Geometry and Large N limits in Laughlin states

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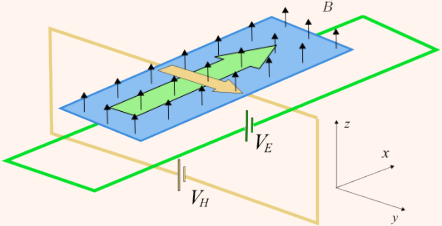
Quantum Hall effect

Observed in two-dimensional electron systems subjected to low temperatures and strong magnetic fields. Hall conductance is quantized $\sigma_H = I/V_H = \nu$, where ν is integer for integer QHE, or a fraction for fractional QHE. Involves many ($N \sim 10^6$) electrons on lowest Landau level, described by a collective (Laughlin) state.



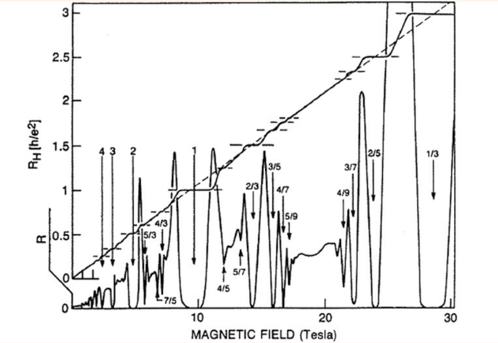
Laughlin state

On the plateaus QHE is described by collective Laughlin state

$$\Psi(z_1,\ldots,z_N)=\prod_{i<j}^{N}(z_i-z_j)^{\beta}e^{-\frac{B}{4}\sum_i |z_i|^2},\quad \beta\in\mathbb{Z}_+$$

[Laughlin'83]

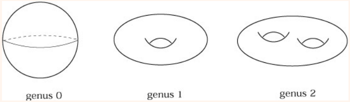
$\beta=1$: Integer QHE,
non-interacting electrons.
$\beta=3,5,7,..$: Fractional QHE,
interacting system.
Planar geometry, strong
constant magnetic field B.
Hall conductance is quantized
$\sigma_H=\nu=1/\beta$.



Other candidate states were proposed for other plateaus (e.g. Pfaffian state $\mathrm{Pf}\frac{1}{z_i-z_j}\Delta(z)^2$, Jain, Read-Rezayi, ...).

Laughlin state

We would like to consider Laughlin state on a compact Riemann surface Σ of genus g



Aim: Study Laughlin state(s) on a genus-g Riemann surface Σ with arbitrary geometry:

- inhomogeneous magnetic field B, with flux $N_\Phi = \frac{1}{2\pi}\int_\Sigma B\sqrt{g}d^2x$
- arbitrary solenoid (Aharonov-Bohm) phases around the cycles (flat connections moduli)
- metric g, and curvature R,
- complex structure moduli J,
- singularities of different types at points $z_1, ..., z_n$.

Main problem: partition function

The Laughlin states have the general form

$$\Psi_r = \frac{1}{\sqrt{Z}} F_r(z_1, .., z_N), \qquad r = 1, .., r_{\mathrm{g}},$$

where F is a holomorphic function. ($r_{\mathrm{g}} = \beta^{\mathrm{g}}$ [Wen-Niu'90])
The partition function is the normalization constant $Z = \langle F_r, F_r \rangle_{L^2}^2$. It is a functional of various geometric parameters

$$Z = Z[g, J, B, \varphi, ...]$$

For example, the partition function for planar Laughlin state is

$$Z = \int_{\mathbb{C}^N} \prod_{i<j}^{N} |z_i - z_j|^{2\beta} e^{-\frac{B}{2} \sum_i |z_i|^2} \prod_{j=1}^{N} d^2 z_j$$

Central object in Log-gases (Coulomb gas, random matrix β-ensemble).

Main goal: Determine $\log Z[g, J, B, \varphi, ...]$ in the limit of large number N of particles.

Why?

Suppose all you ever want to study is the QHE in samples of planar geometry. Still, coupling the system to the metric and studying the dependence of the partition function can help you do that. The prototypical example is conformal field theory, where partition function in a flat metric g_0 and in perturbed metric $g = g_0 e^{2\sigma(z,\bar{z})}$ are related as

$$\log \frac{Z^{CFT}(g)}{Z^{CFT}(g_0)} = \frac{c}{12\pi} S_L(g_0, \sigma),$$

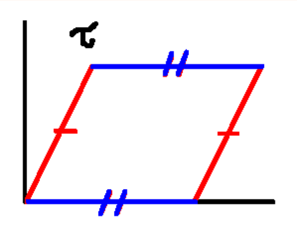
where $c \in \mathbb{R}$ is the central charge of the CFT, and the Liouville action is

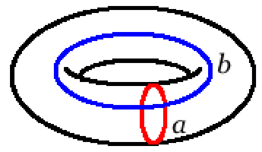
$$S_L(g_0, \sigma) = \int_M (\partial\sigma\bar{\partial}\sigma + R_0\sigma) d^2z,$$

and $R_0 = -\frac{1}{\sqrt{g_0}}\partial\bar{\partial}\log\sqrt{g_0}$ is the scalar curvature of g_0.
This way we learn the value of an important coefficient: central charge.

Why: geometric adiabatic transport

In QHE on Riemann surfaces important coefficients are encoded by *geometric adiabatic transport* [Thouless et.al.; Avron, Seiler, Simon, Zograf, ...], via Berry connection and curvature. Riemann surface (Σ, g, J) provides a natural set of parameters. For example, on a torus we have a complex structure $\tau \in \mathbb{H}$ and AB-phases $\varphi_1, \varphi_2 \in [0,1]^2$, such that the integrals of the magnetic field gauge connection around the a, b-cycles are $\int_a A = \varphi_1; \int_b A = \varphi_2$. The parameter space has complex coordinates $\tau, \varphi = \varphi_1 + \tau\varphi_2$.





Why: geometric adiabatic transport

Degenerate Laughlin states Ψ_r on Riemann surfaces form a vector bundle over the parameter space Y with complex coordinates $y = (\tau, \varphi)$. Moreover, the states depend on parameter space in holomorphic fashion $\Psi_r = \frac{1}{\sqrt{Z(y,\bar{y})}} F_r(y)$. For example, Laughlin states on the torus

$$F_r(z_1, .., z_{N_\Phi}) = \theta_{\frac{r}{\beta},0}(\beta z_c + \varphi, \beta\tau) \prod_{i<j} \left(\frac{\theta_1(z_i - z_j, \tau)}{\eta(\tau)} \right)^\beta e^{-\pi i N_\Phi \frac{(z-\bar{z})^2}{\tau - \bar{\tau}}}$$

$r = 1, .., \beta, \quad z_c = \sum z_j$ [Haldane-Rezayi'85]
Then adiabatic (Berry) connection and curvature are

$$\mathcal{A}_{rr'} = \langle \Psi_r, d_y \Psi_{r'} \rangle_{L^2} = d_y(\langle \Psi_r, \Psi_{r'} \rangle_{L^2}) - \langle d_y \Psi_r, \Psi_{r'} \rangle_{L^2} = \delta_{rr'} \frac{1}{2} d_y \log Z,$$

$$\mathcal{R}_{rr'} = d\mathcal{A}_{rr'} = \delta_{rr'} d_y d_{\bar{y}} \log Z(y, \bar{y}).$$

Geometric adiabatic transport

- Hall conductance σ_H is a first Chern number of the vector bundle of Laughlin states over the space of AB-phases (flat connections moduli) [Thouless et.al.'82'85, Tao-Wu'85, Avron-Seiler'85, Avron-Seiler-Zograf'94]
- Anomalous Hall viscosity η_H, is a first Chern number of the vector bundle of Laughlin states on the moduli space of complex structures of a torus $\mathcal{M}_1$. IQHE: [Avron-Seiler-Zograf'95], FQHE: [Tokatly-Vignale'07, Read'09]
- Transport on higher genus. IQHE: [Levay'97], FQHE [SK-Wiegmann'15].

Another motivation

- Curvature in the experimental sample arises in graphene

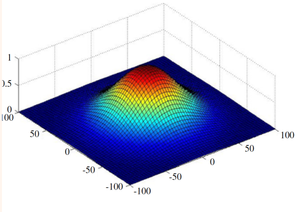


Fig. 6. A smooth curved bump in the graphene sheet.

M.A.H. Vozmediano et al., Physics Reports 496 (2010)

- Recently a QHE-like system was experimentaly realised on a spatial cone Schine et al., Arxiv:1511.07381
  "At the cone tip, we observe that spatial curvature increases the local density of states, and we measure fractional state number excess consistent with the Wen-Zee effective theory, providing the first experimental test of this theory of electrons in both a magnetic field and curved space".

Results: Partition function for IQHE

$$\log Z = \frac{1}{2\pi}\int_\Sigma \left[A_z A_{\bar z} + \frac{1-2s}{2}(A_z\omega_{\bar z} + \omega_z A_{\bar z}) + \left(\frac{(1-2s)^2}{4} - \frac{1}{12}\right)\omega_z\omega_{\bar z}\right] d^2 z \\ + \mathcal{F}[B,R].$$

$$\mathcal{F}[B,R] = -\frac{1}{2\pi}\int_\Sigma \left[\frac{1}{2}B\log B + \frac{2-3s}{12}R\log B + \frac{1}{24}(\log B)\Delta_g(\log B)\right]\sqrt{g}d^2 z \\ + \mathcal{O}(1/B).$$

[SK'13; SK-Ma-Marinescu-Wiegmann'15]

This holds for surfaces of any genus. Terminology: $A_z, A_{\bar z}$ are components of the gauge-connection 1-form for the magnetic field

$$B = g^{z\bar z}(\partial_z A_{\bar z} - \partial_{\bar z} A_z)$$

Also $\omega_z, \omega_{\bar z}$ are components of spin-connection $\omega_z = i\partial_z \log g_{z\bar z}$, and scalar curvature is

$$R = g^{z\bar z}(\partial_z \omega_{\bar z} - \partial_{\bar z}\omega_z)$$

and $s \in \mathbb{Z}/2$ is gravitational spin.

Results: relation to Chern-Simons theory

Note that generating functional $\log Z$ and 3d Chern-Simons (Wen-Zee) actions look similar:

$$\log Z = \frac{1}{2\pi} \int_{\Sigma} \left[A_z A_{\bar{z}} + \frac{1-2s}{2} (A_z \omega_{\bar{z}} + \omega_z A_{\bar{z}}) + \left(\frac{(1-2s)^2}{4} - \frac{1}{12} \right) \omega_z \omega_{\bar{z}} \right] d^2 z$$

$$S_{CS} = \frac{1}{4\pi} \int_{\Sigma \times R} \left[AdA + (1-2s) Ad\omega + \left(\frac{(1-2s)^2}{4} - \frac{1}{12} \right) \omega d\omega \right]$$

[Wen-Zee'92, Abanov-Gromov'15, Gromov-Cho-You-Abanov-Fradkin'15, Son'13, Bradlyn-Read'14]

These two actions are obviously very similar, but one is 2d and another one is in 3d. What is the precise relation?

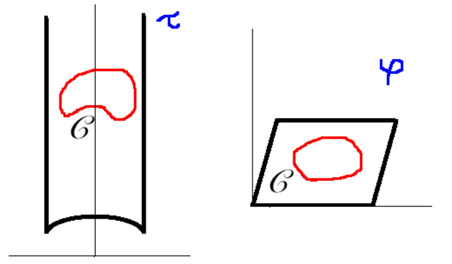
Results: relation to Chern-Simons theory

Consider geometric adiabatic transport of IQHE wave function along a contour $\mathcal{C}$ in the moduli space Y
Define adiabatic connection:

$$\mathcal{A}_y = \langle \Psi, d_y \Psi \rangle_{L^2},$$

and adiabatic phase:

$$e^{i \int_{\mathcal{C}} \mathcal{A}_y}.$$



Theorem ([SK-Ma-Marinescu-Wiegmann'15]):

$$\int_{\mathcal{C}} \mathcal{A}_Y = \frac{1}{4\pi} \int_{\Sigma_Y \times \mathcal{C}} \left[AdA + (1-2s)Ad\omega + \left(\frac{(1-2s)^2}{4} - \frac{1}{12} \right) \omega d\omega \right]$$

(proof relies on Bismut-Gillet-Soulé formula for Quillen anomaly)

Results: Partition function for Laughlin states

$$\log Z = \frac{1}{2\pi\beta}\int_\Sigma \left[A_z A_{\bar z} + \frac{\beta - 2s}{2}(A_z\omega_{\bar z} + \omega_z A_{\bar z}) + \left(\frac{(\beta-2s)^2}{4} - \frac{\beta}{12}\right)\omega_z\omega_{\bar z}\right] d^2 z$$
$$+ \mathcal{F}[B,R].$$

$$\mathcal{F}[B,R] = -\frac{1}{2\pi}\int_\Sigma \left[\frac{2-\beta}{2\beta} B\log B\right]\sqrt{g}d^2z + \ldots$$

[Can-Laskin-Wiegmann'14;Ferrari-SK'14, SK-Wiegmann'15]

Results: Adiabatic curvature

Let's focus on anomalous part

$$\log Z_H = \frac{1}{2\pi}\int_\Sigma \left[\sigma_H A_z A_{\bar z} + 2\eta_H(A_z\omega_{\bar z} + \omega_z A_{\bar z}) - \frac{c_H}{12}\omega_z\omega_{\bar z}\right] d^2 z,$$

which consists of three terms with three coefficients

$$\sigma_H = \frac{1}{\beta}, \quad \eta_H = \frac{\beta - 2s}{4\beta}, \quad c_H = 1 - 3\frac{(\beta-2s)^2}{\beta}$$

The adiabatic curvature for AB-phases on torus $\delta_\varphi A_z = d\varphi, \delta_\varphi A_{\bar z} = d\bar\varphi$

$$\mathcal{R}_{rr'} = d_\varphi d_{\bar\varphi}\log Z_H \delta_{rr'} = \sigma_H d\varphi \wedge d\bar\varphi \delta_{rr'}$$

First Chern number $c_1 = Tr\int \mathcal{R}_{rr'} = \beta\sigma_H = 1.$
Adiabatic curvature for the moduli τ

$$\mathcal{R}_{rr'} = \delta_{rr'} d_\tau d_{\bar\tau}\log Z_H = \eta_H \delta_{rr'}\frac{d\tau\wedge d\bar\tau}{(\tau-\bar\tau)^2}$$

where η_H is anomalous (Hall) viscosity.

New transport coefficient on higher genus

Consider complex structure deformations $g_{z\bar{z}}|dz|^2 \to g_{z\bar{z}}|dz+\mu d\bar{z}|^2$, where Beltrami differential is $\mu = g_{z\bar{z}}^{-1}\sum_{\kappa=1}^{3g-3}\eta_\kappa \delta y_\kappa$ and η_κ is a basis of holomorphic quadratic differentials.
Adiabatic curvature, associated with these deformations is

$$\mathcal{R} = i d_y d_{\bar{y}} \log Z = \left(\eta_H N_\Phi - \frac{c_H}{24}\chi(\Sigma)\right)\Omega_{WP},$$

where $\Omega_{WP} = i\int_M d_y\mu \wedge d_{\bar{y}}\bar{\mu}\; g_{z\bar{z}} d^2 z$ is the Weil-Petersson form on the moduli space. Here [SK-Wiegmann'15] (see also [Bradlyn-Read'15])

$$c_H = 1 - 3\frac{(\beta-2s)^2}{\beta}$$

is a new transport coefficient, transpiring on higher genus surfaces. On singular surfaces (like e.g. cone) it becomes most important!

Lowest Landau level (LLL) and IQHE state

Consider compact connected Riemann surface (Σ, g, J) and positive holomorphic line bundle L^{N_Φ}. The latter corresponds to the magnetic field. The curvature form of the hermitian metric $h^{N_\Phi}(z, \bar{z})$ is given by $F = -i\partial\bar{\partial}\log h^{N_\Phi}$. This is the magnetic field strength of total flux N_Φ though the surface $\frac{1}{2\pi}\int_\Sigma F = N_\Phi$. Magnetic field: $B = g^{z\bar{z}}F_{z\bar{z}}$. On the plane and for constant magnetic field $B = N_\Phi$, this corresponds to $h^{N_\Phi} = e^{-\frac{B}{2}|z|^2}$. LLL wave functions

$$\bar{\partial}\psi = 0$$

are holomorphic sections of L^{N_Φ},

$$\psi_i = s_i(z), \quad i = 1, \ldots, N = N_\Phi + 1 - \mathrm{g}$$

IQHE state: take N_k points on Σ: $z_1, z_2, \ldots, z_{N_k}$. The (holomorphic part F of the) IQHE state is Slater determinant:

$$F(z_1, \ldots, z_{N_k}) = \det[s_i(z_j)]_{i,j=1}^{N}$$

Definition of Laughlin state (FQHE)

Consider now line bundle L^{N_Φ}. But take $N = \frac{1}{\beta} N_\Phi + 1 - \mathrm{g}$ particles, i.e. only fraction of LLL states is occupied (thus *fractional* QHE). The (holomorphic part F of the) Laughlin state satisfies

- $F(z_1, ..., z_N)$ is completely anti-symmetric
- Fix all z_j except one, say z_m. Then $F(\cdot, .., \cdot, z_m, \cdot, .., \cdot)$ is a holomorphic section of L^{N_Φ}.
- Vanishing condition near diagonal $z_i \sim z_j$ in local complex coordinate system on Σ,

$$F(z_1, ..., z_N) \sim \prod_{i<j} (z_i - z_j)^\beta$$

.

Examples

1. Round sphere S^2, constant magnetic field: $h_0^{N_\Phi} = \frac{1}{(1+|z|^2)^{N_\Phi}}$
FQHE state: $F(z_1, ..., z_N) = \prod_{i<j}(z_i - z_j)^\beta$

$$|\Psi(z_1, ..., z_N)|^2 = \prod_{i<j} |z_i - z_j|^{2\beta} \prod_{j=1}^{N} h_0^{N_\Phi}(z_j)$$

[Haldane'83]

2. Flat torus, constant magnetic field: $h_0^{N_\Phi} = e^{-2\pi i N_\Phi \frac{(z-\bar{z})^2}{\tau-\bar{\tau}}}$. FQHE states:

$$F_r(z_1, .., z_N) = \theta_{\frac{r}{\beta},0}(\beta z_c + \varphi, \beta\tau) \prod_{i<j} \left(\frac{\theta_1(z_i - z_j, \tau)}{\eta(\tau)} \right)^\beta$$

[Haldane-Rezayi'85]

3. Higher genus $\Sigma_{\mathrm{g}>1}$: β^{g} states.

Arbitary metric and inhomogeneous magnetic field

The advantage of the language of holomorphic line bundles is that it gives us a clear idea how to put the Laughlin state on Σ with arbitrary metric g and inhomogeneous magnetic field B. Consider some fixed (constant scalar curvature) metric g_0, and constant magnetic field B_0 (and corresponding hermitian metric $h_0^{N_\Phi}(z,\bar z)$). Arbitrary metrics are parameterized by:

- Kähler potential $\phi(z,\bar z)$: $g_{z\bar z} = g_{0\bar z z} + \partial_z \bar\partial_{\bar z}\phi$,
- "magnetic" potential $\psi(z,\bar z)$: $F = F_0 + \partial\bar\partial\psi$, $B = g^{z\bar z}F_{z\bar z}$

Partition function

For the integer QHE ($\beta = 1$), the partition function on arbitrary Σ is

$$Z = \int_{\Sigma^N} |\det s_i(z_j)|^2 \prod_{j=1}^{N} h_0^{N_\Phi}(z_j, \bar{z}_j) e^{-N_\Phi \psi(z_j, \bar{z}_j)} \sqrt{g}^{1-s}(z_j) d^2 z_j$$

For the fractional QHE

$$Z = \sum_{r=1}^{\beta^{\mathrm{g}}} \int_{\Sigma^N} |F_r(z_1, .., z_N)|^2 \prod_{j=1}^{N} h_0^{N_\Phi}(z_j, \bar{z}_j) e^{-N_\Phi \psi(z_j, \bar{z}_j)} \sqrt{g}^{1-s}(z_j) d^2 z_j.$$

Derivation of $\log Z$ in IQHE

For $\beta = 1$ the partition function satisfies determinantal formula:

$$
\begin{aligned}
Z &= \int_{\Sigma^N} |\det s_i(z_j)|^2 \prod_{j=1}^{N} h_0^{N_\Phi}(z_j, \bar z_j) e^{-N_\Phi \psi(z_j, \bar z_j)} \sqrt{g}(z_j) d^2 z_j \\
&= \det \langle s_i, s_j \rangle_{L^2}
\end{aligned}
$$

Denoting $G_{jl} = \langle s_j, s_l \rangle$, we get

$$
\begin{aligned}
\delta \log Z = \delta \operatorname{Tr} \log \langle s_j, s_l \rangle = \\
&= -\frac{1}{2\pi} \sum_{j,l} G^{-1}_{lj} \int_\Sigma \left(\frac{s-1}{2} (\Delta_g \delta\phi) + N_\Phi \delta\psi \right) \bar s_j s_l h^{N_\Phi} \sqrt{g}^{1-s} d^2 z \\
&= -\frac{1}{2\pi} \int_\Sigma \left(\frac{s-1}{2} (\Delta_g B(z, \bar z))\, \delta\phi + N_\Phi B(z, \bar z)\, \delta\psi \right) \sqrt{g}^{1-s} d^2 z,
\end{aligned}
$$

where $B(z, \bar z)$ is the density of states function (Bergman kernel).

Bergman kernel

B is the Bergman kernel on the diagonal. For orthonormal basis of LLL wave functions $\{\psi_j\}$:

$$B(z,\bar{z}) = \sum_{i=1}^{N} |\psi_i|^2 =$$
$$= B + \frac{1-2s}{4} R + \frac{1}{4}\Delta_g \log B + \frac{1}{12}\Delta_g (B^{-1}R) + \mathcal{O}(1/B^2)\,.$$

[Zelditch'98, Catlin'99]

Path integral derivation:

$$B(z,\bar{z}) = \sum_{i=1}^{N} |\psi_i(z)|^2 = \lim_{T\to\infty} \int_{x(0)=z}^{x(T)=z} e^{-\int_0^T (\dot{x}^2 + A\dot{x})dt} \mathcal{D}x(t)$$

[Douglas, SK'09]

$\log Z$ in Integer QHE

$$\log Z = \frac{1}{2\pi}\int_\Sigma \Big[A_z A_{\bar z} + \frac{1-2s}{2}(A_z\omega_{\bar z} + \omega_z A_{\bar z}) + \\ + \left(\frac{(1-2s)^2}{4} - \frac{1}{12}\right)\omega_z\omega_{\bar z}\Big] d^2 z + \mathcal{F}[B,R].$$

The last term is the Liouville action. CFT partition function transforms within the conformal class $g = e^{2\sigma} g_0$

$$\log \frac{Z^{CFT}(g)}{Z^{CFT}(g_0)} = -\frac{c}{12\pi} S_L(\sigma) = -\frac{c}{24\pi}\int_\Sigma \omega_z\omega_{\bar z} d^2 z$$

where c is central charge. What we derived for Laughlin state is the mixed electromagnetic-gravitational anomaly (in Coulomb gas). Since the theory is not conformal (there is a scale, magnetic area: $l^2 \sim V/N_\Phi$) we now have infinite asymptotic expansion.

$\log Z$ for Laughlin states: derivation

The proof is based on the free field representation of Laughlin states

$$\sum_r^{\beta^{\mathrm{g}}} |\Psi_r|^2 = \int e^{i\sqrt{\beta}X(z_1)} \dots e^{i\sqrt{\beta}X(z_N)} e^{-\frac{1}{2\pi}S(g,X)} \mathcal{D}_g X \quad \text{[Moore-Read'91]}$$

where sum goes over all degenerate Laughlin states on Riemann surface and the free field action is

$$S = \int_M \left(\partial X \bar{\partial} X + i\frac{\beta - 2s}{\sqrt{\beta}} X R\sqrt{g} + \frac{i}{\sqrt{\beta}} A \wedge dX\right)$$

for compactified boson: $X \sim X + 2\pi\sqrt{\beta}$.
Novelty: "background charge" $Q = \frac{\beta-2s}{\sqrt{\beta}}$, gauge connection coupling.

$\log Z$ for Laughlin states: derivation

Step 1. The "anomalous part" of the expansion comes from transformation properties under the deformation of the metric and the magnetic field $g_0 \to g = g_0 + \partial_z \partial_{\bar{z}} \phi, A_0 \to A = A_0 + \partial \psi$,

$$\int e^{i\sqrt{\beta}X(z_1)} \dots e^{i\sqrt{\beta}X(z_N)} e^{-\frac{1}{2\pi}S(g,X)} \mathcal{D}_g X$$
$$= e^{S_{\rm ano}} \int e^{i\sqrt{\beta}X(z_1)} \dots e^{i\sqrt{\beta}X(z_N)} e^{-\frac{1}{2\pi}S(g_0,X)} \mathcal{D}_{g_0} X$$

Step 2. The remainder term $\mathcal{F}[R, B]$ of the expansion of $\log Z$ comes from the interacting path integral

$$\frac{1}{\Gamma(s)} \int_0^\infty d\mu \, \mu^{s-1} \int e^{-\frac{1}{2\pi}S(g,X) - \mu \int_M e^{i\sqrt{\beta}X(z)} \sqrt{g} d^2 z} \mathcal{D}_g X \, ,$$

at $s = -N$.

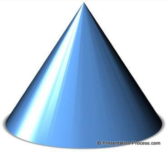
[Ferrari-SK(JHEP2014)]

$\log Z$ for Laughlin states

$$\log Z = \frac{1}{2\pi\beta}\int_\Sigma \left[A_z A_{\bar z} + \frac{\beta - 2s}{2}(A_z\omega_{\bar z} + \omega_z A_{\bar z}) + \left(\frac{(\beta-2s)^2}{4} - \frac{\beta}{12}\right)\omega_z\omega_{\bar z}\right] d^2 z$$
$$- \frac{1}{2\pi}\int_\Sigma \left[\frac{2-\beta}{2\beta} B\log B\right]\sqrt{g}d^2 z + \ldots$$

Singular surfaces

(recent results, see also Laskin-Chiu-Can-Wiegmann, arXiv:1602.04802)
New coefficient $c_H = 1 - 3\frac{(\beta-2s)^2}{\beta}$ transpires also on singular surfaces.



Recall (Cardy-Peschel'88) that the free energy of a 2d system of size L on a surface Σ at criticality has the form

$$F = AL^2 + BL - \frac{c\chi(\Sigma)}{6} \log L + O(1).$$

Moreover, for a cone with angle $0 < \alpha < 1$

$$F = AL^2 + BL - \frac{c\chi(\Sigma)}{12}(\alpha + \frac{1}{\alpha}) \log L + O(1).$$

Singular surfaces

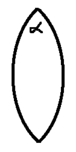
Similar formula holds for $\log Z$ for Laughlin states on singular surfaces

$$\log Z = AN_\Phi^2 + BN_\Phi - \frac{c_H \chi(\Sigma)}{12}(\alpha + \frac{1}{\alpha}) \log N_\Phi + O(1),$$

The $O(1)$ term here is very interesting. Recall that in the smooth case it was given by gravitational anomaly (Liouville action) $O(1) = \frac{c_H}{12\pi} S_L$. Here, in the integer QHE case, it is given by the regularized determinant of laplacian on the cone:

$$O(1) = -\frac{1}{2} \log \det \Delta_{\mathrm{cone}}$$

This can be checked by explicit calculation, e.g. in the case of sphere with two antipodal singularities (american football/spindle)



Question

$$O(1) = -\frac{1}{2}\log\det\Delta_{\mathrm{cone}} \sim \zeta_2'(0,\alpha,1,\alpha),$$

where ζ_2 is Barnes double zeta function.

What is the answer for FQHE (Laughlin state)?

Conjecture: quantum Liouville theory.

Thank you