Übungen (13)

Integralberechnungen

1) Erste Integrale: S. 49, Aufgabe 7

7.
Berechne das Integral.

a) $\int_0^2 (3x^5 - 2x^4 + 1)\,dx$ **d)** $\int_{-2}^{-1} \frac{3 - 2x^2 + 4x^3 - 3x^4}{x^2}\,dx$ **g)** $\int_1^4 \frac{2x^3 - 5\sqrt{x}}{x}\,dx$

b) $\int_1^3 \frac{1 - 5x^4}{4}\,dx$ **e)** $\int_1^2 \left(\frac{2}{x^2} - 7 + 5x^4\right)dx$ **h)** $\int_1^2 \frac{6x^6 + 8x \cdot \sqrt{x} - 1}{2x^2}\,dx$

c) $\int_{-1}^{1} (x^6 + 3x^5 - 2x^4)\,dx$ **f)** $\int_1^4 \left(\frac{3}{\sqrt{x}} - \frac{1}{2}x^4\right)dx$ **i)** $\int_1^2 \frac{2x^2 - 7\sqrt{x}}{x}\,dx$

Lineare Substitution

2) Stammfunktionen: S. 50, Aufgabe 1; S. 51, Aufgabe 3 a)–e)

1.
Bestimme zur Funktion f eine Stammfunktion.

a) $f(x) = (3x - 4)^4$ **b)** $f(x) = \frac{(2 - 5x)^3}{3}$ **c)** $f(x) = \frac{-2}{(5x + 1)^2}$ **d)** $f(x) = \frac{4}{\sqrt{3x - 1}}$

3.
Bestimme zur Funktion f eine Stammfunktion

a) $f(x) = (x - 5)^4$; $f(x) = (2x + 3)^2$; $f(x) = (\frac{1}{2}x - 4)^4$; $f(x) = (2 - 3x)^4$

b) $f(x) = 5 \cdot (3x + 7)^3$; $f(x) = 0{,}5 \cdot (2 - x)^3$; $f(x) = \frac{(2x + 2)^4}{3}$; $f(x) = \sqrt{2} \cdot (\sqrt{3}x + 2)^3$

c) $f(x) = \frac{1}{4}(4x - 1)^3 + \frac{1}{2}(1 - 4x)^2$; $f(x) = \frac{1}{3}(3 - 0{,}5x)^4 - \frac{1}{5}(2x + 0{,}3)^3$

d) $f(x) = \frac{1}{(x - 4)^2}$; $f(x) = \frac{3}{(2x + 7)^2}$; $f(x) = \frac{0{,}5}{(3 - 0{,}5x)^2}$; $f(x) = \frac{\sqrt{5}}{(\sqrt{2} \cdot x + 4)^2}$

e) $f(x) = \frac{1}{\sqrt{x + 4}}$; $f(x) = \frac{2}{\sqrt{5 + 3x}}$; $f(x) = \frac{1}{4\sqrt{1 - x}}$; $f(x) = \frac{3}{\sqrt{2x}}$

3) Integrale: S. 51, Aufgabe 4, a)–g), i)–l)

4.
Wende lineare Substitution an.

a) $\int_1^2 (4x + 1)^2\,dx$ **e)** $\int_{-1}^{-4} \frac{1}{(2x - 1)^2}\,dx$ **i)** $\int_0^2 ((2x + 1)^2 + (2x + 1)^3)\,dx$

b) $\int_{-1}^{+1} (\frac{1}{3}x - 4)^3\,dx$ **f)** $\int_0^4 \frac{1}{\sqrt{2x + 1}}\,dx$ **j)** $\int_0^1 (3(4x - 3)^2 + 2(4x - 3)^4)\,dx$

c) $\int_{-1}^{-4} (-3x + 2)^4\,dx$ **g)** $\int_{-1}^{0} \frac{5}{(3 - 2x)^2}\,dx$ **k)** $\int_0^1 \left(\frac{1}{(3x + 2)^2} + \frac{1}{\sqrt{3x + 2}}\right)dx$

d) $\int_{+1}^{+2} \frac{1}{(3x + 4)^2}\,dx$ **h)** $\int_1^2 \left(\sqrt{6 - 3x}\right)^{-1}dx$ **l)** $\int_1^2 \left(\frac{4}{\sqrt{2x}} - (2x)^4\right)dx$

Welches Problem tritt bei Aufgabe h) auf?

Übungen (13) — Lösungen

1) **S. 49, Aufgabe 7:**

a) $\int_0^2 (3x^5 - 2x^4 + 1)\,dx = \Big[\frac{1}{2}x^6 - \frac{2}{5}x^5 + x\Big]_0^2$
$= \big(\frac{1}{2}\cdot 2^6 - \frac{2}{5}\cdot 2^5 + 2\big) - \big(0\big) = \frac{106}{5},$

b) $\int_1^3 \frac{1-5x^4}{4}\,dx = \Big[\frac{1}{4}\cdot(x - x^5)\Big]_1^3 = \frac{1}{4}(3-3^5) - \frac{1}{4}(1-1^5) = \frac{-240}{4} = -60,$

c) $\int_{-1}^1 (x^6 + 3x^5 - 2x^4)\,dx = \Big[\frac{1}{7}x^7 + \frac{1}{2}x^6 - \frac{2}{5}x^5\Big]_{-1}^1$
$= \big(\frac{1}{7} + \frac{1}{2} - \frac{2}{5}\big) - \big(-\frac{1}{7} + \frac{1}{2} + \frac{2}{5}\big) = \frac{2}{7} - \frac{4}{5} = -\frac{18}{35}.$

Bei den nachfolgenden Aufgabenteilen sind die Integranden nicht überall definiert, aber es liegen keine Definitionslücken im Integrationsintervall, so dass die Integrale wohldefiniert sind.

d) $\int_{-2}^{-1} \frac{3 - 2x^2 + 4x^3 - 3x^4}{x^2}\,dx = \int_{-2}^{-1} (3x^{-2} - 2 + 4x - 3x^2)\,dx$
$= \Big[-\frac{3}{x} - 2x + 2x^2 - x^3\Big]_{-2}^{-1} = \big(3+2+2+1\big) - \big(\frac{3}{2} + 4 + 8 + 8\big) = -\frac{27}{2},$

e) $\int_1^2 \big(\frac{2}{x^2} - 7 + 5x^4\big)\,dx = \int_1^2 (2x^{-2} - 7 + 5x^4)\,dx$
$= \Big[-2x^{-1} - 7x + x^5\Big]_1^2 = (-1 - 14 + 32) - (-2 - 7 + 1) = 25,$

f) $\int_1^4 \big(\frac{3}{\sqrt{x}} - \frac{1}{2}x^4\big)\,dx = \int_1^4 (3x^{-\frac{1}{2}} - \frac{1}{2}x^4)\,dx$
$= \Big[6x^{\frac{1}{2}} - \frac{1}{10}x^5\Big]_1^4 = (12 - \frac{1024}{10}) - (6 - \frac{1}{10}) = -\frac{963}{10},$

g) $\int_1^4 \frac{2x^3 - 5\sqrt{x}}{x}\,dx = \int_1^4 (2x^2 - 5x^{-\frac{1}{2}})\,dx$
$= \Big[\frac{2}{3}x^3 - 10x^{\frac{1}{2}}\Big]1^4 = (\frac{128}{3} - 20) - (\frac{2}{3} - 10) = 32,$

h) $\int_1^2 \frac{6x^6 + 8x\sqrt{x} - 1}{2x^2}\,dx = \int_1^2 (3x^4 + 4x^{-\frac{1}{2}} - \frac{1}{2}x^{-2})\,dx$
$= \Big[\frac{3}{5}x^5 + 8x^{\frac{1}{2}} + \frac{1}{2}x^{-1}\Big]1^2 = (\frac{96}{5} + 8\sqrt{2} + \frac{1}{4}) - (\frac{3}{5} + 8 + \frac{1}{2}) = \frac{207}{20} + 8\sqrt{2},$

i) $\int_1^2 \frac{2x^2 - 7\sqrt{x}}{x}\,dx = \int_1^2 (2x - 7x^{-\frac{1}{2}})\,dx$
$= \Big[x^2 - 14x^{\frac{1}{2}}\Big]_1^2 = (4 - 14\sqrt{2}) - (1 - 14) = 17 - 14\sqrt{2},$

2) **S. 50, Aufgabe 1:**

a) $f(x) = (3x-4)^4, \quad F(x) = \frac{1}{5}(3x-4)^5 \cdot \frac{1}{3} = \frac{1}{15}(3x-4)^5,$

b) $f(x) = \frac{1}{3}(2-5x)^3, \quad F(x) = \frac{1}{3} \cdot \frac{1}{4}(2-5x)^4 \cdot \frac{1}{-5} = -\frac{1}{60}(2-5x)^4,$

c) $f(x) = \frac{-2}{(5x+1)^2} = -2(5x+1)^{-2}, \quad F(x) = -2 \cdot \frac{1}{-1} \cdot (5x+1)^{-1} \cdot \frac{1}{5} = \frac{2}{5(5x+1)},$

d) $f(x) = \frac{4}{\sqrt{3x-1}} = 4(3x-1)^{-\frac{1}{2}}, \quad F(x) = 4 \cdot 2 \cdot (3x-1)^{\frac{1}{2}} \cdot \frac{1}{3} = \frac{8}{3}\sqrt{3x-1},$

S. 51, Aufgabe 3 a)–e):

a) $F(x) = \frac{1}{5}(x-5)^5;$

$F(x) = \frac{1}{3}(2x+3)^3 \cdot \frac{1}{2} = \frac{1}{6}(2x+3)^3;$

$F(x) = \frac{1}{5}(\frac{1}{2}x-4)^5 \cdot \frac{1}{\frac{1}{2}} = \frac{2}{5}(\frac{1}{2}x-4)^5;$

$F(x) = \frac{1}{5}(2-3x)^5 \cdot \frac{1}{-3} = -\frac{1}{15}(2-3x)^5;$

b) $F(x) = \frac{5}{4}(3x+7)^4 \cdot \frac{1}{3} = \frac{5}{12}(3x+7)^4,$

$F(x) = \frac{1}{8}(2-x)^4 \cdot \frac{1}{-1} = -\frac{1}{8}(2-x)^4,$

$F(x) = \frac{1}{15}(2x+2)^5 \cdot \frac{1}{2} = \frac{1}{30}(2x+2)^5,$

c) $F(x) = \frac{1}{4} \cdot \frac{1}{4}(4x-1)^4 \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{3}(1-4x)^3 \cdot \frac{1}{-4} = \frac{1}{64} \cdot (4x-1)^4 - \frac{1}{24}(1-4x)^3,$

$F(x) = \frac{1}{3} \cdot \frac{1}{5}(3-0{,}5x)^5 \cdot \frac{1}{-0{,}5} - \frac{1}{5} \cdot \frac{1}{4}(2x+0{,}3)^4 \cdot \frac{1}{2} =$

$= -\frac{2}{15} \cdot (3-0{,}5x)^5 - \frac{1}{40} \cdot (2x+0{,}3)^4,$

d) $f(x) = \frac{1}{(x-4)^2} = (x-4)^{-2}, \quad F(x) = \frac{1}{-1} \cdot (x-4)^{-1} = -\frac{1}{x-4},$

$f(x) = \frac{3}{(2x+7)^2} = 3(2x+7)^{-2}, \quad F(x) = -3 \cdot (2x+7)^{-1} \cdot \frac{1}{2} = -\frac{3}{2(2x+7)},$

$f(x) = \frac{0{,}5}{(3-0{,}5x)^2} = 0{,}5 \cdot (3-0{,}5x)^{-2},$

$F(x) = -0{,}5 \cdot (3-0{,}5x)^{-1} \cdot \frac{1}{-0{,}5} = \frac{1}{3-0{,}5x},$

$f(x) = \frac{\sqrt{5}}{(\sqrt{2} \cdot x + 4)^2} = \sqrt{5} \cdot (\sqrt{2} \cdot x + 4)^{-2},$

$F(x) = -\sqrt{5} \cdot (\sqrt{2} \cdot x + 4)^{-1} \cdot \frac{1}{\sqrt{2}} = -\frac{\sqrt{5}}{\sqrt{2}} \cdot \frac{1}{\sqrt{2} \cdot x + 4},$

e) $f(x) = \frac{1}{\sqrt{x+4}} = (x+4)^{-\frac{1}{2}}, \quad F(x) = 2 \cdot (x+4)^{\frac{1}{2}} = 2\sqrt{x+4},$

$$f(x) = \frac{2}{\sqrt{5+3x}} = 2\cdot(5+3x)^{-\frac{1}{2}}, \quad F(x) = 2\cdot 2(5+3x)^{\frac{1}{2}}\cdot\frac{1}{3} = \frac{4}{3}\cdot\sqrt{5+3x},$$
$$f(x) = \frac{1}{4\sqrt{1-x}} = \frac{1}{4}\cdot(1-x)^{-\frac{1}{2}}, \quad F(x) = \frac{1}{4}\cdot 2(1-x)^{\frac{1}{2}}\cdot\frac{1}{-1} = -\frac{1}{2}\sqrt{1-x},$$
$$f(x) = \frac{3}{\sqrt{2x}} = 3\cdot(2x)^{-\frac{1}{2}}, \quad F(x) = 3\cdot 2(2x)^{\frac{1}{2}}\cdot\frac{1}{2} = 3\sqrt{2x}.$$

3) **S. 51, Aufgabe 4:**

a) $\int_1^2 (4x+1)^2\,dx = \Big[\frac{1}{3}(4x+1)^3\cdot\frac{1}{4}\Big]_1^2 = \frac{9^3}{12} - \frac{5^3}{12} = \frac{151}{3},$

b) $\int_{-1}^1 (\frac{1}{3}x-4)^3\,dx = \Big[\frac{1}{4}(\frac{1}{3}x-4)^4\cdot 3\Big]_{-1}^1$
$= \frac{3}{4}\cdot\big((\frac{1}{3}-4)^4 - (-\frac{1}{3}-4)^4\big) = -\frac{1160}{9},$

c) $\int_{-1}^{-4} (-3x+2)^4\,dx = \Big[\frac{1}{5}(-3x+2)^5\cdot\frac{1}{-3}\Big]_{-1}^{-4} = -\frac{1}{15}(14^5-5^5) = -\frac{178233}{5},$

d) $\int_1^2 \frac{1}{(3x+4)^2}\,dx = \int_1^2 (3x+4)^{-2}\,dx$
$= \Big[-(3x+4)^{-1}\cdot\frac{1}{3}\Big]_1^2 = -\frac{1}{3}\cdot(\frac{1}{10}-\frac{1}{7}) = \frac{1}{70},$

e) $\int_{-1}^{-4} \frac{1}{(2x-1)^2}\,dx = \int_{-1}^{-4} (2x-1)^{-2}\,dx$
$= \Big[-(2x-1)^{-1}\cdot\frac{1}{2}\Big]_{-1}^{-4} = \frac{1}{2}\cdot(\frac{1}{9}-\frac{1}{3}) = -\frac{1}{9},$

f) $\int_0^4 \frac{1}{\sqrt{2x+1}}\,dx = \int_0^4 (2x+1)^{-\frac{1}{2}}\,dx = \Big[2\cdot(2x+1)^{\frac{1}{2}}\cdot\frac{1}{2}\Big]_0^4 = \sqrt{9}-\sqrt{1} = 2,$

g) $\int_{-1}^0 \frac{5}{(3-2x)^2}\,dx = \int_{-1}^0 5(3-2x)^{-2}\,dx = \Big[5\cdot(-(3-2x)^{-1})\cdot\frac{1}{-2}\Big]_{-1}^0$
$= \frac{5}{2}\cdot\Big[(3-2x)^{-1}\Big]_{-1}^0 = \frac{5}{2}\cdot(\frac{1}{3}-\frac{1}{5}) = \frac{1}{3},$

i) $\int_0^2 \big((2x+1)^2+(2x+1)^3\big)\,dx = \Big[\frac{1}{3}(2x+1)^3\cdot\frac{1}{2}+\frac{1}{4}(2x+1)^4\cdot\frac{1}{2}\Big]_0^2$
$= (\frac{5^3}{6}+\frac{5^4}{8}) - (\frac{1}{6}+\frac{1}{8}) = 5^3\cdot(\frac{1}{6}+\frac{5}{8}) - \frac{7}{24} = \frac{296}{3},$

j) $\int_0^1 \big(3(4x-3)^2+2(4x-3)^4\big)\,dx = \Big[(4x-3)^3\cdot\frac{1}{4}+\frac{2}{5}(4x-3)^5\cdot\frac{1}{4}\Big]_0^1$
$= (\frac{1}{4}+\frac{1}{10}) - (-\frac{3^3}{4}-\frac{3^5}{10}) = \frac{157}{5},$

k) $\int_0^1 (\frac{1}{(3x+2)^2}+\frac{1}{\sqrt{3x+2}})\,dx = \int_0^1 \big((3x+2)^{-2}+(3x+2)^{-\frac{1}{2}}\big)\,dx$
$= \Big[-\frac{1}{3}(3x+2)^{-1}+\frac{2}{3}(3x+2)^{\frac{1}{2}}\Big]_0^1 = (-\frac{1}{15}+\frac{2}{3}\sqrt{5}) - (-\frac{1}{6}+\frac{2}{3}\sqrt{2})$

$$= \frac{1}{10} + \frac{2}{3} \cdot (\sqrt{5} - \sqrt{2}) \approx 0{,}65,$$

l) $$\int_1^2 \Big(\frac{4}{\sqrt{2x}} - (2x)^4\Big)\, dx = \int_1^2 \big(4(2x)^{-\frac{1}{2}} - 16x^4\big)\, dx = \Big[8(2x)^{\frac{1}{2}}\frac{1}{2} - \frac{16}{5}x^5\Big]_1^2$$

$$= (8 - \frac{16 \cdot 32}{5}) - (4\sqrt{2} - \frac{16}{5}) = -\frac{456}{5} - 4\sqrt{2} \approx -96{,}86.$$

h) Der Definitionsbereich des Integranden $f(x) = \big(\sqrt{6-3x}\big)^{-1}$ ist $]-\infty, 2[$, insbesondere ist er bei 2 nicht definiert, das Integral $\int_1^2 f(x)\, dx$ daher nicht definiert. Nachfolgend eine Skizze des Graphen und des Integrationsbereiches.