Übungen (2)

Integralberechnungen

1) Erste Integrale: S. 49, Aufgabe 7

7.
Berechne das Integral.

a) $\int_0^2 (3x^5 - 2x^4 + 1)\,dx$
b) $\int_1^3 \frac{1-5x^4}{4}\,dx$
c) $\int_{-1}^1 (x^6 + 3x^5 - 2x^4)\,dx$
d) $\int_{-2}^{-1} \frac{3-2x^2+4x^3-3x^4}{x^2}\,dx$
e) $\int_1^2 \left(\frac{2}{x^2} - 7 + 5x^4\right)dx$
f) $\int_1^4 \left(\frac{3}{\sqrt{x}} - \frac{1}{2}x^4\right)dx$
g) $\int_1^4 \frac{2x^3 - 5\sqrt{x}}{x}\,dx$
h) $\int_1^2 \frac{6x^6 + 8x\cdot\sqrt{x} - 1}{2x^2}\,dx$
i) $\int_1^2 \frac{2x^2 - 7\sqrt{x}}{x}\,dx$

2) Symmetrische Integrale: S. 49, Aufgabe 8

8.
Berechne das Integral.

a) $\int_{-1}^{+1} (4x^3 + 2x)\,dx$
b) $\int_{-2}^{+2} (2x^2 - 4)\,dx$
c) $\int_{-1}^{+1} (x^5 - 3x^3)\,dx$
d) $\int_{-3}^{+3} (3x^2 + 5x^4)\,dx$
e) $\int_{-2}^{+2} (4x^5 - 3)\,dx$
f) $\int_{-4}^{+4} (x^3 - x^2)\,dx$

Lineare Substitution

3) Stammfunktionen: S. 50, Aufgabe 1; S. 51, Aufgabe 3 a)–e)

1.
Bestimme zur Funktion f eine Stammfunktion.

a) $f(x) = (3x-4)^4$
b) $f(x) = \frac{(2-5x)^3}{3}$
c) $f(x) = \frac{-2}{(5x+1)^2}$
d) $f(x) = \frac{4}{\sqrt{3x-1}}$

3.
Bestimme zur Funktion f eine Stammfunktion

a) $f(x) = (x-5)^4$; $f(x) = (2x+3)^2$; $f(x) = (\frac{1}{2}x - 4)^4$; $f(x) = (2-3x)^4$
b) $f(x) = 5\cdot(3x+7)^3$; $f(x) = 0{,}5\cdot(2-x)^3$; $f(x) = \frac{(2x+2)^4}{3}$; $f(x) = \sqrt{2}\cdot(\sqrt{3}x+2)^3$
c) $f(x) = \frac{1}{4}(4x-1)^3 + \frac{1}{2}(1-4x)^2$; $f(x) = \frac{1}{3}(3-0{,}5x)^4 - \frac{1}{5}(2x+0{,}3)^3$
d) $f(x) = \frac{1}{(x-4)^2}$; $f(x) = \frac{3}{(2x+7)^2}$; $f(x) = \frac{0{,}5}{(3-0{,}5x)^2}$; $f(x) = \frac{\sqrt{5}}{(\sqrt{2}\cdot x+4)^2}$
e) $f(x) = \frac{1}{\sqrt{x+4}}$; $f(x) = \frac{2}{\sqrt{5+3x}}$; $f(x) = \frac{1}{4\sqrt{1-x}}$; $f(x) = \frac{3}{\sqrt{2x}}$

4) Integrale: S. 51, Aufgabe 4, a)–g), i)–l)

4.
Wende lineare Substitution an.

a) $\int_1^2 (4x+1)^2\,dx$
b) $\int_{-1}^{+1} (\frac{1}{3}x - 4)^3\,dx$
c) $\int_{-1}^{-4} (-3x+2)^4\,dx$
d) $\int_{+1}^{+2} \frac{1}{(3x+4)^2}\,dx$
e) $\int_{-1}^{-4} \frac{1}{(2x-1)^2}\,dx$
f) $\int_0^4 \frac{1}{\sqrt{2x+1}}\,dx$
g) $\int_{-1}^0 \frac{5}{(3-2x)^2}\,dx$
h) $\int_1^2 \left(\sqrt{6-3x}\right)^{-1}dx$
i) $\int_0^2 ((2x+1)^2 + (2x+1)^3)\,dx$
j) $\int_0^1 (3(4x-3)^2 + 2(4x-3)^4)\,dx$
k) $\int_0^1 \left(\frac{1}{(3x+2)^2} + \frac{1}{\sqrt{3x+2}}\right)dx$
l) $\int_1^2 \left(\frac{4}{\sqrt{2x}} - (2x)^4\right)dx$

Welches Problem tritt bei Aufgabe h) auf?

Übungen (2) — Lösungen

1) **S. 49, Aufgabe 7:**

a) $\displaystyle\int_0^2 (3x^5 - 2x^4 + 1)\,dx = \Big[\frac{1}{2}x^6 - \frac{2}{5}x^5 + x\Big]_0^2$
$\displaystyle= \big(\frac{1}{2}\cdot 2^6 - \frac{2}{5}\cdot 2^5 + 2\big) - \big(0\big) = \frac{106}{5},$

b) $\displaystyle\int_1^3 \frac{1-5x^4}{4}\,dx = \Big[\frac{1}{4}\cdot(x - x^5)\Big]_1^3 = \frac{1}{4}(3-3^5) - \frac{1}{4}(1-1^5) = \frac{-240}{4} = -60,$

c) $\displaystyle\int_{-1}^1 (x^6 + 3x^5 - 2x^4)\,dx = \Big[\frac{1}{7}x^7 + \frac{1}{2}x^6 - \frac{2}{5}x^5\Big]_{-1}^1$
$\displaystyle= \big(\frac{1}{7} + \frac{1}{2} - \frac{2}{5}\big) - \big(-\frac{1}{7} + \frac{1}{2} + \frac{2}{5}\big) = \frac{2}{7} - \frac{4}{5} = -\frac{18}{35}.$

Bei den nachfolgenden Aufgabenteilen sind die Integranden nicht überall definiert, aber es liegen keine Definitionslücken im Integrationsintervall, so dass die Integrale wohldefiniert sind.

d) $\displaystyle\int_{-2}^{-1} \frac{3-2x^2+4x^3-3x^4}{x^2}\,dx = \int_{-2}^{-1}(3x^{-2} - 2 + 4x - 3x^2)\,dx$
$\displaystyle= \Big[-\frac{3}{x} - 2x + 2x^2 - x^3\Big]_{-2}^{-1} = \big(3+2+2+1\big) - \big(\frac{3}{2}+4+8+8\big) = -\frac{27}{2},$

e) $\displaystyle\int_1^2 \big(\frac{2}{x^2} - 7 + 5x^4\big)\,dx = \int_1^2 (2x^{-2} - 7 + 5x^4)\,dx$
$\displaystyle= \Big[-2x^{-1} - 7x + x^5\Big]_1^2 = (-1-14+32) - (-2-7+1) = 25,$

f) $\displaystyle\int_1^4 \big(\frac{3}{\sqrt{x}} - \frac{1}{2}x^4\big)\,dx = \int_1^4 (3x^{-\frac{1}{2}} - \frac{1}{2}x^4)\,dx$
$\displaystyle= \Big[6x^{\frac{1}{2}} - \frac{1}{10}x^5\Big]_1^4 = (12 - \frac{1024}{10}) - (6 - \frac{1}{10}) = -\frac{963}{10},$

g) $\displaystyle\int_1^4 \frac{2x^3 - 5\sqrt{x}}{x}\,dx = \int_1^4 (2x^2 - 5x^{-\frac{1}{2}})\,dx$
$\displaystyle= \Big[\frac{2}{3}x^3 - 10x^{\frac{1}{2}}\Big]1^4 = (\frac{128}{3} - 20) - (\frac{2}{3} - 10) = 32,$

h) $\displaystyle\int_1^2 \frac{6x^6 + 8x\sqrt{x} - 1}{2x^2}\,dx = \int_1^2 (3x^4 + 4x^{-\frac{1}{2}} - \frac{1}{2}x^{-2})\,dx$
$\displaystyle= \Big[\frac{3}{5}x^5 + 8x^{\frac{1}{2}} + \frac{1}{2}x^{-1}\Big]1^2 = (\frac{96}{5} + 8\sqrt{2} + \frac{1}{4}) - (\frac{3}{5} + 8 + \frac{1}{2}) = \frac{207}{20} + 8\sqrt{2},$

i) $\displaystyle\int_1^2 \frac{2x^2 - 7\sqrt{x}}{x}\,dx = \int_1^2 (2x - 7x^{-\frac{1}{2}})\,dx$
$\displaystyle= \Big[x^2 - 14x^{\frac{1}{2}}\Big]_1^2 = (4 - 14\sqrt{2}) - (1 - 14) = 17 - 14\sqrt{2},$

2) **S. 49, Aufgabe 8:** Diese Integrale kann man genauso wie in der vorangehenden

Aufgabe mit Hilfe der Integralformel berechnen. Es fällt jedoch auf, dass in allen Fällen die Integralgrenzen symmetrisch zu 0 liegen. In solchen Fällen kann man Symmetrien der Integranden ausnutzen (siehe gesondertes Übungsblatt).

a) $\int_{-1}^{1} (4x^3 + 2x)\, dx = \left[x^4 + x^2\right]_{-1}^{1} = 2 - 2 = 0,$

b) $\int_{-2}^{2} (2x^2 - 4)\, dx = \left[\frac{2}{3}x^3 - 4x\right]_{-2}^{2} = (\frac{16}{3} - 8) - (-\frac{16}{3} + 8) = \frac{32}{3} - 16 = -\frac{16}{3}.$

c) $\int_{-1}^{1} (x^5 - 3x^3)\, dx = \left[\frac{1}{6}x^6 - \frac{3}{4}x^4\right]_{-1}^{1} = (\frac{1}{6} - \frac{3}{4}) - (\frac{1}{6} - \frac{3}{4}) = 0,$

d) $\int_{-3}^{3} (3x^2 + 5x^4)\, dx = \left[x^3 + x^5\right]_{-3}^{3} = (3^3 + 3^5) - (-3^3 - 3^5)$

$= 2 \cdot (3^3 + 3^5) = 2 \cdot 3^3 \cdot (1 + 3^2) = 540,$

e) $\int_{-2}^{2} (4x^5 - 3)\, dx = \left[\frac{2}{3}x^6 - 3x\right]_{-2}^{2} = (\frac{128}{3} - 6) - (\frac{128}{3} + 6) = -12,$

f) $\int_{-4}^{4} (x^3 - x^2)\, dx = \left[\frac{1}{4}x^4 - \frac{1}{3}x^3\right]_{-4}^{4} = (4^3 - \frac{1}{3} \cdot 4^3) - (4^3 + \frac{1}{3} \cdot 4^3) = -\frac{128}{3},$

3) **S. 50, Aufgabe 1:**

a) $f(x) = (3x-4)^4, \quad F(x) = \frac{1}{5}(3x-4)^5 \cdot \frac{1}{3} = \frac{1}{15}(3x-4)^5,$

b) $f(x) = \frac{1}{3}(2-5x)^3, \quad F(x) = \frac{1}{3} \cdot \frac{1}{4}(2-5x)^4 \cdot \frac{1}{-5} = -\frac{1}{60}(2-5x)^4,$

c) $f(x) = \frac{-2}{(5x+1)^2} = -2(5x+1)^{-2}, \quad F(x) = -2 \cdot \frac{1}{-1} \cdot (5x+1)^{-1} \cdot \frac{1}{5} = \frac{2}{5(5x+1)},$

d) $f(x) = \frac{4}{\sqrt{3x-1}} = 4(3x-1)^{-\frac{1}{2}}, \quad F(x) = 4 \cdot 2 \cdot (3x-1)^{\frac{1}{2}} \cdot \frac{1}{3} = \frac{8}{3}\sqrt{3x-1},$

S. 51, Aufgabe 3 a)–e):

a) $F(x) = \frac{1}{5}(x-5)^5;$

$F(x) = \frac{1}{3}(2x+3)^3 \cdot \frac{1}{2} = \frac{1}{6}(2x+3)^3;$

$F(x) = \frac{1}{5}(\frac{1}{2}x - 4)^5 \cdot \frac{1}{\frac{1}{2}} = \frac{2}{5}(\frac{1}{2}x - 4)^5;$

$F(x) = \frac{1}{5}(2-3x)^5 \cdot \frac{1}{-3} = -\frac{1}{15}(2-3x)^5;$

b) $F(x) = \frac{5}{4}(3x+7)^4 \cdot \frac{1}{3} = \frac{5}{12}(3x+7)^4,$

$F(x) = \frac{1}{8}(2-x)^4 \cdot \frac{1}{-1} = -\frac{1}{8}(2-x)^4,$

$F(x) = \frac{1}{15}(2x+2)^5 \cdot \frac{1}{2} = \frac{1}{30}(2x+2)^5,$

$F(x) = \sqrt{2} \cdot \frac{1}{4}(\sqrt{3} \cdot x + 2)^4 \cdot \frac{1}{\sqrt{3}},$

c) $F(x) = \frac{1}{4}\cdot\frac{1}{4}(4x-1)^4\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{1}{3}(1-4x)^3\cdot\frac{1}{-4} = \frac{1}{64}\cdot(4x-1)^4 - \frac{1}{24}(1-4x)^3,$

$$F(x) = \frac{1}{3}\cdot\frac{1}{5}(3-0{,}5x)^5\cdot\frac{1}{-0{,}5} - \frac{1}{5}\cdot\frac{1}{4}(2x+0{,}3)^4\cdot\frac{1}{2} = $$
$$= -\frac{2}{15}\cdot(3-0{,}5x)^5 - \frac{1}{40}\cdot(2x+0{,}3)^4,$$

d) $f(x) = \frac{1}{(x-4)^2} = (x-4)^{-2}, \quad F(x) = \frac{1}{-1}\cdot(x-4)^{-1} = -\frac{1}{x-4},$

$$f(x) = \frac{3}{(2x+7)^2} = 3(2x+7)^{-2}, \quad F(x) = -3\cdot(2x+7)^{-1}\cdot\frac{1}{2} = -\frac{3}{2(2x+7)},$$
$$f(x) = \frac{0{,}5}{(3-0{,}5x)^2} = 0{,}5\cdot(3-0{,}5x)^{-2},$$
$$F(x) = -0{,}5\cdot(3-0{,}5x)^{-1}\cdot\frac{1}{-0{,}5} = \frac{1}{3-0{,}5x},$$
$$f(x) = \frac{\sqrt{5}}{(\sqrt{2}\cdot x+4)^2} = \sqrt{5}\cdot(\sqrt{2}\cdot x+4)^{-2},$$
$$F(x) = -\sqrt{5}\cdot(\sqrt{2}\cdot x+4)^{-1}\cdot\frac{1}{\sqrt{2}} = -\frac{\sqrt{5}}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}\cdot x+4},$$

e) $f(x) = \frac{1}{\sqrt{x+4}} = (x+4)^{-\frac{1}{2}}, \quad F(x) = 2\cdot(x+4)^{\frac{1}{2}} = 2\sqrt{x+4},$

$$f(x) = \frac{2}{\sqrt{5+3x}} = 2\cdot(5+3x)^{-\frac{1}{2}}, \quad F(x) = 2\cdot 2(5+3x)^{\frac{1}{2}}\cdot\frac{1}{3} = \frac{4}{3}\cdot\sqrt{5+3x},$$
$$f(x) = \frac{1}{4\sqrt{1-x}} = \frac{1}{4}\cdot(1-x)^{-\frac{1}{2}}, \quad F(x) = \frac{1}{4}\cdot 2(1-x)^{\frac{1}{2}}\cdot\frac{1}{-1} = -\frac{1}{2}\sqrt{1-x},$$
$$f(x) = \frac{3}{\sqrt{2x}} = 3\cdot(2x)^{-\frac{1}{2}}, \quad F(x) = 3\cdot 2(2x)^{\frac{1}{2}}\cdot\frac{1}{2} = 3\sqrt{2x}.$$

4) **S. 51, Aufgabe 4:**

a) $\int_1^2 (4x+1)^2\,dx = \Big[\frac{1}{3}(4x+1)^3\cdot\frac{1}{4}\Big]_1^2 = \frac{9^3}{12} - \frac{5^3}{12} = \frac{151}{3},$

b) $\int_{-1}^1 (\frac{1}{3}x-4)^3\,dx = \Big[\frac{1}{4}(\frac{1}{3}x-4)^4\cdot 3\Big]_{-1}^1$

$$= \frac{3}{4}\cdot\big((\frac{1}{3}-4)^4 - (-\frac{1}{3}-4)^4\big) = -\frac{1160}{9},$$

c) $\int_{-1}^{-4} (-3x+2)^4\,dx = \Big[\frac{1}{5}(-3x+2)^5\cdot\frac{1}{-3}\Big]_{-1}^{-4} = -\frac{1}{15}(14^5-5^5) = -\frac{178233}{5},$

d) $\int_1^2 \frac{1}{(3x+4)^2}\,dx = \int_1^2 (3x+4)^{-2}\,dx$

$$= \Big[-(3x+4)^{-1}\cdot\frac{1}{3}\Big]_1^2 = -\frac{1}{3}\cdot(\frac{1}{10}-\frac{1}{7}) = \frac{1}{70},$$

e) $\int_{-1}^{-4} \frac{1}{(2x-1)^2}\,dx = \int_{-1}^{-4} (2x-1)^{-2}\,dx$

$$= \Big[-(2x-1)^{-1}\cdot\frac{1}{2}\Big]_{-1}^{-4} = \frac{1}{2}\cdot(\frac{1}{9}-\frac{1}{3}) = -\frac{1}{9},$$

f) $\displaystyle\int_0^4 \frac{1}{\sqrt{2x+1}}\,dx = \int_0^4 (2x+1)^{-\frac{1}{2}}\,dx = \Big[2\cdot(2x+1)^{\frac{1}{2}}\cdot\frac{1}{2}\Big]_0^4 = \sqrt{9}-\sqrt{1} = 2,$

g) $\displaystyle\int_{-1}^0 \frac{5}{(3-2x)^2}\,dx = \int_{-1}^0 5(3-2x)^{-2}\,dx = \Big[5\cdot(-(3-2x)^{-1})\cdot\frac{1}{-2}\Big]_{-1}^0$

$\displaystyle= \frac{5}{2}\cdot\Big[(3-2x)^{-1}\Big]_{-1}^0 = \frac{5}{2}\cdot(\frac{1}{3}-\frac{1}{5}) = \frac{1}{3},$

i) $\displaystyle\int_0^2 \big((2x+1)^2+(2x+1)^3\big)\,dx = \Big[\frac{1}{3}(2x+1)^3\cdot\frac{1}{2}+\frac{1}{4}(2x+1)^4\cdot\frac{1}{2}\Big]_0^2$

$\displaystyle= (\frac{5^3}{6}+\frac{5^4}{8}) - (\frac{1}{6}+\frac{1}{8}) = 5^3\cdot(\frac{1}{6}+\frac{5}{8}) - \frac{7}{24} = \frac{296}{3},$

j) $\displaystyle\int_0^1 \big(3(4x-3)^2+2(4x-3)^4\big)\,dx = \Big[(4x-3)^3\cdot\frac{1}{4}+\frac{2}{5}(4x-3)^5\cdot\frac{1}{4}\Big]_0^1$

$\displaystyle= (\frac{1}{4}+\frac{1}{10}) - (-\frac{3^3}{4}-\frac{3^5}{10}) = \frac{157}{5},$

k) $\displaystyle\int_0^1 (\frac{1}{(3x+2)^2}+\frac{1}{\sqrt{3x+2}})\,dx = \int_0^1 \big((3x+2)^{-2}+(3x+2)^{-\frac{1}{2}}\big)\,dx$

$\displaystyle= \Big[-\frac{1}{3}(3x+2)^{-1}+\frac{2}{3}(3x+2)^{\frac{1}{2}}\Big]_0^1 = (-\frac{1}{15}+\frac{2}{3}\sqrt{5}) - (-\frac{1}{6}+\frac{2}{3}\sqrt{2})$

$\displaystyle= \frac{1}{10}+\frac{2}{3}\cdot(\sqrt{5}-\sqrt{2}) \approx 0{,}65,$

l) $\displaystyle\int_1^2 (\frac{4}{\sqrt{2x}}-(2x)^4)\,dx = \int_1^2 \big(4(2x)^{-\frac{1}{2}}-16x^4\big)\,dx = \Big[8(2x)^{\frac{1}{2}}\frac{1}{2}-\frac{16}{5}x^5\Big]_1^2$

$\displaystyle= (8-\frac{16\cdot 32}{5}) - (4\sqrt{2}-\frac{16}{5}) = -\frac{456}{5} - 4\sqrt{2} \approx -96{,}86.$

h) Der Definitionsbereich des Integranden $f(x) = \big(\sqrt{6-3x}\big)^{-1}$ ist $]-\infty, 2[$, insbesondere ist er bei 2 nicht definiert, das Integral $\int_1^2 f(x)\,dx$ daher nicht definiert. Nachfolgend eine Skizze des Graphen und des Integrationsbereiches.