Talk

Mittag-Leffler

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Geometry and large N asymptotics in quantum Hall states

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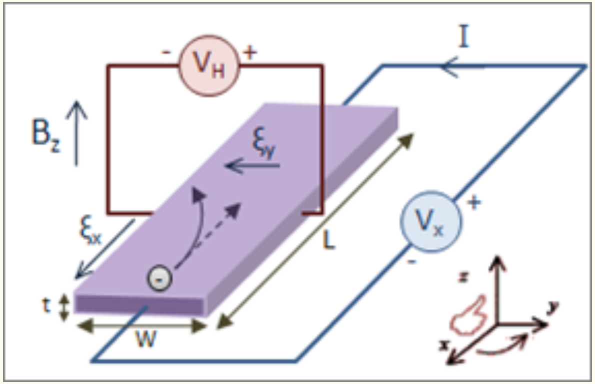
Classical Hall effect

Appearance of electric potential difference (V_H) perpendicular to the direction of the current (I) in conductors placed in magnetic field (B_z)

$$V_H = \frac{I \cdot B_z}{n \cdot e}$$



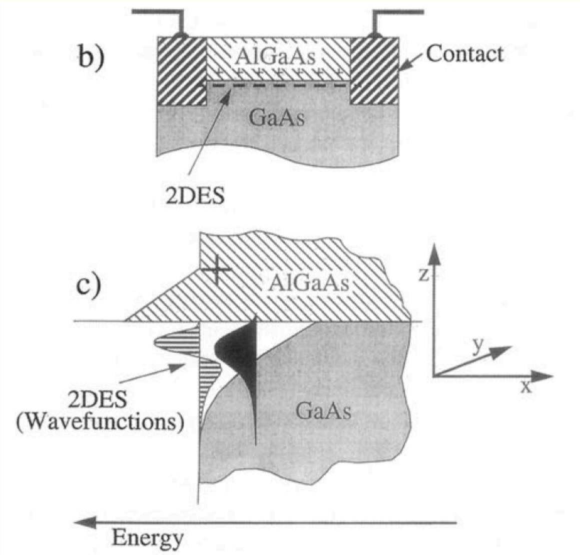
where n is the density of electrons of charge e.

Hall resistance $R_H = \frac{B_z}{ne}$

grows linearly with the magnetic field.

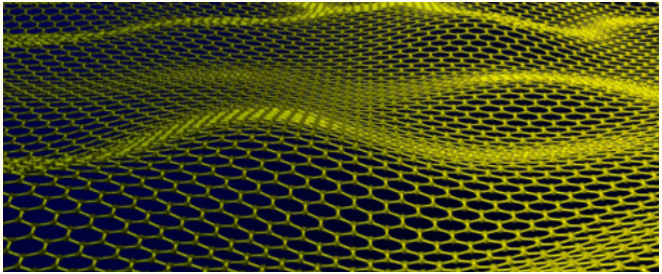
Quantum Hall effect (QHE)

Observed in 2d electron systems, subjected to low temperatures and strong magnetic fields.



Gallium-Arsenide

Graphene

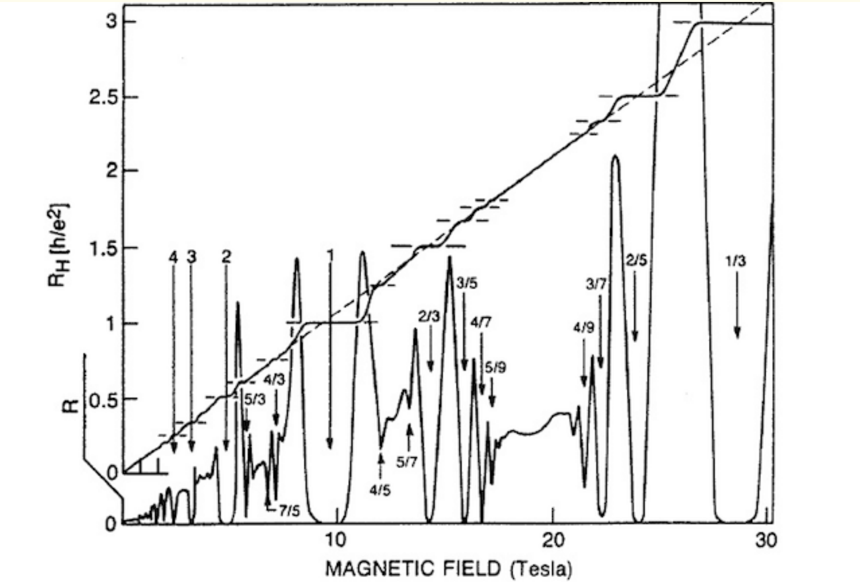


Hall resistance R_H undergoes a series of plateaux, where it is quantized (in units of e^2/h).

$$\sigma_H := \frac{1}{R_H} = \begin{cases} n \in \mathbb{Z}_+ & \rightsquigarrow \text{Integer QHE} \\ p/q \in \mathbb{Q} & \rightsquigarrow \text{Fractional QHE} \end{cases}$$

von Klitzing 1981

Tsui, Stormer, Gossard 1982



precision measurement of fine structure constant $\alpha = e^2/hc = 1/137.035999173(35)$

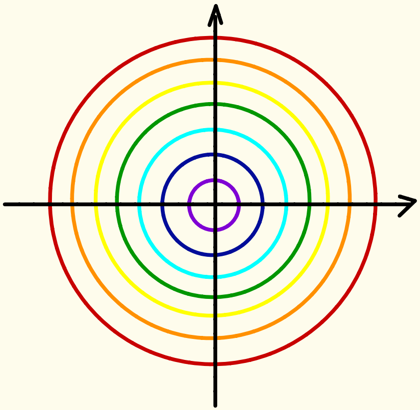
Integer QHE plateaux are explained by one-particle wave functions (non-interacting)

$$\Psi(z_1, \ldots z_N) = \det \psi_n(z_m)\Big|_{n,m=1}^{N}$$

"Slater determinant".

$$\psi_n = z^n \cdot e^{-\frac{B}{4}|z|^2}$$

lowest Landau level (LLL)



~ degenerate ground states in strong magnetic field.

Fractional QHE

Strongly-interacting (via Coulomb forces) system.

Laughlin 1983: Assign a trial wave function ("state") to each plateau.

$$\Psi(z_1, \ldots, z_N)$$

* holomorphic
* vanishing conditions

$\Psi = 0$

Laughlin state

$$\underbrace{\Psi(z_1, \dots, z_N)}_{\mathbb{C}^N} = c \cdot \prod_{n<m}^{N} (z_n - z_m)^\beta \cdot e^{-\frac{B}{4} \sum_{n=1}^{N} |z_n|^2}$$

$$\beta \in \mathbb{Z}_+$$

Hall conductance $\quad \sigma_H = 1/\beta$

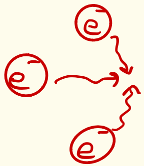
Another famous QHE state

Moore-Read 1991:

$$\Psi_{MR}(z_1,\dots,z_N) = c\cdot \mathrm{Pf}\left(\frac{1}{z_n - z_m}\right)\cdot \prod_{n<m}^{N}(z_n - z_m)\cdot e^{-\frac{B}{4}\sum_n |z_n|^2}$$

Pfaffian of anti-sym. matrix

$$\sigma_H = 5/2$$



$\Psi = 0$

Mathematically, QHE wave functions define sequences of probability measures on $\mathbb{C}^N$ (actually, $\mathbb{C}^N/S_N$)

$$\mu_N := \frac{1}{N!} \left|\Psi(z_1, \dots, z_N)\right|^2 \cdot \prod_{n=1}^{N} d^2 z_n$$

Total mass of μ_N is the L^2-norm of Ψ. For Laughlin:

$$Z = \frac{1}{N!} \int_{\mathbb{C}^N} \exp\left[-\frac{B}{2} \sum_n |z_n|^2 + \beta \sum_{n \neq m} \log |z_n - z_m|\right] \prod_{n=1}^{N} d^2 z_n$$

2D Coulomb gas partition function.

More generally,

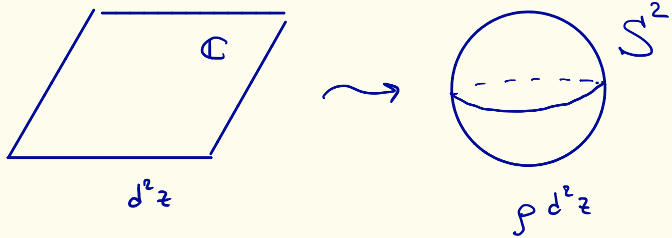
$$Z = \int_{\mathbb{C}^N} \exp\left\{ -N \sum_{n=1}^{N} V(z_n, \bar{z}_n) + \beta \sum_{n \neq m} \log |z_n - z_m| \right\} \cdot \prod_{n=1}^{N} d^2 z_n$$

grav. "spin" (s=1 in pure Coulomb gas)

$$V = \phi(z, \bar{z}) - \frac{1-s}{N} \log \rho(z, \bar{z})$$

"magnetic potential"

volume form $\rho\, d^2 z$ on $\mathbb{C}$



State-of-the-art

Thm (Leblé-Serfaty 2017)

$$\log Z = -\beta N^2 I_V(\mu_V) + \frac{\beta}{2} N \log N - N\, C(\beta) - N\left(1-\frac{\beta}{2}\right)\int_{\mathbb{C}} \mu_V \log \mu_V + o(N)$$

where $I_V = -\iint_{\mathbb{C}\times\mathbb{C}} \log|z-w|\, d\mu(z)\, d\mu(w) + \int_{\mathbb{C}} V\, d\mu$

and μ_V its unique minimizer ("equilibrium measure")

[in fact, corollary of a stronger large deviations result]

$\mathcal{O}(1)$ term

Prop

Can, Laskin, Wiegmann 2014
F. Ferrari, SK 2014

$$\log Z = -\beta N^2 I_V(\mu_V) - N\left(s - \frac{\beta}{2}\right)\int_\Sigma \mu_V \log \mu_V$$
$$- \frac{c_H}{12}\int_\Sigma \left(|\partial \log \mu_V|^2 - 2\partial\bar\partial \log \mu_V\right) + \text{const} + R_{1/N}$$

remainder terms

$$c_H = 1 - 3\left(\sqrt{\beta} - \frac{2s}{\sqrt{\beta}}\right)^2$$

($s=1$ in pure Coulomb gas)

"central charge"

Remainder term via

gaussian free field (GFF) Ferrari-SK 2014

Def (e.g. Sheffield, math/0312099) $D \subset \mathbb{R}^d$

GFF h is a formal sum $h = \sum_j \alpha_j f_j$

α_j – i.i.d. gaussians

f_j – orthonormal basis in the Hilbert space $H(D)$ w.r.t. inner product $(f_1, f_2)_\nabla = \int_D \nabla f_1 \cdot \nabla f_2$

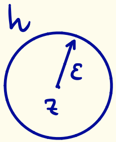
$(h, \cdot)_\nabla$ applied to a test fn is a random variable

Exponent of GFF "$e^{\gamma h}$"

Def (e.g. Duplantier, Sheffield 2009) $0<\gamma<2$

$$e^{\gamma h} := \lim_{\varepsilon \to 0} \varepsilon^{\gamma^2/2} \, e^{\gamma h_\varepsilon(z)}$$

where $h_\varepsilon(z)$ is the circle mean value



J-P. Kahane, “Sur le chaos multiplicatif”, Ann. Sci. Math. Québec, 1985
R. Rhodes, V. Vargas 2016

$$\mathbb{E}\left[e^{\gamma h(z_1)} \cdots e^{\gamma h(z_N)} \right]$$

Representation of the remainder term

$$R_{1/N} = \log E_{\mu_v} \left(\int e^{i\sqrt{\beta} h} d^2 z \right)^N - \log E_{\mu_0} \left(\int e^{i\sqrt{\beta} h} d^2 z \right)^N$$

<u>Conjecture</u> (Ferrari-SK) : $\quad R_{1/N} = \mathcal{O}(1/N)$

as $N \to \infty$

(proof involves path integrals)

Applications

anomalous (Hall) viscosity

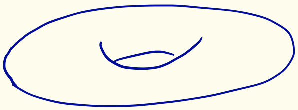
Avron, Seiler, Zograf 1995

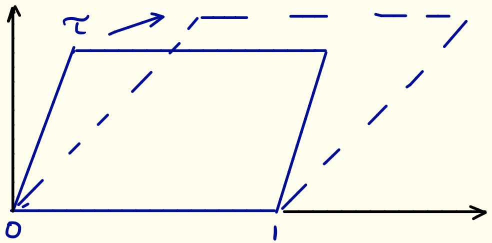
viscosity tensor

$$\sigma_{\alpha\beta} = - \eta_{\alpha\beta\gamma\delta}\, \dot{u}_{\gamma\delta}$$

stress   strain-rate

QHE states on torus





"Hall" viscosity

$$\eta_H = \frac{\beta}{4} N_\phi - \frac{c_H}{24} \chi(\Sigma)$$

Tokatly, Vignale; N. Read 2008

SK-Wiegmann 2015

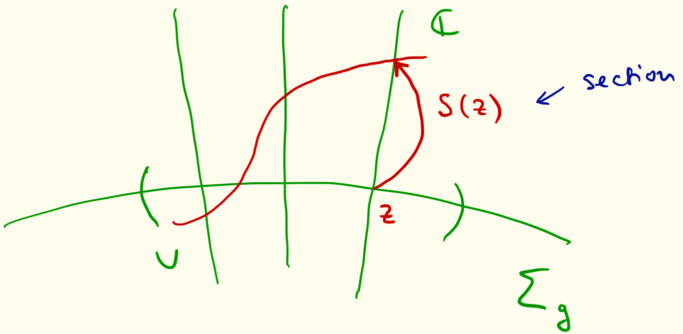
QH states on Riemann surfaces

Recall, that the one-particle wave functions on $\mathbb{C}$ are $\psi_n = z^n e^{-\frac{B}{4}|z|^2}$.

What is an analog of holomorphic polynomials on a compact Σ_g? $z^n \rightsquigarrow S_n(z)$, holomorphic sections of a holomorphic line bundle L.

$$\boxed{\bar\partial_L S(z) = 0}$$

$$\bar\partial_L : C^\infty(\Sigma, L) \to \Omega^{0,1}(\Sigma, L)$$



+ holomorphic transition functions t_{UV} on $U \cap V$

Consider a compact oriented genus-g Riemann surface (Σ, g, J)

metric

complex structure

and a <u>positive</u> holomorphic line bundle L on Σ

of <u>degree</u> $N_\phi = \deg L$.

Hermitian metric $h(z, \bar{z})$, $\| s(z) \|_h^2 \in \mathbb{R}_+$

Its curvature $F = -i \partial \bar{\partial} \log h \in \Omega^{1,1}(\Sigma)$, $F > 0$ (positivity)

$$\frac{1}{2\pi} \int_\Sigma F = N_\phi \quad (\text{degree, "flux" of magnetic field})$$

Equivalently, holomorphic sections of L will have

N_ϕ zeroes on Σ.

Integer QH state ("Slater determinant")

Consider $\Sigma^N = \underbrace{\Sigma \times \dots \times \Sigma}_{N}$, $N = \dim H^0(\Sigma, L)$

$$\Psi(z_1, \dots, z_N) = \det S_n(z_m)\Big|_{n,m=1}^{N}$$

Its L^2-norm is given by

$$Z = \int_{\Sigma^N} \| \det S_n(z_m) \|_h^2 \cdot \prod_{n=1}^{N} \sqrt{g}\, d^2 z_n$$

Consider an arbitrary hermitian metric and arbitrary Riemannian metric

$$h = h_0 \, e^{-N_\phi \psi(z,\bar z)}, \quad F(h) > 0$$

$$g = g_0 + \partial\bar\partial \phi(z,\bar z) > 0$$

$\boxed{\text{Thm}}$ SK 2013, SK-Ma-Marinescu-Wiegmann 2015

Asymptotic large N f-la for log Z

$$\log Z = -N_\phi^2 S_2(\psi) + \frac{1}{2} N_\phi S_1(\phi,\psi) + \frac{1}{6} S_0(\phi) + O(1/N_\phi)$$

$$S_2(\psi) = \frac{1}{2\pi} \int_\Sigma |\partial\psi|^2 + \psi g_0$$

$$S_0(\phi) = \frac{1}{2\pi} \int \left(\left|\partial \log \frac{g}{g_0}\right|^2 + R_0 \log \frac{g}{g_0} \right)$$

$$S_1(\phi,\psi) = \frac{1}{2\pi} \int_\Sigma \left(-\frac{1}{2} \psi R_0 + F \log \frac{g}{g_0} \right)$$

Proof Variational f-la

$$\delta \log Z = -\frac{1}{2\pi}\int_{\Sigma}\left(N_\phi B_{N_\phi}\cdot\delta\phi - \frac{1}{2}\Delta B_{N_\phi}\cdot\delta\phi\right)\sqrt{g}\,d^2z$$

where Bergman kernel for the $H^0(\Sigma, L)$

$$B_{N_\phi}(z,\bar z) = \sum_{n=1}^{N}\|s_n(z)\|_h^2 \simeq N_\phi + \frac{1}{2}R(g) + \mathcal{O}(1/N_\phi)$$

Complete asymptotic expansion at large N_ϕ

(Boutet de Monvel - Sjostrand, Zelditch, Catlin, …)

$\square$

Laughlin states on Σ_g

$$\Psi(z_1, \dots, z_N) = \prod_{n<m} (z_n - z_m)^\beta \, e^{-\frac{\beta}{4} \sum_n |z_n|^2}$$

$\beta \in \mathbb{Z}_+$

<u>Def</u> Consider (Σ, g, J) and holomorphic line bundle (L, h) of degree N_φ. Consider $\Sigma^N = \Sigma \times \dots \times \Sigma$

$N = \frac{1}{\beta} N_\varphi + 1 - g$ (assume $N_\varphi | \beta$)

* $\pi_n \Psi \in H^0(\Sigma, L)$ (projection on n-th factor in $\Sigma \times \dots \times \Sigma$)

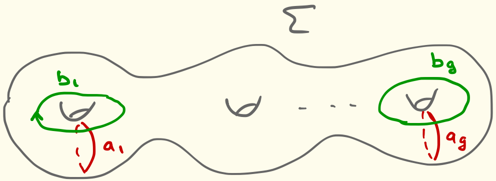
* $\pi_{nm} \Psi = (z_n - z_m)^\beta$, near diagonal $(z_n \sim z_m)$

Prop (SK 2017) Basis of β^g Laughlin states (Wen-Niu 91 conjecture)

($g=1$: Haldane-Rezayi 85)

$r \in (1, \dots, \beta)^g$

$$\boxed{\Psi_r = \Theta\begin{bmatrix} r/\beta \\ 0 \end{bmatrix}\left(\beta \sum_{n=1}^{N} z_n - \beta\Delta_0 - \beta\,\mathrm{div}\,L, \beta\tau\right) \cdot \prod_{n<m}^{N} E(z_n, z_m)^{\beta} \cdot \prod_{n=1}^{N} \sigma(z_n)^{\frac{1}{g}\deg L - \beta}}$$



$\omega_j \in H^1(\Sigma, \mathbb{Z})$
$j = 1, \dots, g$

$\tau_{ij} = \int_{b_i} \omega_j$

Abel map:

$I : \Sigma \to \mathbb{C}^g / \Lambda$

$\Lambda = \{ m + m'\tau,\ m, m' \in \mathbb{Z}^g \}$

Prime form (analog of $z-y$ on Σ)

$$E(z, y) \simeq \frac{z-y}{\sqrt{dz}\sqrt{dy}} \quad \text{as } z \sim y$$

Divisor of L, $\mathrm{div}\,L :=$ sum of zeroes of $s_n(z)$ (w/ multiplicities)

Geometric adiabatic transport

QHE wave functions are typically degenerate (β^g Laughlin states on genus-g surface) and depend on parameter spaces $\mathcal{M}$ (e.g. complex structure moduli of Σ).

Thus we have a Hilbert bundle $H \to \mathcal{M}$

Conjecture

N.Read 2008

This bundle is projectively flat, at least asymptotically as $N \to \infty$.

Berry connection

Def A connection on Hilbert bundle $H \to M$ is a linear map $\nabla_\xi : C^\infty(M, H) \to \Omega^1(C^\infty(M,H))$ for $\xi \in \mathrm{Vect}\, M$, s.t. it respects L^2 structure $\langle\, , \rangle$ on H

$$\xi \langle \psi, \psi' \rangle = \langle \nabla_\xi \psi, \psi' \rangle + \langle \psi, \nabla_\xi \psi' \rangle$$

Berry connection is such ∇_ξ that $\langle \psi, \nabla_\xi \psi' \rangle = 0$ $\forall \xi \in \mathrm{Vect}\, M$.

Then assuming some trivialisation we can write

$$\nabla_\xi \psi = d_\xi \psi + A_\xi \psi \implies \langle \psi_r, A_\xi \psi_{r'} \rangle_{L^2} = - \langle \psi_r, d_\xi \psi_{r'} \rangle_{L^2}$$

Projective flatness conjecture

Similar question was asked by Axelrod- della Pietra- Witten '91 and Hitchin '91 for the bundles $H^0(M, L^k) \to \mathcal{T}$ where M is moduli space of flat $SU(N)$ connections on Σ and $\mathcal{T}$ is Teichmüller space.

They construct Hitchin - KZ (Knizhnik- Zamolodchikov) connection $\nabla^{KZ} s$, $s \in H^0(M, L^k)$ preserving fibers, which is projectively flat

By contrast, for bundles of QH states the L^2- inner product is on Σ^N (<u>not</u> on M), which their connection does not preserve.

Transport on moduli space of complex structures $\mathcal{M}_g$

Deformations of complex structure on Σ

$$g_{z\bar z}|dz|^2 \longrightarrow g_{z\bar z}|dz+\mu\, d\bar z|^2$$, where μ is $(1,-1)$ - differential

$$\mu = g_{z\bar z}^{-1} \sum_{k=1}^{3g-3} \eta_k\, \delta y_k$$, where $\{\eta_k\}$ is a basis of holom. quadratic diffs.

* <u>Integer</u> QH state ("determinant line bundle" over $\mathcal{M}_g$)

Berry curvature $\quad R = i\,\partial_y \bar\partial_y \log Z$

$$= i\,\partial_y\bar\partial_y \log \frac{Z}{\det' \bar\partial_L^+ \bar\partial_L} + \underbrace{i\,\partial_y\bar\partial_y \log \det' \bar\partial_L^+\bar\partial_L}_{\to\, 0 \text{ as } \deg L \to \infty} =$$

$$= \left(\frac{1}{4} N_\phi + \frac{1}{12}\chi(\Sigma)\right)\Omega_{wp}$$

Weil-Peterson (1,1) form

$$\Omega_{wp} = \int_\Sigma |\mu|^2 R \sqrt{g}\, d^2 z$$

SK-Wiegmann 2015

For Laughlin states (rank - β^g vector bundle over $\mathcal{M}_g$)

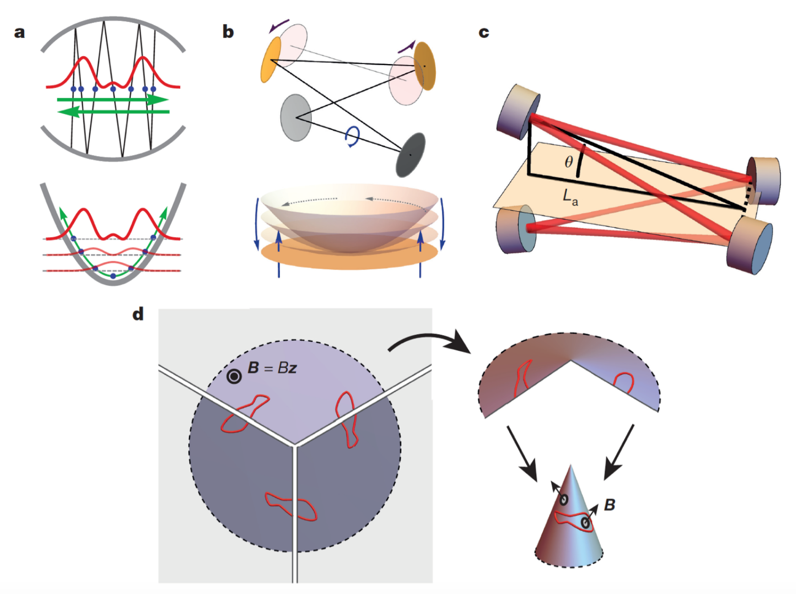
$$R_{rr'} = \left(\frac{1}{4} N_\phi - \frac{c_H}{24} \chi(\Sigma) \right) \Omega_{WP} \cdot \delta_{rr'}$$

$$\boxed{c_H = 1 - 3 \left(\sqrt{\beta} - \frac{2s}{\sqrt{\beta}} \right)^2}$$

Synthetic Landau levels for photons

Schine et. al., Nature 2016

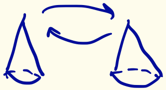
Experimental test of QHE in curved space



Anyon (quasi-hole) braiding

$$\Psi \to e^{i\frac{\pi}{\beta}}\,\Psi$$

Cone ("genon") braiding



$$\Psi \to e^{i\pi\frac{c_H}{12}\alpha_1\alpha_2}\,\Psi$$

The End