Quantum Hall effect and Kähler metrics

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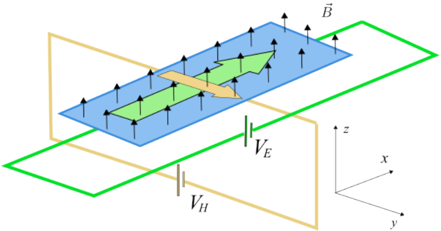
Plan

- Motivation
- Landau levels on curved backgrounds
- Generating functional in integer QHE
- Bergman kernel
- Random geometry, induced by QHE

Quantum Hall effect

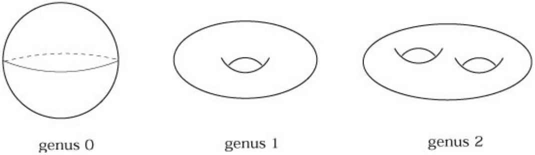
Observed in two-dimensional electron systems subjected to low temperatures and strong magnetic fields. Hall conductance is quantized $\sigma_H = I/V_H = \nu$, where ν is integer for integer QHE, or a fraction for fractional QHE. Involves many ($N \sim 10^6$) electrons on lowest Landau level, described by a collective (Laughlin) state.

$$\Psi(\{z_i\}) = \prod_{i<j}^{N} (z_i - z_j)^\beta e^{-\frac{B}{2}\sum_i |z_i|^2}, \quad \beta = 1/\nu \in \mathbb{Z}_+$$



Motivation

Quantum Hall effect happens in a planar sample. Here we will ask, what happens when QHE is considered on a curved geometry.



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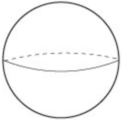
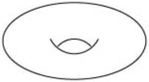


genus 0



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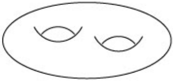

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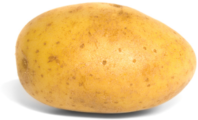
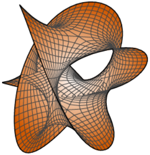




Also, QHE can be defined on Kähler manifolds

Motivation

In physics one can obtain important information about the system by putting it on a manifold with a Riemannian metric

$$ds^2 = g(z, \bar{z}) dz d\bar{z}$$

Prototypical example (Polyakov, 1981): CFT partition function on a compact Riemann surface (M, g_0) has the following behavior under the transformation of the reference metric g_0 to a new metric $g = e^{2\sigma} g_0$

$$\log \frac{Z^{CFT}(g)}{Z^{CFT}(g_0)} = \frac{c}{12\pi} S_L(g_0, \sigma),$$

where $c \in \mathbb{R}$ is the central charge of the CFT, and the Liouville action is

$$S_L(g_0, \sigma) = \int_M (\partial\sigma\bar{\partial}\sigma + R_0\sigma) d^2 z,$$

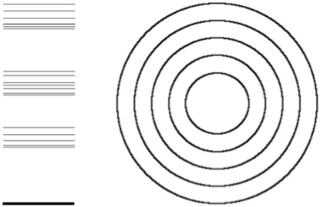
and $R_0 = -\frac{1}{\sqrt{g_0}} \partial\bar{\partial} \log \sqrt{g_0}$ is the scalar curvature of g_0.

Landau levels

Charged particle on the plane in constant magnetic field $B = \partial\bar{A} - \bar{\partial}A$. Hamiltonian

$$\hat{H} = \frac{1}{2m}\frac{1}{\sqrt{g}}\left(|i\hbar\partial + eA|^2 + eB\right)$$

Infinite tower of energy levels, each level is highly degenerate. Structure:



Lowest level $n = 0$

Landau levels

The lowest energy level is special, since the Hamilatonian factorizes and wave functions satisfy first order equation

$$\left[\frac{\partial}{\partial \bar{z}} + \bar{A}\right] \psi_k(z) = 0$$

For example on the two-plane with constant magnetic field $\bar{A} = Bz$, in complex coordinates $z = re^{i\phi}$, the wave functions are

$$\psi_k(z, \bar{z}) = z^k e^{-B|z|^2}, \quad k = 0, 1 \ldots \infty (\text{or total flux} \int B),$$

Lowest Landau level on a curved manifold

What is the analog of this picture (rich LLL) in a more general setup, e.g. for compact manifolds, inhomogeneous magnetic fields, any space dimension?

Lowest Landau level on a curved manifold

What is the analog of this picture (rich LLL) in a more general setup, e.g. for compact manifolds, inhomogeneous magnetic fields, any space dimension? Conditions:

- Manifold admits complex coordinates z^a, $\bar{z}^{\bar{a}}$ $(a, \bar{a} = 1, ..., n)$
- ... and Kähler metric $g_{a\bar{a}} = \partial_a \bar{\partial}_{\bar{a}} K$
- Magnetic field is "holomorphic", i.e. only $F_{a\bar{a}}$ components are non-zero.

Then Hamiltonian (magnetic Schrödinger operator) factorizes:

$$H = g^{a\bar{a}} D_a \bar{D}_{\bar{a}} + g^{a\bar{a}} F_{a\bar{a}}$$

provided $g^{a\bar{a}} F_{a\bar{a}} = \textit{constant}$, which is equivalent to Maxwell equation $(\nabla F = 0)$.

Lowest Landau level on a Riemann surface

Constant magnetic field: $B_0 = kg_0$. Shrödinger equation for the lowest energy level reduces to

$$(\bar{\partial} + A_{\bar{z}})\psi = 0, \quad \text{where } A_{\bar{z}} = -k\bar{\partial}\log h_0$$

with many solutions $\psi_i^0(z,\bar{z}) = s_i(z)h_0^k(z,\bar{z})$. Mathematically, magnetic field is described by the holomorphic line bundle $L^k \to M$, $s_i(z)$ is the basis of holomorphic sections ($i = 1,..,N_k$) and constant magnetic field $F = kg_0$ means choice of "polarization", implying positivity: $F > 0$ since the metric shall be positive definite everywhere on M. Examples:

- S^2, $s_i(z) = z^{i-1}$, $i = 1,..,k+1$.
- T^2, $s_i(z) = \theta_{\frac{i}{k},0}(kz, k\tau)$, $i = 1,..,k$
- on a surface of genus h there are $N_k = k - h + 1$ sections.

Kähler metrics

In two dimensions we are used to parameterizing the metric in conformal class.

$$g|dz|^2 = e^{2\sigma(z,\bar{z})} g_0 |dz|^2$$

Landau levels prefer Kähler parameterization (Kähler class)

$$g|dz|^2 = (g_0 + \partial\bar{\partial}\phi)|dz|^2$$

where the scalar function $\phi(z,\bar{z})$ is called Kähler potential. QHE droplet is incompressible – deformations have the same area, and all metrics in the Kähler class have the same area.
If $\psi_i(z,\bar{z}) = s_i(z) h_0^k(z,\bar{z})$ are LLL wave functions for the magnetic field $B_0 = kg_0$, then wave functions for the magnetic field $B = k(g_0 + \partial\bar{\partial}\phi)$ are

$$\psi_i(z,\bar{z}) = s_i(z) h_0^k(z,\bar{z}) e^{-k\phi(z,\bar{z})}$$

Laughlin wave function on Riemann surface

The Laughlin wave function of N_k non-interacting fermions (integer QHE) is given by Slater determinant

$$\Psi(z_1, \ldots, z_{N_k}) = \frac{1}{\sqrt{N_k!}} \det \psi_i(z_j)$$

$$= \frac{1}{\sqrt{N_k!}} [\det s_i(z_j)] \cdot \prod_{j=1}^{N_k} h_0^k(z_j) \cdot e^{-k \sum_j \phi(z_j)}$$

where I plugged LLL wavefunctions on curved metric

$$\psi_i(z, \bar{z}) = s_i(z) h_0^k(z, \bar{z}) e^{-k\phi(z, \bar{z})}$$

The Laughlin wave function for the fractional QHE is

$$\Psi_\beta(z_1, \ldots, z_{N_k}) = \frac{1}{\sqrt{N_k!}} \left(\det \psi_i(z_j)\right)^\beta$$

Partition function for integer QHE

Partition function (generating functional)

$$Z^{QHE}(g_0, g) = \int_{M^{\otimes N_k}} |\Psi(z_1, \ldots, z_{N_k})|^2 \prod_{i=1}^{N_k} \sqrt{g(z_i)} d^2 z_i =$$

$$= \frac{1}{N_k!} \int_{M^{\otimes N_k}} |\det s_i(z_j)|^2 e^{-k \sum_i \phi(z_i)} \prod_{i=1}^{N_k} h_0^k(z_i) \sqrt{g(z_i)} d^2 z_i .$$

Varying $Z^{QHE}(g_0, g)$ with respect to $\delta\phi(z)$ allows to define density correlation functions $\rho(z) = \frac{1}{k} \sum_i \delta(z - z_i)$:

$$\langle \rho(x) \rho(y) \ldots \rho(z) \rangle .$$

Partition function for integer QHE

On the two-sphere or on the plane, Wiegmann-Zabrodin (2006):

$$Z^{FQHE,S^2}(W) = \frac{1}{N!} \int_{\mathbb{C}^{\otimes N}} |\Delta(z)|^{2\beta} e^{-N\sum_i W(z_i)} \prod_{i=1}^{N} d^2 z_i,$$

For $\beta = 1$ this is partition function of the ensemble of random normal matrices. For any beta this is called beta-ensemble.
Wiegmann-Zabrodin derived first three terms large k expansion, using loop equation, for arbitrary W

$$\log Z^{FQHE,S^2}(W) = N^2 S_0(W) + NS_1(W) + S_0(W) + \mathcal{O}(1/N)$$

Derivation

The integer QHE partition function enjoys determinantal representation

$$Z^{QHE}(g_0, g) = \det_{ij} \int_M \bar{s}_i s_j h_0^k e^{-k\phi} \sqrt{g} d^2 z,$$

studied by Donaldson (2004). Variation of the free energy wrt $\delta\phi(z)$

$$\delta \log Z^{QHE}(g_0, g) = \int_M (-k\rho_k + \Delta\rho_k)\delta\phi \sqrt{g} d^2 z.$$

is controlled by the density of states function

$$\rho_k(z) = \sum_{i=1}^{N_k} \bar{\psi}_i(z)\psi_i(z) = \sum_{i=1}^{N_k} |s_i(z)|^2 h_0^k e^{-k\phi}.$$

In math this is known as the Bergman kernel on the diagonal.

Bergman kernel

Density function (Bergman kernel) has a local expansion for large k

$$\rho_k(z) = \sum_{i=1}^{N_k} \bar{\psi}_i(z)\psi_i(z) = \sum_{i=1}^{N_k} |s_i(z)|^2 h_0^k e^{-k\phi}$$

$$= k^n \left(1 + \frac{1}{2k} R(z) + \frac{1}{3k^2}\Delta R + ...\right)$$

known as Tian-Yau-Zelditch expansion (Tian' 90, Zelditch' 98). Total

number of states is $N_k = \int_M \rho_k(z) g d^2 z$, in complex dimension $n = 1$ we get

$$N_k = k + \frac{2-2h}{2}$$

(Riemann-Roch)

Bergman kernel from path integral

In physics the Bergman kernel expansion can be derived from the path integral for a particle in the magnetic field $F = dA = kg$ (Douglas, S.K., 2008), taking large time limit of

$$\rho_k(z) = \lim_{T\to\infty} \langle z|e^{-TH}|z\rangle =$$

$$= \lim_{T\to\infty} \int_{z(0)=z}^{z(T)=z} e^{-\frac{1}{\hbar}\int_0^T dt\,(g_{a\bar{a}}\dot{z}^a\dot{\bar{z}}^{\bar{a}}+A_a\dot{z}^a)} \prod_{0<t<T} \sqrt{g(z(t))}\,\mathcal{D}z(t)\mathcal{D}\bar{z}(t) =$$

$$= k^n\left[1 + \frac{\hbar}{2k}R + \frac{\hbar^2}{k^2}\left(\frac{1}{3}\Delta R + \frac{1}{24}|\mathrm{Riem}|^2 - \frac{1}{6}|\mathrm{Ric}|^2 + \frac{1}{8}R^2\right) + \ldots\right].$$

Interlude: Balanced confuguration

The confuguration of particles on LLL with constant density function, “’knows” about constant curvature:

$$\rho_k(z) = k^n \left(1 + \frac{1}{2k} R(z) + \dots\right) = \text{constant} \approx k^n + \frac{c}{2} k^{n-1} + ...$$

Therefore

$$R(z) \approx c + \mathcal{O}(1/k)$$

Such configuration is called “balanced”. In a certain sense, it is the configuration with maximal entropy. When $k \to \infty$, balanced metric becomes exact constant scalar curvature metric (that’s why this setup is central in complex geometry).

Derivation, cont'd

Using the expansion of ρ_k we can integrate out the free energy order by order in k (in principle to all orders)

$$\delta \log Z^{QHE}(g_0, g) = \int_M (-k\rho_k + \Delta\rho_k)\delta\phi\, g d^2 z$$

$$\log Z^{QHE}(g_0, g) = \frac{k^2}{2\pi} S_{AY}(g_0, \phi) + \frac{k}{4\pi} S_M(g_0, \phi) + \frac{1}{6\pi} S_L(g_0, \phi) + \mathcal{O}(1/k)$$

where the following functionals appear

$$S_{AY}(g_0, \phi) = \int_M (\frac{1}{2}\phi\partial\bar{\partial}\phi + \phi g_0) d^2 z \qquad \text{Aubin – Yau}$$

$$S_M(g_0, \phi) = \int_M (-\phi R_0 + g \log \frac{g}{g_0}) d^2 z \qquad \text{Mabuchi}$$

$$S_L(g_0, \sigma) = \int_M (\partial\sigma\bar{\partial}\sigma + R_0\sigma) d^2 z, \qquad \text{Liouville}$$

Derivation, cont'd

All these functionals satisfy one-cocycle condition on the space of metrics: $S(g_0, g_2) = S(g_0, g_1) + S(g_1, g_2)$. Starting from order $1/k$ this becomes easy, since $S(g_0, g) = S(g) - S(g_0)$ (exact one-cocycle):

$$\log Z^{QHE}(g_0, g) = -\frac{k^2}{2\pi} S_{AY}(g_0, \phi) + \frac{k}{4\pi} S_M(g_0, \phi) + \frac{1}{6\pi} S_L(g_0, \phi)$$
$$-\frac{5}{96\pi k}\left(\int_M R^2 g d^2 z - \int_M R_0^2 g_0 d^2 z\right) + \ldots$$

Derivation, cont'd

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Conjecture: all remainder terms (starting order $1/k$ and lower) are exact one-cocycles.

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$$+\frac{1}{2880\pi k^2}\left(\int_M (29R^3 - 66 R\Delta R)\, g d^2 z - \right.$$
$$\left. - \int_M (29 R_0^3 - 66 R_0 \Delta_0 R_0)\, g_0 d^2 z\right) + \mathcal{O}(1/k^3)$$

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$$
\begin{aligned}
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\end{aligned}
$$

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✓

Physical interpretation

There is a long-term effort to understand the effective theory of QHE systems. One interpretation is that electrons in QHE form an incompressible "QHE liquid" (Girvin-MacDonald-Platzman 1986, see also recent papers by Son, Hoyos, Wiegmann, Abanov etc)

$$\mathcal{F}^{FQHE} = -\beta \frac{k^2}{2\pi} S_{AY}(g_0, \phi) + \beta \frac{k}{4\pi} S_M(g_0, \phi) + \left(\frac{1}{3} + \frac{\beta - 1}{2}\right) \frac{1}{2\pi} S_L(g_0, \phi) + \ldots$$

Can-Laskin-Wiegmann (2014). The first coefficient is inverse Hall conductance. The response to the curvature is interpreted as an anomalous Hall viscosity

$$\eta = \frac{\delta \rho_k}{\delta R} \quad \text{where} \quad (\delta S_M = R \qquad \delta S_L = \Delta R)$$

The Liouville term is related to heat transport (Cardy)

(more details in P. Wiegmann's talk)

Beta-deformed Bergman kernel

Begman kernel is ubiquitous in Kähler geometry. Analogs of Z^{QHE} have been studied by Donaldson (2004) and R. Berman (2008-) on Kähler M - derived leading order term in the large k expansion of Z^{QHE}. Darboux-Cristoffel formula for the Bergman kernel

$$\rho_k(z_1) = \int_{M^{N_k-1}} |\det s_i(z_j)|^2 e^{-k\sum_i \phi(z_i)} \prod_{i=2}^{N_k} h_0^k(z_i) g^n(z_i) d^{2n} z_i$$

Define "β-deformed" Bergman kernel ⚠

$$\rho_k^{\beta}(z_1) = \int_{M^{N_k-1}} |\det s_i(z_j)|^{2\beta} e^{-\beta k\sum_i \phi(z_i)} \prod_{i=2}^{N_k} h_0^{\beta k}(z_i) g^n(z_i) d^{2n} z_i$$

This is a new object in Kähler geometry.

FQHE and off-diagonal Bergman kernel

Statement: β-deformed Bergman kernel has large k local asymptotic expansion (no rigorous proof available at the moment).
Conjecture: there exists asymptotic expansion

$$\rho_k^\beta = \beta \sum_{s=0}^{\infty} (\beta k)^{n-s} P_s(\beta)(R^s + ...),$$

where $P_s(\beta)$ is a polynomial in β of degree s, and R^s schematically denotes curvature invariants of degree s.
Another representation for the FQHE partition function $\mathcal{Z}^\beta(g)$

$$\mathcal{Z}^\beta(g) = \frac{1}{N_k!} \int_{M^{N_k}} (\det B_k(z_i, z_j))^{2\beta} \prod_{i=1}^{N_k} g^n(z_i) d^{2n} z_i$$

Where $B_k(z_i, z_j) = \sum_i \bar{\psi}(z_i)\psi(z_j)$ - off-diagonal Bergman kernel.

$$B_k(z_i, z_j) = e^{-k|z_i - z_j|^2}(k^n + ...)$$

QHE and random geometry

When Polyakov derived his formula $Z^{CFT}(g) = e^{\frac{c}{12\pi} S_L(g_0,\sigma)} Z^{CFT}(g_0)$, he immediately realized that it can be used to define the probability measure on two dimensional metrics

$$d\mu_g^L = e^{\frac{c}{12\pi} S_L(g_0,\sigma)} \mathcal{D}g,$$

thus defining "random geometry", induced by free fields. This gave rise to a beautiful subject of Liouville theory and non-critical string theory. Following the same logic, the QHE effect induces its own "random geometry":

$$d\mu_g^{QHE} = Z^{QHE}(g, g_0)\mathcal{D}g = e^{\frac{\kappa}{12\pi} S_M(g_0,\sigma)} \mathcal{D}g,$$

$$d\mu_g^L = \frac{Z^{FQHE(g,g_0)}}{(Z^{QHE}(g, g_0))^\beta} \mathcal{D}g = e^{\frac{\beta-1}{12\pi} S_L(g_0,\sigma)} \mathcal{D}g, \quad \beta = 3, 5, 7...$$

with the appropriate choice of the measure. This type of measures can be studied in the framework of "random Bergman metric", for more details see: F. Ferrari, S.K., S. Zelditch (2012-2014)

Thank you