \int \delta\phi\,\sqrt{g} d^2 z$

$$S_{AY}(g,\phi) = \int_M \frac{1}{2}\phi\partial\bar{\partial}\phi + \phi\sqrt{g_0} d^2 z,$$

Mabuchi: $\delta S_M = \int R\,\delta\phi\,\sqrt{g} d^2 z$

$$S_M(g_0,\phi) = \int_M \left(-\phi R_0\sqrt{g_0} + \sqrt{g}\log\frac{\sqrt{g}}{\sqrt{g_0}}\right) d^2 z$$

Liouville: $\delta S_L = \int (\Delta R)\,\delta\phi\,\sqrt{g} d^2 z$

$$S_L(g_0,\sigma) = \int_M \left(\frac{1}{2}\partial\sigma\bar{\partial}\sigma + \sigma R_0 g_0\right)$$

in conformal gauge $e^\sigma\sqrt{g_0} = \sqrt{g_0} + \partial\bar{\partial}\phi$.

Liouville action is a hallmark of gravitational (conformal) anomaly. For two metrics the conformal class $g = e^{2\sigma} g_0$, there is Polyakov formula

$$S_L(g_0, g) = \log \frac{\det' \Delta_g}{\det' \Delta_{g_0}}$$

More generally, CFT partition function transforms

$$\log \frac{Z^{CFT}(g)}{Z^{CFT}(g_0)} = \frac{c}{12\pi} S_L(\sigma) = \frac{c}{6\pi} \int_\Sigma \omega_z \omega_{\bar{z}} d^2 z$$

where c is central charge. Our result is the mixed electromagnetic-gravitational anomaly

$$\log Z = \frac{1}{2\pi} \int_\Sigma \left[A_z A_{\bar{z}} + \frac{1-2s}{2} (A_z \omega_{\bar{z}} + \omega_z A_{\bar{z}}) + \right. \\ \left. + \left(\frac{(1-2s)^2}{4} - \frac{1}{12} \right) \omega_z \omega_{\bar{z}} \right] d^2 z + \mathcal{F}[B, R].$$

Geometric adiabatic transport

The integer QHE wavefunction $F(z_1, .., z_{N_k}) = \det s_i(z_j)$ is a section $\mathcal{S}$ of the determinant line bundle $\mathcal{L}$ over the parameter space $Y = Jac(\Sigma) \times \mathcal{M}_{\mathrm{g}}$, $\mathcal{L} = \det H^0(\Sigma, L^k \otimes K^s)$. For the basis $\{s_j\}$ of $H^0(\Sigma, L^k \otimes K^s)$, the Quillen metric of $\mathcal{S}$ of $\mathcal{L}$ is given by

$$\|\mathcal{S}\|^2 = \frac{\det\langle s_j, s_l\rangle}{\det' \Delta_{L^k \otimes K^s}} = \frac{Z}{\det' \Delta_{L^k \otimes K^s}} ,$$

Then adiabatic curvature is related to the curvature $\Omega^{\mathcal{L}}$ of the Quillen metric as

$$\mathcal{R} = -d_Y \bar{d}_Y \log Z = \Omega^{\mathcal{L}} - d_Y \bar{d}_Y \log \det{}' \Delta_{L^k \otimes K^s}$$

where the last term is vanishhing as $k \to \infty$.

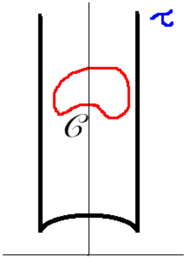
Relation to Chern-Simons theory

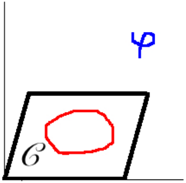
Consider geometric adiabatic transport of IQHE wave function along a contour $\mathcal{C}$ in the moduli space $Y = Jac(\Sigma) \times \mathcal{M}_g$
Define adiabatic connection:

$$\mathcal{A}_Y = \langle \Psi, d_Y \Psi \rangle_{L^2},$$

and adiabatic phase:

$$e^{i \int_{\mathcal{C}} \mathcal{A}_Y}.$$





([SK-Ma-Marinescu-Wiegmann'15]):

$$\int_{\mathcal{C}} \mathcal{A}_Y = \frac{1}{4\pi} \int_{\Sigma_Y \times \mathcal{C}} \left[\sigma_H A dA + 2\eta_H A d\omega - \frac{1}{12} c_H \omega d\omega \right]$$

(proof relies on Quillen-Bismut et al theory)

Bismut-Gillet-Soulé curvature formula

X is "universal curve" (union of all Riemann surfaces Σ_y over Y), and E is line bundle over X, which is the "union" of all line bundles $(L^k \otimes K^s)_y \to \Sigma_y$.

$$\Omega^{\mathcal{L}} = -2\pi i \int_{X|Y} \left[\mathrm{Ch}(E)\mathrm{Td}(TX|Y)\right]_{(4)}.$$

Here the integrand is a form of mixed degree on X. The subscript 4 means that only 4-form component of the full expression is retained, so that the result of the integration is a 2-from. The notation $X|Y$ means that the integration goes over the fibers, i.e., over the spaces Σ_y at y fixed. Explicitly:

$$\Omega^{\mathcal{L}} = \frac{i}{4\pi} \int_{X|Y} \left[\mathrm{F} \wedge \mathrm{F} + (1-2s)\,\mathrm{F} \wedge \mathrm{R}_{TX|Y} + \left(\frac{(1-2s)^2}{4} - \frac{1}{12}\right) \mathrm{R}_{TX|Y} \wedge \mathrm{R}_{TX|Y}\right]$$

$$\int_{\mathcal{C}} \mathcal{A} = \frac{1}{4\pi} \int_{\sigma^{-1}(\mathcal{C})} A \wedge dA + \frac{1-2s}{2}(A \wedge d\omega + dA \wedge \omega) + \left(\frac{(1-2s)^2}{4} - \frac{1}{12}\right) \omega \wedge d\omega\,.$$

$\sigma : X \to Y$ is projection. Chern-Simons effective theory of QHE [Fröhlich-Studer'92;Wen-Zee'92,Abanov-Gromov-Fradkin'15].

Partition function of Laughlin state

For the FQHE Laughlin state we have the following results:

$$\log Z = \frac{1}{2\pi}\int_\Sigma \Big[\sigma_H A_z A_{\bar z} + 2\eta_H (A_z\omega_{\bar z} + \omega_z A_{\bar z}) + \\ - \frac{1}{12} c_H \omega_z \omega_{\bar z}\Big] d^2 z + \mathcal{F}[B, R].$$

where

$$\mathcal{F} = -\frac{1}{2\pi}\int_\Sigma \left[\frac{2-\beta}{2\beta} B \log B\right] \sqrt{g} d^2 z + \mathcal{O}(1/k).$$

Hall conductance $\sigma_H = \frac{1}{\beta}$,
"Hall viscosity" $\eta_H = \frac{1}{4}(1 - \frac{2s}{\beta})$, [Avron-Seiler-Zograf 1995]
"Hall central charge" $c_H = 1 - 3\frac{(\beta-2s)^2}{\beta}$.

Beta-deformed Bergman kernel

Darboux-Cristoffel formula for Bergman kernel

$$B_k(z_1) = \int_{\Sigma^{N_k-1}} |\det s_i(z_j)|^2 \prod_{j=2}^{N_k} h_0^k(z_j,\bar{z}_j) e^{-k\psi(z_j,\bar{z}_j)} \sqrt{g}(z_j) d^2 z_j$$

(integrate all z_j's but z_1). Up to overall combinatorial constant.
Beta-deformed Bergman "kernel" (no longer a kernel): density of electrons in Laughlin state

$$B_k(z_1) = \int_{\Sigma^{N_k-1}} |\det s_i(z_j)|^{2\beta} \prod_{j=2}^{N_k} h_0^{\beta k}(z_j,\bar{z}_j) e^{-\beta k\psi(z_j,\bar{z}_j)} \sqrt{g}(z_j) d^2 z_j$$

Also admits large k expansion:

$$B_k(z) = \sum_{m=0}^{\infty} c_m(\beta) k^{1-m}$$

where $c_m(\beta)$ are invariants of B, R with coefficients depending on β.

Adiabatic curvature for Laughlin states

Adiabatic connection and curvature for Laughlin states on Y is actually abelian (this is still a conjecture for $\mathrm{g} > 1$).

$$\mathcal{A}_{rs} = \langle \Psi_r, d_Y \Psi_s \rangle = \mathcal{A}\delta_{rs}, \quad \mathcal{R}_{rs} = \mathcal{R}\delta_{rs}.$$

The adiabatic phase for the transport along the smooth closed contour $\mathcal{C} \in Y$ is

$$\int_{\mathcal{C}} \mathcal{A} = \frac{1}{4\pi} \int_{\sigma^{-1}(\mathcal{C})} \sigma_H A \wedge dA + 2\eta_H (A \wedge d\omega + dA \wedge \omega) - \frac{1}{12} c_H \omega \wedge d\omega \,.$$

$\sigma : X \to Y$ is projection. Correspondingly, there should be a β-deformation of the BGS curvature formula, for the adiabatic curvature of Laughlin states (conjecture).

Free field representation

The proof is based on the free field representation of Laughlin states

$$\sum_r^{n_\beta} |\Psi_r|^2 = \int e^{i\sqrt{\beta}X(z_1)} \dots e^{i\sqrt{\beta}X(z_{N_k})} e^{-\frac{1}{2\pi}S(g,X)}$$

[Moore-Read'91]

where sum goes over all degenerate Laughlin states on Riemann surface and the free field action is

$$S = \int_M \left(\partial X \bar{\partial} X + i\sqrt{\beta} R X + \frac{i}{\sqrt{\beta}} A \wedge dX\right)$$

[Ferrari-SK'14]

Moore-Read famously proposed this representation on the plane $(R = 0)$. The novel feature of Ferrari-SK'14 is the background charge term $i\sqrt{\beta}RX$ and magnetic field coupling $\frac{i}{\sqrt{\beta}} A \wedge dX$. We then develop large k techniques to tackle the path integral above.

New transport coefficient

Consider complex structure deformations $g_{z\bar{z}}|dz|^2 \to g_{z\bar{z}}|dz + \mu d\bar{z}|^2$, where Beltrami differential is $\mu = g_{z\bar{z}}^{-1} \sum_{\kappa=1}^{3g-3} \eta_\kappa \delta y_\kappa$ and η_κ are a basis of holomorphic quadratic differentials.
Berry curvature, associated with these deformations is

$$\mathcal{R} = i\mathbf{d}\bar{\mathbf{d}} \log Z = \big(\eta_H k + \frac{c_H}{12}\chi(M)\big)\Omega_{WP},$$

where $\Omega_{WP} = i\int_M \mathbf{d}\mu \wedge \bar{\mathbf{d}}\bar{\mu}\, g_{z\bar{z}} d^2 z$ is the Weil-Petersson form on the moduli space. Here $c_H = 1 - 3\frac{(\beta-2s)^2}{\beta}$ is a new quantized transport coefficient, it can only be seen on higher genus surfaces, since on torus $\chi(M) = 0$.

[SK-Wiegmann'15]

[Bradlyn-Read'15]

Singular surfaces

Riemann surfaces with conic singularities or cusps.

- Curvature in the experimental sample arises in graphene

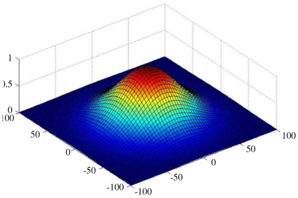


Fig. 6. A smooth curved bump in the graphene sheet.

M.A.H. Vozmediano et al., Physics Reports 496 (2010)

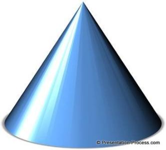
- Recently a QHE-like system was experimentaly realised on a spatial cone Schine et al., Arxiv:1511.07381
- There are proposals to experimentally realize QHE on a higher-genus Riemann surface as multi-layer system with n defects (branched covering of complex plane with n singular points). Barkeshli, Qi 2013.

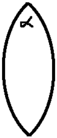
The gravitational term (Liouville action) becomes most important in presence of singularities. The large k expansion of $\log Z$ breaks down at $\mathcal{O}(\log k)$. Instead we encounter zeta function for the scalar laplacian on the cone (Cheeger, Brüning, Lesch, Melrose, Müller, Vertman, ...). This is what replaces the smooth-case expansion (conjecture)

$$\log Z = -\frac{k^2}{2\pi} S_{AY}(g_0,\phi) + \frac{k}{4\pi} S_M(g_0,\phi) + (\zeta(0,\Delta_{\mathrm{cone}}) - \zeta(0,\Delta_0) \log k$$

$$-\frac{1}{2}\log\frac{\det\Delta_{\mathrm{cone}}}{\det\Delta_0} + \mathcal{O}(1/k)$$

Working examples: flat cone over S^1, Troyanov spindle (sphere with antipodal singularities).





$$-\frac{1}{2}\log\det\Delta_{\mathrm{cone}}\sim\zeta_2'(0,\alpha,1,\alpha),$$

where ζ_2 is Barnes double zeta function. (There are few explicit formulas, somewhat unreliable literature. See Spreafico 2004,2009)

What is the answer for FQHE (Laughlin state)?

Conjecture: quantum Liouville theory. (but this is long story)