Talk

SISSA, Trieste

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# Geometry and large N asymptotics in quantum Hall states

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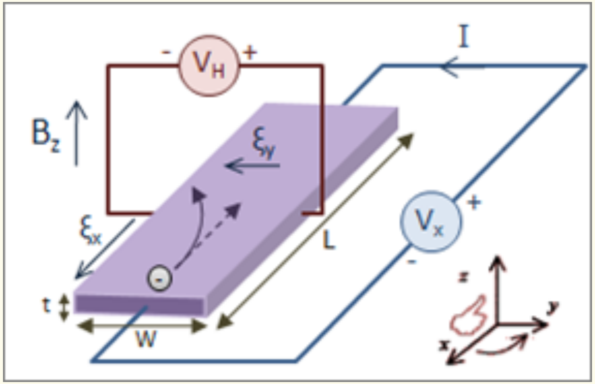
# Classical Hall effect

Appearance of electric potential difference ( $V_H$ ) perpendicular to the direction of the current ( $I$ ) in conductors placed in magnetic field ( $B_z$ )

$$V_H = \frac{I \cdot B_z}{n \cdot e}$$

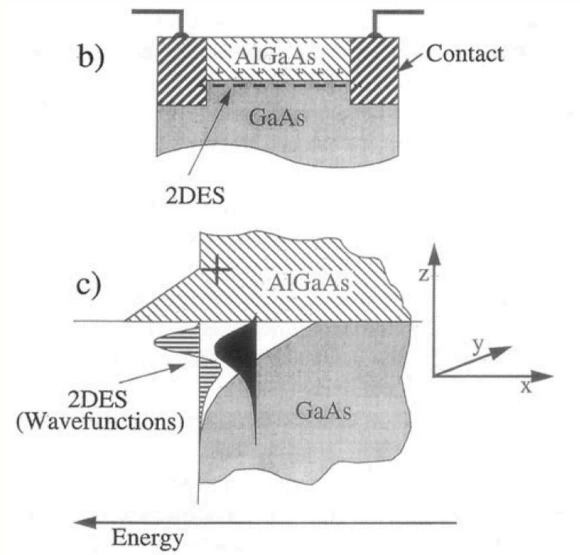



where $n$ is the density of electrons of charge $e$.

Hall resistance $R_H = \frac{B_z}{n e}$

grows linearly with the magnetic field.

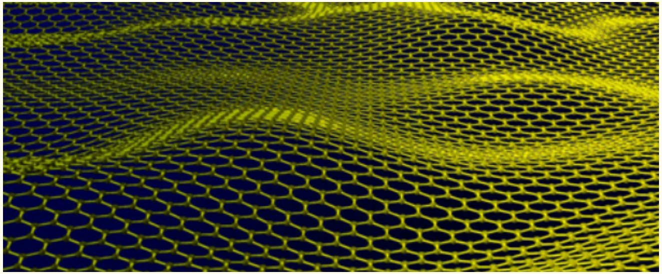
# Quantum Hall effect (QHE)




Gallium-Arsenide

Observed in 2d electron systems, subjected to low temperatures and strong magnetic fields.

Graphene

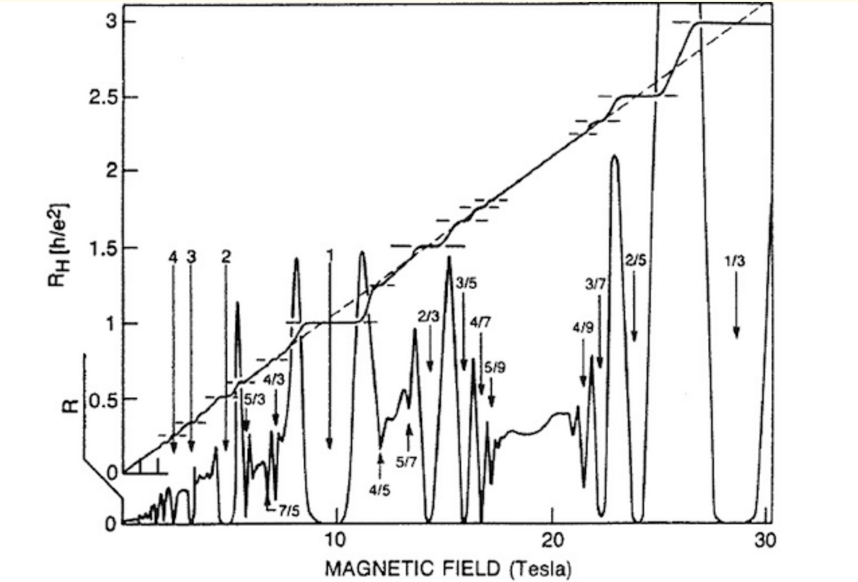


Hall resistance $R_H$ undergoes a series of plateaux, where it is quantized (in units of $e^2/h$).

$$\sigma_H := \frac{1}{R_H} = \begin{cases} n \in \mathbb{Z}_+ & \rightsquigarrow \text{Integer QHE} \\ p/q \in \mathbb{Q} & \rightsquigarrow \text{Fractional QHE} \end{cases}$$

von Klitzing 1981

Tsui, Stormer, Gossard 1982




precision measurement of fine structure constant $\alpha = e^2/hc =$

$= 1/137.035999173(35)$

Integer QHE plateaux are explained by one-particle wave functions (non-interacting)

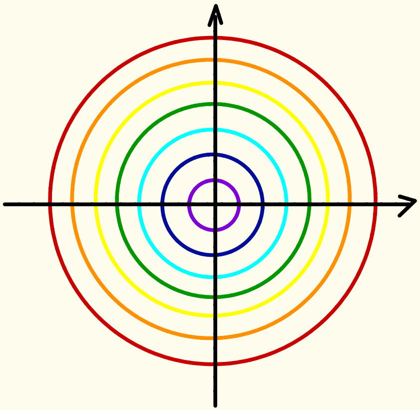
$$\Psi(z_1, \dots z_N) = \det \psi_n(z_m)\Big|_{n,m=1}^{N}$$

"Slater determinant".

$$\psi_n = z^n \cdot e^{-\frac{B}{4}|z|^2}$$

lowest Landau level (LLL)

~ degenerate ground states in strong magnetic field.



# Fractional QHE

Strongly-interacting (via Coulomb forces) system.

Laughlin 1983: Assign a trial wave function ("state") to each plateau.

$$\Psi(z_1, \dots, z_N)$$

* holomorphic
* vanishing conditions

$\Psi = 0$

# Laughlin state

$$\underbrace{\Psi(z_1, \ldots, z_N)}_{\mathbb{C}^N} = c \cdot \prod_{n<m}^{N} (z_n - z_m)^{\beta} \cdot e^{-\frac{B}{4} \sum_{n=1}^{N} |z_n|^2}$$

$$\beta \in \mathbb{Z}_+$$

Hall conductance $\quad \sigma_H = 1/\beta$

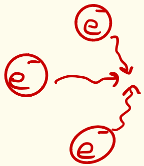
# Another famous QHE state

Moore-Read 1991:

$$\Psi_{MR}(z_1, \dots, z_N) = c \cdot \mathrm{Pf}\left( \frac{1}{z_n - z_m} \right) \cdot \prod_{n<m}^{N} (z_n - z_m) \cdot e^{-\frac{B}{4} \sum_n |z_n|^2}$$

Pfaffian of anti-sym. matrix

$$\sigma_H = 5/2$$




$\Psi = 0$

QHE wave functions define sequences of probability measures on $\mathbb{C}^N$ ( actually, $\mathbb{C}^N/S_N$ )

$$\mu_N := \frac{1}{N!} \left|\Psi(z_1, \dots, z_N)\right|^2 \cdot \prod_{n=1}^{N} d^2 z_n$$

Total mass of $\mu_N$ is the $L^2$- norm of $\Psi$. For Laughlin:

$$Z = \frac{1}{N!} \int_{\mathbb{C}^N} \exp\left[-\frac{B}{2}\sum_n |z_n|^2 + \beta \sum_{n \neq m} \log|z_n - z_m|\right] \prod_{n=1}^{N} d^2 z_n$$

2D Coulomb gas partition function.

More generally,

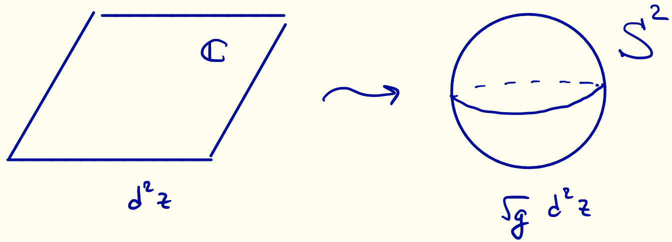
$$Z = \int_{\mathbb{C}^N} \exp\left\{ -N\sum_{n=1}^{N} V(z_n,\bar z_n) + \beta \sum_{n\neq m} \log|z_n - z_m| \right\} \cdot \prod_{n=1}^{N} d^2 z_n$$

grav. "spin" (s=1 in pure Coulomb gas)

$$V = \phi(z,\bar z) - \frac{1-s}{N}\log\sqrt{g}(z,\bar z)$$

"magnetic potential"

volume form $\sqrt{g}\,d^2z$ on $\mathbb{C}$




Coulomb gas
Random normal/complex matrices
Beta ensembles

# State-of-the-art

**Thm** (Leblé-Serfaty 2017)

$$\log Z = -\beta N^2 I_V(\mu_V) + \frac{\beta}{2} N \log N - N\, C(\beta) - N\left(1 - \frac{\beta}{2}\right) \int_{\mathbb{C}} \mu_V \log \mu_V + o(N)$$

where $I_V = -\iint_{\mathbb{C}\times\mathbb{C}} \log|z-w|\, d\mu(z)\, d\mu(w) + \int_{\mathbb{C}} V\, d\mu$

and $\mu_V$ its unique minimizer ("equilibrium measure")

[ in fact, corollary of a stronger large deviations result ]

# $\mathcal{O}(1)$ term

**Prop**

Can, Laskin, Wiegmann 2014
F. Ferrari, SK 2014

$$\log Z = -\beta N^2 I_V(\mu_V) - N\left(s - \frac{\beta}{2}\right)\int_\Sigma \mu_V \log \mu_V$$

$$- \frac{c_H}{12} \int_\Sigma \left( |\partial \log \mu_V|^2 - 2\partial\bar{\partial} \log \mu_V \right) + \text{const} + R_{1/N}$$

Liouville action

remainder terms

$$c_H = 1 - 3\left(\sqrt{\beta} - \frac{2s}{\sqrt{\beta}}\right)^2$$

( $s = 1$ in pure Coulomb gas )

"central charge"

( earlier work:
Zabrodin - Wiegmann '06 )

# Free field representation

$\sigma(z,\bar{z})$ - free field on $(\Sigma, g)$, $B$ is magnetic field

$$S(\sigma) = \frac{1}{2\pi}\int_\Sigma \left(\partial\sigma\bar{\partial}\sigma + \frac{i}{4}\left(\sqrt{\beta} - \frac{2s}{\sqrt{\beta}}\right)\sigma R\sqrt{g} + \frac{i}{\sqrt{\beta}}\sigma B\sqrt{g}\right)d^2z$$

$$|\Psi_L(z_1,\dots,z_N)|^2 \simeq \left\langle e^{i\sqrt{\beta}\sigma(z_1)} \cdots e^{i\sqrt{\beta}\sigma(z_N)}\right\rangle$$

[Moore-Read '91
Fubini '91]

Remainder term:

$$R_{1/N} = \int D_g\sigma\; e^{-S(\sigma) + N\log\int_\Sigma e^{i\sqrt{\beta}\sigma}\sqrt{g}\,d^2z} -$$

$$- \int D_{g_0}\sigma\; e^{-S(\sigma) + N\log\int_\Sigma e^{i\sqrt{\beta}\sigma}\sqrt{g_0}\,d^2z}$$

$$\simeq O\left(\frac{1}{N}\right)$$

# Adiabatic transport on moduli spaces

anomalous (Hall) viscosity

viscosity tensor

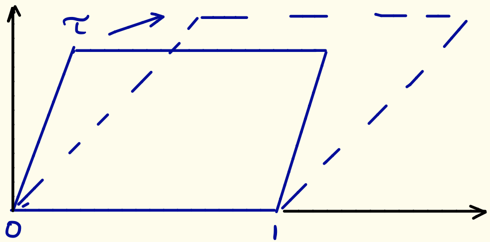
$$\sigma_{\alpha\beta} = - \eta_{\alpha\beta\gamma\delta}\, \dot{u}_{\gamma\delta}$$

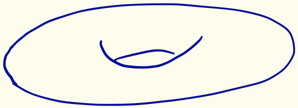
stress

strain-rate

Avron, Seiler, Zograf 1995

QHE states on torus



"Hall" viscosity

$$\eta_H = \frac{1}{4} N_\phi - \frac{c_H}{24}\chi(\Sigma)$$

Tokatly, Vignale; N. Read 2008

SK-Wiegmann 2015

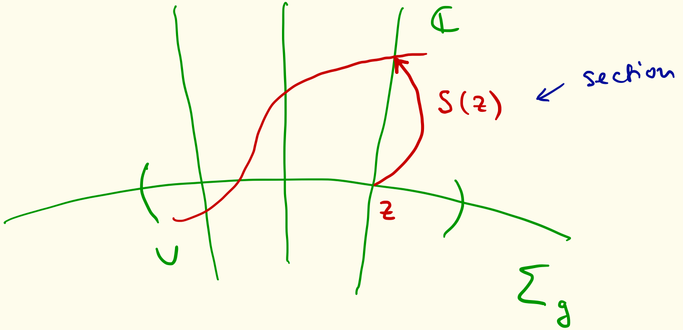
# QH states on Riemann surfaces

Recall, that the one-particle wave functions on $\mathbb{C}$ are $\psi_n = z^n e^{-\frac{B}{4}|z|^2}$.

What is an analog of holomorphic polynomials on a compact $\Sigma_g$? $z^n \rightsquigarrow S_n(z)$, holomorphic sections of a holomorphic line bundle $L$.

$$\boxed{\bar\partial_L S(z) = 0}$$

$$\bar\partial_L : C^\infty(\Sigma, L) \to \Omega^{0,1}(\Sigma, L)$$




+ holomorphic transition functions $t_{UV}$ on $U \cap V$

Consider a compact oriented genus-$g$ Riemann surface $(\Sigma, g, J)$
metric
complex structure

and a positive holomorphic line bundle $L$ on $\Sigma$

of degree $N_\phi = \deg L$.

Hermitian metric $h(z, \bar{z})$, $\| s(z) \|_h^2 \in \mathbb{R}_+$

Its curvature $F = -i \partial \bar{\partial} \log h \in \Omega^{1,1}(\Sigma)$, $F > 0$ (positivity)

$$\frac{1}{2\pi} \int_\Sigma F = N_\phi$$ ( degree, "flux" of magnetic field )

Equivalently, holomorphic sections of $L$ have $N_\phi$ zeroes

on $\Sigma$ (divisor description of $L$) : $\operatorname{div} L = \sum_{\alpha=1}^{N_\phi} q_\alpha$

Dimension of the vector space $H^0(\Sigma, L)$ of holomorphic sections is

$$\dim H^0(\Sigma, L) = N_\phi + 1 - g \qquad (\text{Riemann-Roch thm.})$$

Basis $\{ s_n(z) \}$, $n = 1, \dots, \dim H^0$. Examples:

$\boxed{S^2}$

$$s_n(z) = z^n, \quad n = 0, \dots, N_\phi \qquad h = \frac{1}{(1+|z|^2)^{N_\phi}}$$

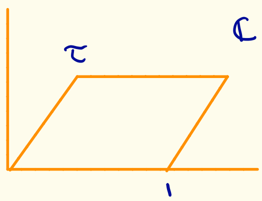
$$\int_{S^2} \| s_n(z) \|_h^2 \sqrt{g}\, d^2 z < \infty$$

$\boxed{T^2}$

$$s_n(z) = \Theta \begin{bmatrix} n/N_\phi \\ 0 \end{bmatrix} (N_\phi z, N_\phi \tau)$$

$n = 1, \dots, N_\phi$

"level-$N_\phi$" theta functions



# Integer QH state ("Slater determinant")

Consider $\Sigma^N = \underbrace{\Sigma \times \dots \times \Sigma}_{N}$ , $N = \dim H^0(\Sigma, L)$

$$\Psi(z_1, \dots, z_N) = \det S_n(z_m)\Big|_{n,m=1}^{N}$$

Its $L^2$-norm is given by

$$Z = \int_{\Sigma^N} \| \det S_n(z_m) \|_h^2 \cdot \prod_{n=1}^{N} \sqrt{g}\, d^2 z_n$$

Consider an arbitrary hermitian metric and arbitrary Riemannian metric

$$h = h_0 e^{-N_\phi \psi(z,\bar z)}, \quad F(h) > 0$$

$$g = g_0 + \partial\bar\partial \phi(z,\bar z) > 0$$

**Prop.** SK 2013, SK-Ma-Marinescu-Wiegmann 2015

Asymptotic large N f-la for $\log Z$

$$\log Z = -N_\phi^2 S_2(\psi) + \frac{1}{2} N_\phi S_1(\phi,\psi) + \frac{1}{6} S_0(\phi) + O(1/N_\phi)$$

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$$S_2(\psi) = \frac{1}{2\pi}\int_\Sigma |\partial\psi|^2 + \psi g_0$$

$$S_0(\phi) = \frac{1}{2\pi}\int\left(\left|\partial \log\frac{g}{g_0}\right|^2 + R_0 \log\frac{g}{g_0}\right)$$

← Liouville action

$$S_1(\phi,\psi) = \frac{1}{2\pi}\int_\Sigma\left(-\frac{1}{2}\psi R_0 + F\log\frac{g}{g_0}\right)$$

Proof    Variational f-la

$$\delta \log Z = -\frac{1}{2\pi}\int_\Sigma \left( N_\phi B_{N_\phi}\cdot\delta\phi - \frac{1}{2}\Delta B_{N_\phi}\cdot\delta\phi \right)\sqrt{g}\, d^2z$$

where Bergman kernel for the $H^0(\Sigma, L)$

$$B_{N_\phi}(z,\bar{z}) = \sum_{n=1}^{N} \| s_n(z)\|_h^2 \simeq N_\phi + \frac{1}{2}R(g) + \mathcal{O}(1/N_\phi)$$

Complete asymptotic expansion at large $N_\phi$

( Boutet de Monvel - Sjostrand, Zelditch, Catlin, ...

Douglas - SK'09 : path integral derivation ) $\square$

# Laughlin states on $\Sigma_g$

$$\Psi(z_1, \dots, z_N) = \prod_{n<m} (z_n - z_m)^\beta \, e^{-\frac{\beta}{4} \sum_n |z_n|^2}$$

$$\beta \in \mathbb{Z}_+$$

**Def** Consider $(\Sigma, g, J)$ and holomorphic line bundle $(L, h)$ of degree $N_\varphi$. Consider $\Sigma^N = \Sigma \times \dots \times \Sigma$

$$N = \frac{1}{\beta} N_\varphi + 1 - g \qquad (\text{assume } N_\varphi | \beta)$$

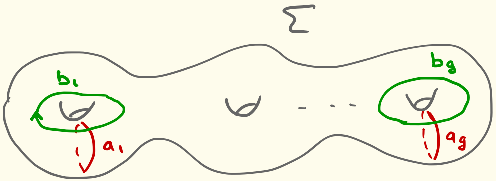
* $\pi_n \Psi \in H^0(\Sigma, L)$ (restriction to $n$-th factor in $\Sigma \times \dots \times \Sigma$)
* $\pi_{nm} \Psi \simeq (z_n - z_m)^\beta$, near diagonal $(z_n \sim z_m)$
* Completely sym / asym for $\beta \in$ even/odd

**Prop** (SK 2017) Basis of $\beta^g$ Laughlin states (Wen-Niu 91 conjecture)

($g=1$: Haldane-Rezayi 85)

$r \in (1,\dots,\beta)^g$

$$\boxed{\Psi_r = \Theta\begin{bmatrix} r/\beta \\ 0 \end{bmatrix}\left(\beta \sum_{n=1}^{N} z_n - \beta\Delta_0 - \beta \operatorname{div} L,\ \beta\tau\right) \cdot \prod_{n<m}^{N} E(z_n, z_m)^{\beta} \cdot \prod_{n=1}^{N} \sigma(z_n)^{\frac{1}{s}\deg L - \beta}}$$




$\omega_j \in H^1(\Sigma, \mathbb{Z})$ $\quad j = 1,\dots,g$

$\tau_{ij} = \int_{b_i} \omega_j$

Abel map:

$I : \Sigma \to \mathbb{C}^g/\Lambda$

$\Lambda = \{m + m'\tau,\ m, m' \in \mathbb{Z}^g\}$

Prime form (analog of $z-y$ on $\Sigma$)

$$E(z,y) \simeq \frac{z-y}{\sqrt{dz}\sqrt{dy}} \quad \text{as } z \sim y$$

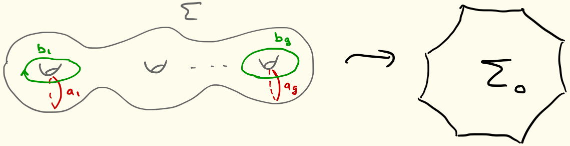
$\operatorname{div} L :=$ sum of zeroes of $S_n(z)$ (w/ multiplicities)

# Free field representation of Laughlin states

$$S(\sigma) = \frac{1}{2\pi}\int_\Sigma \left(\partial\sigma\bar\partial\sigma + \frac{i}{4}\left(\sqrt\beta - \frac{2s}{\sqrt\beta}\right)\sigma R\sqrt g + \frac{i}{\sqrt\beta}\sigma B\sqrt g\right)d^2z$$

$$\langle e^{i\sqrt\beta\,\sigma(z_1)}\cdots e^{i\sqrt\beta\,\sigma(z_N)}\rangle \simeq \sum_{r=1}^{\beta^g}\sum_{\#}(-1)^{\#}\left|\Psi_r^{\#}(z_1,\dots,z_N)\right|^2$$

$\nwarrow$ spin structures on $\Sigma$, $L\otimes S_\#$




$$\int\sigma B\sqrt g\, d^2z := \int_{\Sigma_0}\sigma B\sqrt g\, d^2z - \int_{\partial\Sigma_0}\sigma\Lambda$$

$$d\Lambda = F - \sum_{\mathrm{div}\,L}\delta(q_\alpha)$$

Gerasimov, Marshakov, Morozov et al 1990

At $\beta=1$ this reduces to higher-genus bosonisation f-la

$$\langle e^{i\sigma(z_1)}\cdots e^{i\sigma(z_N)}\rangle = \frac{\det'\bar\partial_L^+\bar\partial_L}{\det\langle s_n,s_m\rangle_{L^2}}\,\|\det s_n(z_m)\|_h^2$$

Verlinde, Verlinde'87, Alvarez-Gaume, Bost, Moore, Nelson, Vafa'97

# Geometric adiabatic transport

QHE wave functions are typically degenerate ($\beta^g$ Laughlin states on genus-$g$ surface) and depend on parameter spaces $\mathcal{M}$ (e.g. moduli space $\mathcal{M}_{g,n}$)

Thus we have a Hilbert bundle $V_{QH} \to \mathcal{M}$

## Conjecture

N.Read 2008 (for $g>0$)

$V_{QH}$ is projectively flat (at least as $N \to \infty$)

(i.e. Berry curvature is $\mathcal{R} = c \cdot \mathbb{1}$, or equivalently adiabatic transport is independent of the path in $\mathcal{M}$, up to $U(1)$ phase)

# Berry connection

**Def** A connection on Hilbert bundle $H \to M$ is a linear map $\nabla_\xi : C^\infty(M, H) \to \Omega^1(C^\infty(M,H))$ for $\xi \in \mathrm{Vect}\, M$, s.t. it respects $L^2$ structure $\langle\, ,\, \rangle$ on $H$

$$\xi \langle \psi, \psi' \rangle = \langle \nabla_\xi \psi, \psi' \rangle + \langle \psi, \nabla_\xi \psi' \rangle$$

Berry connection is such $\nabla_\xi$ that $\langle \psi, \nabla_\xi \psi' \rangle = 0$ $\forall \xi \in \mathrm{Vect}\, M$.

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Then assuming some trivialisation we can write

$$\nabla_\xi \psi = d_\xi \psi + A_\xi \psi \quad \Rightarrow \quad \langle \psi_r, A_\xi \psi_{r'} \rangle_{L^2} = - \langle \psi_r, d_\xi \psi_{r'} \rangle_{L^2}$$

# Projective flatness conjecture

Similar question was asked by Axelrod- della Pietra- Witten '91 and Hitchin '91 for the bundles $H^0(M, L^k) \to \mathcal{T}$ where $M$ is moduli space of flat $SU(N)$ connections on $\Sigma$ and $\mathcal{T}$ is Teichmüller space.

They construct Hitchin / KZ connection

$\nabla^{KZ} s = 0$, $\quad s \in H^0(M, L^k)$ preserving fibers, which is projectively flat

$$|s(A)|^2 = \int DA \, e^{ik S_{CS}} \, , \qquad \langle s', s \rangle = \int DA \, e^{\int_\Sigma A \bar{A}} \, \bar{s}' s$$

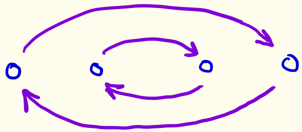
By contrast, for bundles of QH states the $L^2$- inner product is on $\Sigma^N$ (<u>not</u> on $M$).

# Projective flatness of $V_{QH}$

* Quasi-hole (anyonic) states : $\Psi_{an}(\eta \mid z_1, \dots z_N) = \prod_i (z_i - \eta)\, \Psi_{\text{ground st.}}$

For Pfaffian (Moore-Read) states $V_{QH}^{Pf}$ has rank 2 on $\mathcal{M}_{0,4}$.

Asymptotic projective flatness : Bonderson, Gurarie, Nayak 2012



* Laughlin and Pfaffian states are projectively flat on $\mathcal{M}_{1,0}$

* One-particle states are asymptotically p. f. on $\mathcal{M}_{g,0}$

(w/ N. Nemkov)

# Transport on moduli space of complex structures $\mathcal{M}_g$

SK-Ma-Marinescu-Wiegmann

Deformations of complex structure on $\Sigma$

$$g_{z\bar z}\,|dz|^2 \longrightarrow g_{z\bar z}\,|dz+\mu\, d\bar z|^2$$ , where $\mu$ is $(1,-1)$ - differential

$$\mu = g_{z\bar z}^{-1} \sum_{k=1}^{3g-3} \eta_k\, \delta y_k$$ , where $\{\eta_k\}$ is a basis of holom. quadratic diffs.

* Integer QH state ("determinant line bundle" over $\mathcal{M}_g$)

Berry curvature $\quad R = i\,\partial_y \bar\partial_y \log Z$

$$= i\,\partial_y \bar\partial_y \log \underbrace{\frac{Z}{\det' \bar\partial_L^{+} \bar\partial_L}}_{\text{Quillen metric}} + i\,\underbrace{\partial_y \bar\partial_y \log \det' \bar\partial_L^{+} \bar\partial_L}_{\to\, 0 \text{ as } \deg L \to \infty} =$$

$$= \left(\frac{1}{4} N_\phi + \frac{1}{12}\chi(\Sigma)\right) \Omega_{WP} + \dots$$

Weil-Peterson (1,1) form

$$\Omega_{WP} = \int_\Sigma |\mu|^2 R \sqrt{g}\, d^2 z$$

SK-Wiegmann 2015

For Laughlin states ( rank - $\beta^g$ vector bundle over $\mathcal{M}_g$ )

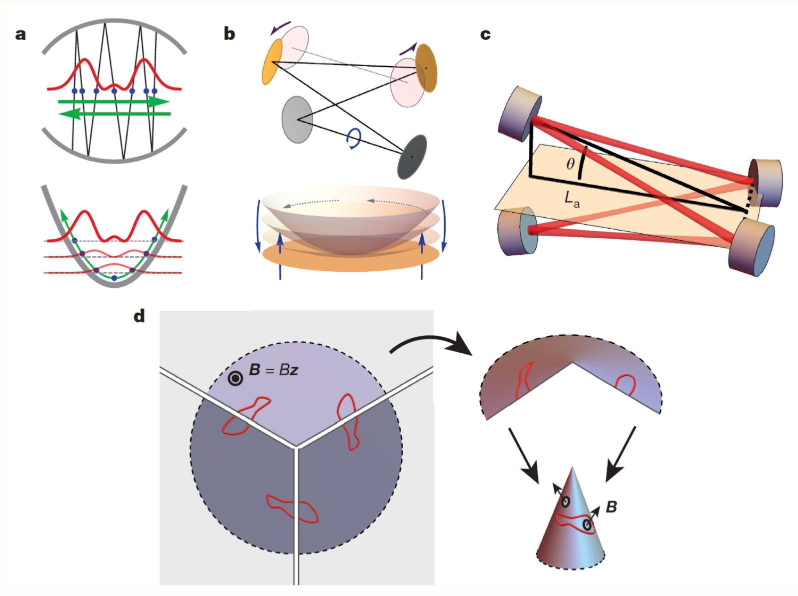
$$R_{rr'} = \left( \frac{1}{4} N_\phi - \frac{c_H}{24} \chi(\Sigma) \right) \Omega_{WP} \cdot \delta_{rr'} + O(1/N_\phi)$$

$$\boxed{c_H = 1 - 3 \left( \sqrt{\beta} - \frac{2s}{\sqrt{\beta}} \right)^2}$$

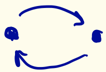
# Synthetic Landau levels for photons

Schine et. al., Nature 2016, 2018

## Experimental test of QHE in curved space

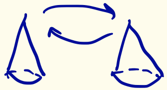



Anyon braiding



$$\Psi \to e^{i\frac{\pi}{\beta}}\Psi$$

Cone ("genon") braiding



$$\Psi \to e^{i\pi \frac{c_H}{12}\alpha_1\alpha_2}\Psi$$

The End