Kähler metrics, 2d gravity and complex random matrices

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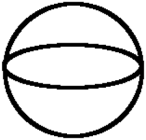
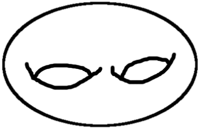
Motivation

Random complex geometry, originating from Polyakov's 1981 paper on 2d gravity, gave rise to many fruitful developments, most recent examples being SLE, AGT conjecture etc. It is reasonable to expect more surprises along the lines where complex geometry meets random processes. At the same time, due to the work of Yau, Tian, Donaldson and others, there has been a lot of progress in understanding the space of Kähler metrics. The main idea is to look at the random metrics from the point of view of Kähler geometry.
For most of the talk we focus on metrics on two dimensional compact surfaces.





Brief reminder: 2d gravity

Start with the path integral over all embeddings $X^i(z,\bar z), i = 1, ..., d$ and all metrics $g_{\mu\nu}$ of closed compact surfaces

$$\int \mathcal{D}g_{\mu\nu}\mathcal{D}_g X^i \exp\left(\int_M \sqrt{\det g}\, d^2z\, X^i \Delta_g X^i\right),$$

with standard L^2 metric on the space of embeddings $||\delta X^i||^2 = \int \sqrt{\det g}\, d^2z\, (\delta X^i)^2$. The X^i integral is gaussian and can be performed to give a power of $\det \Delta_g$. Two-dimensional metric can be put in Weyl form (conformal gauge) with help of diffeos $g_{z\bar z} = e^\sigma g_{0z\bar z}$. Then the dependence of $\det \Delta_g$ on the conformal factor σ can be determined

$$\delta_\sigma \log\det\Delta_g = -\frac{1}{6\pi}\int_M g\, d^2z\, R(g)\delta\sigma = -\frac{1}{12\pi}\delta_\sigma S_L(\sigma)$$

where $R(g) = -g^{-1}\partial\bar\partial \log g$ is the scalar curvature of metric g, and the Liouville action is

$$S_L(\sigma) = S_L(g_0, e^\sigma g_0) = \int_M g_0\, d^2z\, (-\sigma\Delta_0\sigma + 2R_0\sigma)$$

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As a functional of two metrics S_L satisfies the cocycle identity (background independence)

$$S_L(g_0, g_2) = S_L(g_0, g_1) + S_L(g_1, g_2)$$

Under the constant area constraint $A = \int_M g\, d^2 z = \int_M g_0 d^2 z$ eq. of motion of Liouville theory is the equation for constant scalar curvature (csc) $R(g) = \mathrm{const}$. Integral over metrics splits into integral over diffeos, which decouples, and the integral over conformal factor

$$\int \mathcal{D}_{\sigma}\sigma \exp(-S_L(\sigma))$$

Note, that the L^2 metric on the space of 2d metrics leads to non-free-field metric on the space of confomal factors: $||\delta\sigma||^2 = \int e^{\sigma} g_0\, d^2 z\, (\delta\sigma)^2$

Kähler metrics: brief intro

Now we would like to see what happens in Kähler gauge. Let M be n-dimensional compact Kähler manifold. A Kähler metrics is given by its Kähler form $\omega = \sqrt{-1} g_{a\bar{b}} dz^a \wedge d\bar{z}^{\bar{b}}$. Kähler form is closed $d\omega$, so it defines cohomology class $[\omega]$ in $H^{(1,1)}(M)$. Also, the metric $g_{a\bar{b}}$ must be positive definite Hermitian, therefore any metric in the class of $[\omega]$ can be written as $\omega_\phi = \omega_0 + \partial\bar{\partial}\phi > 0$, for a representative 'background' metric ω_0. All metrics in a given class have same volume $V = \int_M \omega_0^n = \int_M \omega_\phi^n$. The space of Kähler metrics in the given class is equivalent to the space of Kähler potentials

$$\mathcal{K} = \{\phi \in C^\infty(M) : \;\; \omega_\phi = \omega_0 + \partial\bar{\partial}\phi > 0\}$$

(modulo constant ϕ's). Formally, $\mathcal{K}$ is an infinite dimensional space. It is endowed with a natural 'Donaldson-Semmes-Mabuchi' metric

$$||\delta\phi||^2 = \int_M \omega_\phi^n \, (\delta\phi)^2,$$

in which $\mathcal{K}$ is a negatively curved locally symmetric space.

In complex dimension one, we have the equivalence between conformal and Kähler gauges

$$e^{\sigma}\omega_0 = \omega_0(1 + \Delta_0\phi) = \omega_\phi,$$

where fixed area constraint in conformal gauge is assumed. For simplicity, here and in the following ω will stand for g. Note that L^2 metric and Mabuchi metric are mutually non-local

$$||\delta\sigma||^2 = \int e^{\sigma}\omega_0\,(\delta\sigma)^2 \qquad ||\delta\phi||^2 = \int_M \omega_\phi\,(\delta\phi)^2$$

Mabuchi K-energy

Mabuchi K-energy is the most natural action functional associated with Kähler metrics. It has csc metrics as critical points

$$\delta S_M(\phi) = -\int_M (R(\omega_\phi) - \bar{R})\delta\phi\, \omega_\phi^n,$$

($\bar{R}$ is average scalar curvature= Euler characteristic). Cocycle identity

$$S_M(\omega_0, \omega_\phi) = S_M(\omega_0, \omega_\psi) + S_M(\omega_\psi, \omega_\phi)$$

is imposed and the variation formula can be integrated to give ($n = 1$)

$$S_M(\omega_0, \omega_\phi) = \int_M \left[\frac{\bar{R}}{2} \phi \partial\bar{\partial}\phi + \omega_\phi \log \frac{\omega_\phi}{\omega_0} + \phi(\bar{R}\omega_0 - Ric(\omega_0)) \right]$$

Note the wrong sign of kinetic term. In dimension n, the expression is

$$S_M(\omega_0, \omega_\phi) = \int_M \left[\frac{\bar{R}\ \phi\omega_\phi^n}{n(n+1)} + \omega_\phi^n \log \frac{\omega_\phi^n}{\omega_0^n} + \phi \left(\frac{\bar{R}\omega_0}{n(n+1)} - Ric(\omega_0) \right) \sum_{i=0}^{n-1} \omega_\phi^i \omega_0^{n-1-i} \right]$$

(T. Mabuchi'87, explicit form is due to G. Tian'98)

We claim that Mabuchi energy plays a role in 2d gravity. Let's return to the anomaly of the conformal laplacian. The zeta-function on M is defined as

$$\zeta(s) = \sum_{\lambda_k \neq 0} \lambda_k^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} dt \, (\mathrm{Tr}\, e^{t\Delta_g} - 1)$$

where λ_k's are eigenvalues of conformal laplacian $\Delta_g u_k + \lambda_k u_k = 0$. Then $\log\det \Delta_g$ corresponds to $\zeta'(0)$. However, a nice variation formula exists as well for the Green's function $\mathrm{Tr}\Delta_g^{-1} = \zeta(1)$. A caveat is that divergence at $s = 1$ is subtracted

$$\tilde{\zeta}(s) = \zeta(s) - \frac{1}{s-1} \frac{A}{4\pi\Gamma(s)}$$

the variation for S^d has been computed (C. Morpurgo'96)

$$\delta\tilde{\zeta}(1) = \delta \int_{S^n} g_0\left(\frac{g}{g_0} - 1\right)\Delta_g^{-1}\left(\frac{g}{g_0} - 1\right) + \frac{2}{n!} g \log \frac{g}{g_0}$$

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In the Kähler gauge $g/g_0 = \omega_\phi/\omega_0 = (1 + \Delta_0\phi)$, we recover Mabuchi energy

$$\delta\tilde{\zeta}(1) = \int_M \left[\frac{\bar{R}}{2}\phi\partial\bar{\partial}\phi + \omega_\phi \log\frac{\omega_\phi}{\omega_0}\right]$$

The Liouville energy is positive definite $S_L \geq 0$ (Onofri-Moser-Trudinger inequality), and invariant under conformal transformations $z \to f(z), \;\; \sigma \to \sigma - \log|f'|^2$.
Mabuchi energy is also positive definite $S_M \geq 0$ and also conformally invariant $z \to f(z), \;\; \omega_\phi/\omega_0 \to \omega_\phi/\omega_0 \cdot |f'|^{-2}$.
Consider the path integral:

$$\int \mathcal{D}\phi\, e^{-S_M(\phi)}$$

Question: does it make sense, and what is the relation to Liouville gravity?

Legendre transform

Kinetic terms and interaction terms of Mabuchi and Liouville actions are Legendre dual. The following relations hold

$$\log \int_M e^{\sigma}\omega_0 = \sup_{\phi}\left(\int_M \sigma\omega_\phi - \int_M \omega_\phi \log\frac{\omega_\phi}{\omega_0}\right)$$

$$\int_M \omega_\phi \log\frac{\omega_\phi}{\omega_0} = \sup_{\sigma}\left(\int_M \sigma\omega_\phi - \log(\int_M e^{\sigma}\omega_0)\right)$$

and

$$\int_M \frac{1}{2}\phi\partial\bar\partial\phi + \phi\omega_0 = \sup_{\sigma}\left(\int_M \sigma\omega_\phi - \int_M \sigma\partial\bar\partial\sigma\right)$$

Conjecture: quantum theories are dual in the sense that

$$e^{-S_M(\phi)} = \int \mathcal{D}\sigma e^{-S_L(\sigma)+\int_M \sigma\omega_\phi}$$

(compare with $\langle e^{-\int_M(\mu T(z)+\bar\mu\bar T(\bar z))}\rangle$, studied by Takhtajan and collaborators)

Matrix model

Consider the following matrix model (β-ensemble)

$$Z_N = \int |\Delta(z_i)|^{2\beta} \prod_{i=1}^{N} e^{\frac{1}{\hbar} W(z_i)} \, d^2 z_i$$

At $\beta = 1$ this arises from the matrix integral for normal random matrices or for $GL(N)$ random matrices with quasiharmonic potential. Following Zabrodin-Wiegmann (2006), we want to study its large-N limit $\log Z_N$ in terms of the density

$$\rho(z) = \hbar \sum_i \delta(z - z_i) = \pi \hbar \, \partial \bar{\partial} \log |z - z_i|^2$$

The idea is that at large N, ρ approximates a smooth function, so one could trade integration over infinite number of discrete variables z_i to a path integral over one function $\rho(z)$.

$$Z_N \sim \int \mathcal{D}\rho \, e^{-\frac{1}{\hbar^2} S(\rho)}$$

$$Z_N \sim \int \mathcal{D}\rho \, e^{-\frac{1}{\hbar^2} S(\rho, W)}$$

(Jevicki-Sakita collective field theory). One can look at this as an integral over metrics $\rho d^2 z$ on $\mathbb{CP}^1$. The action here is

$$-S(\rho, W) = \beta \int \int \rho(z) \log |z-w| \rho(w) d^2 z d^2 w -$$

$$-\int W(z)\rho(z) d^2 z - \frac{2-\beta}{2} \int \rho \log \rho d^2 z$$

First term here is the logarithm of Vandermonde, which becomes regular kinetic term for the field ϕ, defined as $\rho = \Delta\phi$. Second term comes from the potential, and third term is 'entropy term'

$$\prod_i d^2 z_i = J[\rho]\mathcal{D}\rho = e^{-\int \rho \log \rho d^2 z} \mathcal{D}\rho$$

It comes from the integration measure and from Vandermonde at coincident points. This is 'modified' Mabuchi action.

Bergman metrics

The infinite-dimensional space of Kähler potentials can be approximated by a finite dimensional space as follows.
Consider M compact Kähler manifold, with a line bundle L and its k-th power L^k. For a hermitian metric h on L the metric on L^k is h^k, its Ricci curvature $R(h) = -\partial\bar{\partial} \log h^k$. If L is positive ($\equiv R(h)$ is positive definite) we can take the metric in the cohomology class $c_1(L)$ to be proportional to curvature

$$\omega = -\frac{1}{k}\partial\bar{\partial} \log h^k$$

Consider a basis $[s_0(z), ..., s_{N_k}(z)]$ in the space of global holomorphic sections $H^0(M, L^k)$ (think holomorphic polynomials of degree k). Since s_α's are defined up to multiplication by complex number, the choice of basis defines a 'Kodaira embedding' of M into big projective space of sections $z \to s_\alpha(z) \in \mathbb{CP}^{N_k}$.

Now, there is a natural Fubini-Study metric on the projective space $\mathbb{CP}^{N_k}$, which we can pull-back to M to define a Bergman metric on M

$$\omega|_{FS} = \frac{1}{k}\partial\bar{\partial}\log\left(\sum_{\alpha=0}^{N_k}|s_\alpha|^2\right)$$

The basis of s_α is not necessarily normalized. Different choice of basis leads to different embedding $M \to \mathbb{CP}^{N_k}$, and therefore different Bergman metric. The space $\mathcal{K}_k$ of Bergman metrics of level k is therefore

$$\mathcal{K}_k = GL(N_k+1)/U(N_k+1)$$

Now we show that this finite dimensional space approximates the space of Kähler metrics $\mathcal{K}$ in a fixed class.

TYZ theorem

The Bergman metric can be rewritten as

$$\omega|_{FS} = \frac{1}{k}\partial\bar\partial \log\left(\sum_{\alpha=0}^{N_k} |s_\alpha|^2\right) = -\partial\bar\partial \log h + \frac{1}{k}\partial\bar\partial \log\left(h^k \sum_{\alpha=0}^{N_k} |s_\alpha|^2\right)$$

Inside the logarithm we have a global 'density of states' function, called Bergman kernel. It allows for asymptotic $1/k$-expansion

$$\rho_k[\omega_\phi] = h^k \sum_{\alpha=0}^{N_k} |s_\alpha|^2 = k^n + \frac{k^{n-1}}{2}R + k^{n-2}\left(\frac{1}{3}\Delta R + \frac{1}{24}|\mathrm{Riem}|^2 - \frac{1}{6}|\mathrm{Ric}|^2 + \frac{1}{8}R^2\right) + \ldots$$

Therefore the difference between the metric ω and the Bergman metric is

$$\omega|_{FS} - \omega \sim \mathcal{O}(1/k^2)$$

Therefore we can approximate any metric ω by a Begman metric in the same Kähler class (Tian-Yau-Zelditch theorem): $\mathcal{K} = \lim_{k\to\infty} \mathcal{K}_k$.

Bergman kernel from Path integral

Bergman kernel is the density matrix of the magnetic Schrödinger operator, projected on lowest Landau level. It can be represented as long time limit of heat kernel of

$$\rho_k(z) = \lim_{T\to\infty} \langle z|e^{-TH}|z\rangle = e^{-TE_0} \sum_{LLL} \psi_n \psi_n^*$$

Path integral representation of the Bergman kernel (M. Douglas, SK'08)

$$\rho_k(z) = \int_{z(0)=z}^{z(T)=z} \prod_{0<t<T} d^{2n}z(t) \det g(z(t)) \cdot e^{-\frac{1}{\hbar}S}$$

with the action for particle in magnetic field

$$S = \int_0^T dt \left(g_{a\bar{a}} \dot{z}^a \dot{\bar{z}}^{\bar{a}} + A_a \dot{z}^a \right)$$

Hilb map

Let $\omega_0 = -\partial\bar{\partial}\log h_0$ represent a Kähler class. Any other metric ω_ϕ in $[\omega_0]$ can be written as $\omega_\phi = -\partial\bar{\partial}\log h_0 e^{-\phi} = \omega_0 + \partial\bar{\partial}\phi$. The *Hilb* map gives a positive hermitian matrix in $\mathcal{K}_k = GL(N_k+1)/U(N_k+1)$

$$Hilb_{\alpha\beta}(\phi) = \frac{1}{V}\int_M \bar{s}_\alpha s_\beta h_0^k e^{-k\phi}\,\omega_\phi^n.$$

Interestingly, det *Hilb* is a matrix model-like integral. Indeed,

$$\det Hilb(\phi) = \int_M \ldots \int_M \cdot |\det s_\alpha(z_\beta)|^2 \cdot \prod_{\alpha=1}^{N_k} e^{-k\phi(z_\alpha)} h_0^k \omega^n(z_\alpha)$$

For $\mathbb{CP}^1$, $|\det s_\alpha(z_\beta)|^2$ is the square of Vandermonde, for the basis $s_\alpha = z^{\alpha-1}$.

One can proof by taking a variation wrt ϕ,

$$\delta \log \det Hilb(\phi) = \int (-k\rho_k + \Delta\rho_k)\delta\phi\, \omega_\phi^n$$

and using TYZ expansion that for 2d surfaces the following formula holds (Donaldson'02)

$$\frac{1}{k}\log \det Hilb(\phi) = N_k F(\phi) + \frac{1}{2}S_M(\phi) + \frac{1}{6k}S_L(\phi) + \mathcal{O}(1/k^2)$$

here S_L is Liouville action, S_M is Mabuchi action and $F(\phi) = \int_M (\frac{1}{2}|\partial\phi|^2 - \phi\omega)$ is free field (Aubin-Yau) action.

Random Bergman metrics

The idea is to define probability measures on $\mathcal{K}$ by taking $k \to \infty$ limit of probability measures on finite-dimensional Bergman spaces $\mathcal{K}_k$. Let's pick a basis of sections s_α once and for all. Then Bergman metric can be explicitly parametrized by a $GL(N_k+1)$ matrix A as

$$\omega = \frac{1}{k}\partial\bar{\partial} \log \bar{s} A^\dagger A s$$

then $H = A^\dagger A$ is positive hermitian matrix. We want to study finite dimensional matrix model of the form

$$\int_{\mathcal{K}_k} ... \mu_k(H) \to \int_{\mathcal{K}} ... \mu(\omega_\phi), \qquad \text{as} \;\; k \to \infty$$

with the goal e.g. to find μ_k, which approximates Polyakov's 2d gravity measure. Another question is whether there are any other interesting measures possible.

Eigenvalue models

It is interesting e.g. to look at correlation functions of the form $\langle\omega(z_1)...\omega(z_n)\rangle$, $\langle R(z)\rangle$ etc.
A natural set of measures on $GL(N)$ matrix A can be represented as $A = V\Lambda U$, where U, V are unitary and Λ is diagonal
$H = A^\dagger A = U^\dagger\Lambda^2 U$. Then one can consider measures of the form

$$|\Delta(\lambda)|^2 e^{-\sum_\alpha V(\lambda_\alpha)} \prod_\alpha d\lambda_\alpha$$

For eigenvalue measures the distribution on the space of metric is peaked on the background metric

$$\langle\omega(z)\rangle = \frac{1}{k}\partial\bar\partial\langle\log(\bar{s}Hs)\rangle = \omega_0$$

where $\omega_0 = \frac{1}{k}\partial\bar\partial\log(\bar{s}\cdot s)$.

Wishart distibution

$$\mu(A) = e^{-\mathrm{Tr}A^\dagger A}(\det A^\dagger A)^a \, dA^\dagger dA$$

allows for explicit solution for the 2-point function

$$\langle \log(\bar{s}Hs)|_z \log(\bar{s}Hs)|_w \rangle = \int \log(\bar{s}Hs)|_z \log(\bar{s}Hs)|_w e^{-\mathrm{Tr}A^\dagger A}(\det A^\dagger A)^a \, dA^\dagger dA$$

This integral involves nontrivial integration over unitary group, but is not harder than Harish-Chandra-Itzykson-Zuber integral. The answer

$$\langle \log(\bar{s}Hs)|_z \log(\bar{s}Hs)|_w \rangle = -\int_0^1 \frac{dx}{x}(1-x)^{N_k+a-1} \ln(1 - x\rho(z,w)).$$

Here

$$\rho(z,w) = \frac{(\bar{s}_z, s_w)(\bar{s}_w, s_z)}{|s_z|^2 \cdot |s_w|^2} = e^{-kD(z,w)},$$

where $D(z,w)$ is the Calabi's diastasic function defined in the reference metric ω_0. If $d(z,w)$ is a geodesic distance between the points z and w in the reference metric, then

$$D(z,w) = d^2(z,w) + \mathcal{O}(d^4(z,w)).$$

Short-time behavior of the two-point function of the metrics

$$\langle \omega(z)\omega(w)\rangle = \frac{1}{k^2}\partial\bar{\partial}|_z\partial\bar{\partial}|_w\langle \log(\bar{s}Hs)|_z \log(\bar{s}Hs)|_w\rangle \sim \frac{1}{|z-w|^{4-2\beta}}$$

where $\beta = N_k + a > 0$, unless the integral diverges. In Liouville theory

$$\langle e^{\gamma\sigma(z)}e^{\gamma\sigma(w)}\rangle \sim \frac{1}{|z-w|^{\gamma^2}}, \qquad \gamma \leq 2$$

So the singularity is of the same order. Further analysis shows that this property holds for more general eigenvalue potentials. However, these measures are not background independent (except for $\log\det H$), and conformal invariance is broken. The challenge remains to built a proper discretization of Liouville energy.