2D gravity and Kähler metrics

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Summary of the talk

- Liouville theory of 2d quantum gravity was originally introduced as the formal path integral over the space of all the metrics in a given conformal class. This path integral can be properly defined using the methods of CFT, and leads to results, consistent with the matrix model description of 2d gravity.
- A good understanding of the geometry of the space of all metrics has been developed recently, for the metrics in the Kähler class (in 2d, Conformal and Kähler classes are almost equivalent), using so-called Bergman metrics (Yau-Tian-Donaldson program).
- We propose the program of using Bergman approximations to define the theory of random metrics.
- We propose certain new interesting measures on the space of Kähler metrics, which are closely related to the Liouville measure.

Plan

- 2d gravity: reminder
- Space of Kähler metrics
- Random Bergman metrics
- Functionals on the Kähler metrics: Liouville, Mabuchi, Aubin-Yau
- Mabuchi action and Liouville theory

Liouville theory

In two dimensions, we have (Polyakov, 1981) a successful theory of quantum gravity, i.e. a theory of random metrics. The Einstein action is trivial ($\int_M R\sqrt{g} = 2\pi\chi(M)$, the Euler number), but the nontrivial metric dependence appears from the coupling to gravity the theory of free conformal matter fields X^i with central charge c

$$S(X^i; g_{ab}) = \int R\sqrt{g}d^2z - \mu\int_M \sqrt{g}d^2z - \int_M X^i\Delta_g X^i\sqrt{g}d^2z.$$

The gaussian integral over X^i can be performed

$$\int e^{\int_M X^i\Delta_g X^i\sqrt{g}d^2z}\mathcal{D}_g X^i = \left(\det{}'\Delta_g\right)^{-c/2}.$$

Two-dimensional metric can be put into conformal form $g_{z\bar{z}} = e^{\sigma}g_{0z\bar{z}}(\tau)$, and the space of conformal classes is parameterized by some moduli space of τ.

Then the dependence of $\det'\Delta_g$ on the conformal factor σ can be determined

$$\delta_\sigma \log\det\Delta_g = -\frac{1}{6\pi}\int_M R(g)\delta\sigma\sqrt{g}d^2z = -\frac{1}{12\pi}\delta_\sigma S_L(\sigma)$$

where $R(g) = -\frac{1}{\sqrt{g}}\partial\bar{\partial}\log\sqrt{g}$ is the scalar curvature of metric g, and the Liouville action is

$$S_L(\sigma) = S_L(g_0, e^\sigma g_0) = \int_M (-\sigma\Delta_0\sigma + 2R_0\sigma)\sqrt{g_0}d^2z$$

Therefore, in 2d quantum gravity we would like to make sense of the path integral

$$\int exp\left[-\int R\sqrt{g}d^2z - \mu\int_M e^\sigma\sqrt{g_0}d^2z - \frac{26-c}{24\pi}S_L(g_0,\sigma)\right]\mathcal{D}\sigma$$

over all the metrics in a fixed conformal class $g_{ab} = e^\sigma g_{0ab}$.

- Under the constant area constraint $A = \int_M g\, d^2z = \int_M g_0 d^2z$ eq. of motion of Liouville theory is the equation for constant scalar curvature (csc) metric $R(g) = \mathrm{const}$.
- The crucial property of the Liouville action S_L satisfies the cocycle identity

$$S_L(g_0, g_2) = S_L(g_0, g_1) + S_L(g_1, g_2)$$

- There is a natural L^2 metric (DeWitt) on the space of conformal metrics $||\delta\sigma||^2 = \int e^\sigma g_0\, d^2z\, (\delta\sigma)^2$, which generates the formal path integral measure $\mathcal{D}\sigma$.
- The gravity-matter theory is background independent (independent of the background metric g_0)

$$Z^{\mathrm{matter}}[g_0] \int \exp\left(-\frac{26-c}{24\pi} S_L(g_0, \sigma)\right) \mathcal{D}\sigma$$

The space of Kähler metrics

Now we would like to change the point of view, and consider what happens when instead of the conformal class we consider Kähler class. Let M be n-dimensional compact Kähler manifold. A Kähler metrics is given by its Kähler form $\omega = \sqrt{-1} g_{a\bar{b}} dz^a \wedge d\bar{z}^{\bar{b}}$. Kähler form is closed $d\omega$, so it defines cohomology class $[\omega]$ in $H^{(1,1)}(M)$. Also, the metric $g_{a\bar{b}}$ must be positive definite Hermitian, therefore any metric in the class of $[\omega]$ can be written as $\omega_\phi = \omega_0 + \partial\bar{\partial}\phi > 0$, for a representative 'background' metric ω_0. All metrics in a given class have same volume $V = \int_M \omega_0^n = \int_M \omega_\phi^n$. The space of Kähler metrics in the given class is equivalent to the space of Kähler potentials

$$\mathcal{K} = \left\{\phi \in C^\infty(M) : \ \ \omega_\phi = \omega_0 + \partial\bar{\partial}\phi > 0\right\}$$

(modulo constant ϕ's). Formally, $\mathcal{K}$ is an infinite dimensional space. It is endowed with a natural 'Donaldson-Semmes-Mabuchi' metric

$$||\delta\phi||^2 = \int_M \omega_\phi^n \, (\delta\phi)^2,$$

in which $\mathcal{K}$ is a negatively curved locally symmetric space.

Kähler metrics in 2d

In complex dimension one, we have the equivalence between conformal and Kähler gauges

$$e^{\sigma}\omega_0 = \omega_0(1+\Delta_0\phi) = \omega_\phi,$$

where fixed area constraint in conformal gauge is assumed. For simplicity, here and in the following ω will stand for g. Note that the Mabuchi metric and DeWitt metric are mutually non-local

$$||\delta\sigma||^2 = \int e^{\sigma}\omega_0\,(\delta\sigma)^2 \qquad ||\delta\phi||^2 = \int_M \omega_\phi\,(\delta\phi)^2$$

In fact, while the first one has constant negative curvature, the second one has constant positive curvature.

Bergman metrics

The infinite-dimensional space of Kähler potentials can be approximated by a finite dimensional space as follows.
Consider M compact Kähler manifold, with a line bundle L and its k-th power L^k. For a hermitian metric h on L the metric on L^k is h^k, its Ricci curvature $R(h) = -\partial\bar{\partial}\log h^k$. If L is positive ($\equiv R(h)$ is positive definite) we can take the metric in the cohomology class $c_1(L)$ to be proportional to curvature

$$\omega = -\frac{1}{k}\partial\bar{\partial}\log h^k$$

Consider a basis $[s_1(z), ..., s_{N_k}(z)]$ in the space of global holomorphic sections $H^0(M, L^k)$ (think holomorphic polynomials of degree k).
Since s_α's are defined up to multiplication by complex number, the choice of basis defines a 'Kodaira embedding' of M into the projective space of sections $z \to s_\alpha(z) \in \mathbb{CP}^{N_k - 1}$.

Now, there is a natural Fubini-Study metric on the projective space $\mathbb{CP}^{N_k}$, which we can pull-back to M to define a Bergman metric on M

$$\omega|_{FS} = \frac{1}{k}\partial\bar{\partial}\log\left(\sum_{\alpha=1}^{N_k}|s_\alpha|^2\right)$$

The basis of s_α is not necessarily normalized. Different choice of basis leads to different embedding $M \to \mathbb{CP}^{N_k-1}$, and therefore different Bergman metric. The space $\mathcal{B}_k$ of Bergman metrics of level k is therefore

$$\mathcal{B}_k = GL(N_k)/U(N_k)$$

Now we show that this finite dimensional space approximates the space of Kähler metrics $\mathcal{K}$ in a fixed class.

Tian-Yau-Zelditch theorem

The Bergman metric can be rewritten as

$$\omega|_{FS} = \frac{1}{k}\partial\bar{\partial}\log\left(\sum_{\alpha=0}^{N_k}|s_\alpha|^2\right) = -\partial\bar{\partial}\log h + \frac{1}{k}\partial\bar{\partial}\log\left(h^k\sum_{\alpha=0}^{N_k}|s_\alpha|^2\right)$$

Inside the logarithm we have a global 'density of states' function, called Bergman kernel. It allows for asymptotic $1/k$-expansion

$$\rho_k[\omega_\phi] = h^k\sum_{\alpha=0}^{N_k}|s_\alpha|^2 = k^n + \frac{k^{n-1}}{2}R + k^{n-2}\left(\frac{1}{3}\Delta R + \frac{1}{24}|\mathrm{Riem}|^2 - \frac{1}{6}|\mathrm{Ric}|^2 + \frac{1}{8}R^2\right) + \ldots$$

Therefore the difference between the metric ω and the Bergman metric is

$$\omega|_{FS} - \omega \sim \mathcal{O}(1/k^2)$$

Therefore we can approximate any metric ω by a Begman metric in the same Kähler class (Tian-Yau-Zelditch theorem): $\mathcal{K} = \lim_{k\to\infty}\mathcal{B}_k$.

Hilb and *FS* map

There are explicit maps between the infinite-dimensional space $\mathcal{K}$ of Kähler potentials, and the symmetric space $\mathcal{B}_k = GL(N_k)/U(N_k)$. Let $\omega_0 = -\partial\bar{\partial}\log h_0$ represent a Kähler class. Consider an orthonormal basis of sections s_α: $\int \bar{s}_\alpha s_\beta h_0^k \omega_0 = \delta_{\alpha\beta}$. Then for any other metric ω_ϕ in the class $[\omega_0]$ define the *Hilb* map as

$$Hilb_{\alpha\beta}(\phi) = \frac{1}{V}\int_M \bar{s}_\alpha s_\beta h_0^k e^{-k\phi}\,\omega_\phi.$$

$Hilb_{\alpha\beta}$ is clearly a positive hermitian matrix, i.e it belongs to $\mathcal{B}_k = GL(N_k)/U(N_k)$. The map back from $GL(N_k)/U(N_k)$ to $\mathcal{K}$ is called FS map. For a given positive hermitian matrix H, we can built a metric

$$\omega_\phi = \mathrm{FS}(H) = \frac{1}{k}\partial\bar{\partial}\log\sum_{\alpha,\beta}\bar{s}_\alpha H_{\alpha\beta} s_\beta$$

The composition $FS \circ Hilb(\phi) = \omega_\phi + \mathcal{O}(1/k)$ for large k.

Random Bergman metrics

We would like to define probability measures on $\mathcal{K}$ by taking $k \to \infty$ limit of probability measures $\mu_k(H)$ on finite-dimensional Bergman spaces $\mathcal{B}_k$. This will provide a regularization of the formal path integral over the space of metrics, in the sense, that

$$\int_{\mathcal{K}} ...\mu(\omega_\phi) := \lim_{k\to\infty} \int_{\mathcal{B}_k} ...\mu_k(H)$$

E.g., using the Hilb map, the invariant metric on $GL(N_k)/U(N_k)$ converges to Mabuchi metric on the space $\mathcal{K}$ of Kähler potentials

$$\mathrm{Tr}(H^{-1}\delta H)^2 = k^3 \int_M (\delta\phi)^2 \omega_\phi + \mathcal{O}(1/k),$$

so are the corresponding volume forms.

Random Bergman metrics, cont'd

Two possibilities:

- one can start with some 'empirical' measure on H and derive the corresponding continuous measure at $k \to \infty$, if any good limit exists.
- one can start with a formal continuous measure of the type

$$\mu(\omega_\phi) = e^{-S(\omega_\phi)}\mathcal{D}\phi$$

  on the space of potentials $\mathcal{K}$, and pull it back to $\mathcal{B}_k$ using Hilb map, or push forward to $\mathcal{B}_k$, using FS map.

What is the regularized measure, corresponding to 2d gravity? Are there any other interesting action functionals on the metrics?

Action functionals in the Kähler gauge

What are the natural functionals on Kähler metrics?
Recall, that in conformal class the simplest functional was the area: $\delta A = \int \delta\sigma \sqrt{g} d^2 z$. Analogously, for the Kähler metrics the simplest functional is the Aubin-Yau energy

$$\delta F_{AY}(\omega_0, \phi) = \int_M \delta\phi \, \omega_\phi, \qquad F_{AY}(\omega_0, \phi) = \int_M \frac{1}{2}\phi\partial\bar{\partial}\phi + \phi\omega_0$$

The second nontrivial functional in conformal class is the Liouville energy, whose variation gave scalar curvature $\delta S_L(\omega_0, \sigma) = \int_M \delta\sigma R(g)\sqrt{g} d^2 z$. The analog in Kähler class is the Mabuchi energy has csc metrics as critical points

$$\delta S_M(\phi) = \int_M (\bar{R} - R(\omega_\phi))\delta\phi \, \omega_\phi,$$

$\bar{R} = \frac{1}{A}\int R\sqrt{g} d^2 z$ is the average scalar curvature.

Mabuchi energy

The variational formula for Mabuchi energy can be integrated to give explicitly

$$S_M(\omega_0, \omega_\phi) = \int_M \left[\frac{\bar{R}}{2} \phi \partial \bar{\partial} \phi + \phi(\bar{R}\omega_0 - Ric(\omega_0)) + \omega_\phi \log \frac{\omega_\phi}{\omega_0} \right]$$

(T. Mabuchi'87, explicit form is due to G. Tian'98). The cocycle identity

$$S_M(\omega_0, \omega_\phi) = S_M(\omega_0, \omega_\psi) + S_M(\omega_\psi, \omega_\phi)$$

is satisfied both by Mabuchi and Aubin-Yau functionals.
The Mabuchi action on the space of metrics in the Kähler class plays a similar role to that of the Liouville action in the conformal class. Indeed
- both actions are 1-cocycles,
- both actions are positive definite convex functionals on the corresponding spaces of metrics,
- have constant scalar curvature metrics as critical points,
- another analogy comes from the relation to zeta-function on manifold.

The zeta-function on M is defined as

$$\zeta(s) = \sum_{\lambda_k \neq 0} \lambda_k^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} dt \, (\mathrm{Tr}\, e^{t\Delta_g} - 1)$$

where λ_k's are eigenvalues of the laplacian $\Delta_g u_k + \lambda_k u_k = 0$. Then $\zeta'(0) = \log\det\Delta_g$ and its variation gives the Liouville action.
A nice variation formula exists (C. Morpurgo'96) as well for the Green's function $\mathrm{Tr}\Delta_g^{-1} = \zeta(1)$, defined by subtracting divergence at $s = 1$

$$\tilde{\zeta}(s) = \zeta(s) - \frac{1}{s-1}\frac{A}{4\pi\Gamma(s)}$$

Then in the Kähler class we recover Mabuchi energy

$$\delta\tilde{\zeta}(1) = \int_M \left[\frac{\bar{R}}{2}\phi\partial\bar{\partial}\phi + \omega_\phi \log\frac{\omega_\phi}{\omega_0}\right]$$

Summing up, all the above suggests that the following path integral should be introduced

$$\int e^{-\gamma S_M(\omega_0,\phi)}\mathcal{D}\phi$$

Although there is yet no derivation of the Mabuchi path integral from the first principles, there is a deeper relation with Liouville gravity.

Relation with Liouville theory

Classically, kinetic terms and interaction terms of Mabuchi and Liouville actions are Legendre dual. The following relations hold

$$\log \int_M e^{\sigma}\omega_0 = \sup_{\phi}\left(\int_M \sigma\omega_\phi - \int_M \omega_\phi \log\frac{\omega_\phi}{\omega_0}\right)$$

$$\int_M \omega_\phi \log\frac{\omega_\phi}{\omega_0} = \sup_{\sigma}\left(\int_M \sigma\omega_\phi - \log(\int_M e^{\sigma}\omega_0)\right)$$

and

$$\int_M \frac{1}{2}\phi\partial\bar{\partial}\phi + \phi\omega_0 = \sup_{\sigma}\left(\int_M \sigma\omega_\phi - \int_M \sigma\partial\bar{\partial}\sigma\right)$$

Quantum mechanically, we should consider generating functional

$$e^{-S_M(\phi)} = \int \mathcal{D}\sigma e^{-S_L(\sigma)+\int_M \sigma\omega_\phi}$$

Relation with Liouville theory, cont'd

The generating functional is well-defined in the CFT formulation of Liouville theory due to David-Distler-Kawai

$$e^{-W_\beta(\omega_0,\omega_\phi)} = \int e^{-\frac{1}{4\pi b^2} S_L^q(\omega_0,\sigma) - \mu \int_M e^\sigma \omega_0 + \frac{\beta}{4\pi} \int_M \sigma\omega_\phi} \mathcal{D}_{\omega_0}\sigma \cdot Z^{\text{matter}}[\omega_0]$$

Here deformed Liouville action $S_L^q(\omega_0,\sigma) = \int_M (-\sigma\Delta_0\sigma + 2qR_0\sigma)\omega_0$, and $q = b^2 + 1$, is required for the background independence of the theory. Now the measure is usual shift-invariant field theory measure (not DeWitt). At $\beta_c = \frac{q\bar{R}}{b^2}$ we get exact relation

$$W_{\beta_c}(\omega_0,\omega_\phi) = W_{\beta_c}(\omega_0,\omega_0) + \frac{q^2\bar{R}}{4\pi b^2}\int_M \left(\frac{\bar{R}}{2}\phi\partial\bar{\partial}\phi + \phi(\bar{R}\omega_0 - R_0\omega_0)\right).$$

Here on the rhs stands the 'free field' part of the Mabuchi energy.

Relation with Liouville theory, cont'd

Interaction term of the Mabuchi energy appears at large β

$$W_\beta(\omega_0, \omega_\phi) = -\frac{q\beta}{4\pi}\int_M \omega_\phi \log\frac{\omega_\phi}{\omega_0}, \qquad \beta \to \infty$$

In fact, for any β we get the following exact Ward identity

$$W_\beta(\omega_0, \omega_\phi) = W_\beta(\omega_\phi, \omega_\phi) - \frac{q\beta}{4\pi}\int_M \omega_\phi \log\frac{\omega_\phi}{\omega_0},$$

As a consequence of these relations we can introduce the following model, which interpolates between two theories

$$\int e^{-\frac{1}{4\pi b^2} S_L^q(\omega_0,\sigma) + \frac{\beta}{4\pi}\int_M \sigma\omega_0 - \mu\int_M e^{\sigma - b^2\beta\phi}\omega_0 - \frac{\beta b^2}{4\pi} S_M^\beta(\omega_0,\phi)} \mathcal{D}_{\omega_0}\sigma\, \mathcal{D}\phi \cdot Z^{\rm matter}[\omega_0]$$

The main feature is that this path integral is independent of a choice of ω_0 (background independence).

Mabuchi theory as a matrix integral

To build the approximation of the Mabuchi path integral, using $\mathcal{B}_K$ consider again the Hilb map: $\phi \to Hilb(\phi) \in \mathcal{B}_k$ The following asymptotic formula holds (Donaldson'02)

$$\frac{1}{k}\log\det Hilb(\phi) = -N_k F_{AY}(\omega_0,\phi) + \frac{1}{2}S_M(\omega_0,\phi) + \frac{1}{6k}S_L(\omega_0,\phi) + \mathcal{O}(1/k^2)$$

Here on the rhs $F_{AY}(\phi)$ is Aubin-Yau functional, S_M is Mabuchi action and S_L is the Liouville action,. The proof is by taking a variation wrt ϕ,

$$\delta \log\det Hilb(\phi) = \int(-k\rho_k + \Delta\rho_k)\delta\phi\,\omega_\phi^n$$

and using the asymptotic expansion of the Bergman kernel ρ_k for large k. Therefore, up to $1/k$ corrections

$$\mu(\omega_\phi) = e^{-\gamma S_M}\mathcal{D}\phi = [\det Hilb(\phi)]^{-2\gamma/k}\, e^{-\gamma k N_k F_{AY}(\omega_0,\phi)}\mathcal{D}\phi$$

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Therefore we propose the following definition of the path integral measure

$$\mu(\omega_\phi) = e^{-\gamma S_M}\mathcal{D}\phi := \lim_{k\to\infty}[\det H]^{-2\gamma/k}\, e^{-\gamma k N_k F_{AY}(FS(H))}[dH]_{Haar}$$

Here, instead of 'pull-back' of the Aubin-Yau $F_{AY}(\phi)$ we take its 'push-forward' under the FS map: $H \to \phi = FS(H) \in \mathcal{K}$, since the composition of these two maps $FS \circ Hilb = Id$ is identity map for large k. The Haar measure $GL(N_k)/U(N_k)$ corresponds to the invariant metric, which converges to the Mabuchi metric in the continuous limit. This is a complicated matrix integral, since the $U(N_k)$ does not decouple (not an eigenvalue integral). However, we need only large N limit.

Eigenvalue model

On $\mathbb{CP}^1$ we can choose a basis of section $s_\alpha = z^\alpha$, number of section is $N_k = k + 1$. Then det Hilb can be written as an eigenvalue matrix integral

$$\det Hilb(\phi) = \prod_{\alpha=0}^{N_k} \left[\int_{\mathbb{CP}^1} \frac{d^2 z_\alpha}{(1 + |z_\alpha|^2)^2} \right] |\Delta(z)|^2 \, e^{\sum_\alpha W(z_\alpha)},$$

with the potential $W(z) = -k\phi + \log \frac{\omega_\phi}{\omega_0}$. Consider β-ensemble

$$Z_\beta = \prod_{\alpha=0}^{N_k} \left[\int_{\mathbb{CP}^1} \frac{d^2 z_\alpha}{(1 + |z_\alpha|^2)^2} \right] |\Delta(z)|^{2\beta} \, e^{\sum_\alpha W(z_\alpha)}$$

There is large N_k expansion of this integral (Wiegmann, Zabrodin'06)

$$\log Z_\beta = -N_k^2 F_{AY}(\omega_0, \phi) + N_k S_M^\beta(\omega_0, \phi) + \ldots$$

Eigenvalue model, cont'd

Therefore at large N_k, we have

$$\int e^{-\gamma S_M^\beta(\omega,\phi)}\mathcal{D}\phi := \prod_{\alpha=0}^{N_k}\left[\int_{\mathbb{CP}^1}\frac{d^2 z_\alpha}{(1+|z_\alpha|^2)^2}\right]|\Delta(z)|^{2\beta}\cdot$$

$$\cdot\int e^{\sum_\alpha W(z_\alpha)}e^{-\gamma N_k^2 F_{AY}(\omega_0,\phi)}\mathcal{D}\phi,$$

with the potential $W(z)=-k\phi+\log\frac{\omega_\phi}{\omega_0}$. At large k the expression inside the multiple integral is a correlation function in the theory with the (gaussian) Aubin-Yau action, but nontrivial integration measure $\mathcal{D}\phi$.