09/02/2017 Banff

Atsushi Ito: Remark on higher syzygies on abelian surfaces

over $\mathbb{C}$

§ Intro

X smooth projective variety

L very ample line bundle

$$X \hookrightarrow \mathbb{P}(H^0(X,L)) = \mathbb{P}^N$$

$S = \mathrm{Sym}\, H^0(X,L) \cong \mathbb{C}[x_0, \dots, x_N]$

section ring

$$\dots \longrightarrow E_1 \longrightarrow E_0 \longrightarrow \bigoplus_{m \geq 0} H^0(X, mL) \longrightarrow 0$$

minimal graded free resolution as graded S-module

Def: $p \geq 0$, (N_p) holds for $L \overset{def}{\iff} E_0 = S$
and $E_i = S(-i-1)^{\oplus \mathrm{rk}\, E_i}$ $1 \leq i \leq p$

e.g. $p = 0$, (N_0) holds $\iff$ proj. normal

$p = 1$, (N_1) holds $\iff$ $I_{X/\mathbb{P}^n}$ is generated by quadrics

[Green] X smooth proj. curve

$\deg L \geq 2g+1+p \Rightarrow (N_p)$ holds for L

$$\left(\begin{array}{lll} \geq 2g & \Rightarrow & L \text{ bpf} \\ \geq 2g+1 & \Rightarrow & L \text{ very ample} \end{array}\right)$$

<u>Theorem</u> (Küronya-Lozovanu 2015)

X abelian surface & L ample line bundle with $(L^2) \geq 5(p+2)^2$ & $p \geq 0$.

Then (N_p) holds $\Leftrightarrow$ $X \not\supset C$ elliptic curve with $(L.C) \leq p+2$

intersection no.

<u>Main remark</u>: $(L^2) > 4(p+2)^2$ is enough

§ Multiplier ideal method

<u>Theorem</u> (Lazarsfeld-Pareschi-Popa 2011)

$p \geq 0$, X abelian variety, L ample

If ex. effective $\mathbb{Q}$-divisor D s.t. $\frac{1}{p+2}L - D$ ample

and $\mathcal{J}(X,D) = m_o$ $\quad o \in X$, Then (N_p) holds

$\quad \shortparallel$

$f_*\mathcal{O}_Y(K_{Y/X} - f^*D)$ $\quad f: Y \to X$

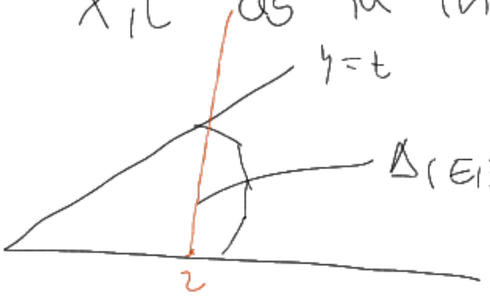
In particular, (N_p) holds if $\underbrace{\varepsilon(X, L; 0)}_{\text{Seshadri constant}} > (p+2)\dim X$

$$\begin{array}{ccc} \pi : \tilde{X} & \longrightarrow & X \\ \cup & & \psi \\ E & & 0 \end{array}$$

$$\varepsilon(X, L; 0) = \max\{ t > 0 \mid \pi^* L - tE \text{ is nef} \}$$

<u>Outline of proof</u> (of Thm [KL]) "$\Leftarrow$"

X, L as in Thm



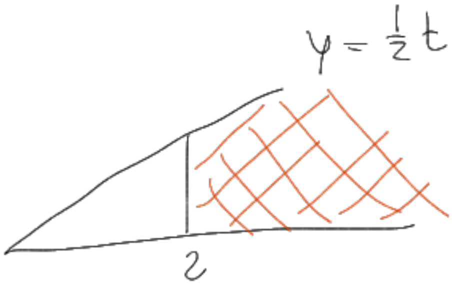
$\Delta_{(E,z)}(\pi^*(\frac{1}{p+2} L)) = \Delta$, $z \in E$

They show: if length$(\Delta \cap (\{z\} \times \mathbb{R})) > 1$

for $\exists\, z \in E$ then $\exists\, D$ in Thm [LPP]

$\rightsquigarrow$ (N_p) holds

X general surface
X^{ψ} B ample $\mathbb{Q}$-divisor

int$(\Delta(\pi^* B) \cap$  $) \neq \emptyset$

$\Longrightarrow$ $\exists\, D \equiv cB$ $\quad 0 < c < 1$

$\mathcal{J}(X, D) = m_x$ around $x \in X$

§ Sketch of proof of main remark "$\Leftarrow$"

Enough to show

Lemma: X abelian surface, B ample $\mathbb{Q}$-divisor

If
- $(B^2) > 4$ (or $\pi^* B - 2E$ is big)
- $(B \cdot C) > 1$ $\forall$ $C \subset X$ elliptic curve

then ex. effective $\mathbb{Q}$-divisor D s.t.
- $B - D$ is ample
- $\mathcal{J}(X, D) = m_0$.

$\Longrightarrow$ $B = \frac{1}{p+2} L$

$\because$) take $t > 2$, $\pi^* B - tE$ big
$t \in \mathbb{Q}$

take gen. eff. $\mathbb{Q}$-div. $\tilde{D}_0 \equiv \pi^* B - tE$
$D_0 := \pi_* \tilde{D}_0 \equiv B$

$1 > \frac{2}{t} \geq c := \mathrm{lct}(X, D) := \max\{c' > 0 \mid \underline{(X, c' D_0) \text{ is lc}}\}$

$\Updownarrow$

$\exists f : Y \to X$ $K_Y = f^*(K_X + c' D_0) + \sum a_i E_i$

$\forall$ $a_i \geq -1$

If $\exists$ E_i and $a_i = -1$ for (X, cD_0)

If $\exists$ E_i and $a_i = -1$. $f(E_i) = pt \in X$ $\otimes$

parallel translation of small perturbation of $cD_0 =: D$

almost $\equiv$ to cB

$\rightsquigarrow$ $B - D$ is ample

If $\not\exists$ E_i with $\underbrace{a_i = -1}$ and $\pi(E_i) = C \subset X$ curve

i.e. $cD_0 = C +$ eff. $\mathbb{Q}$-div.

- C must be smooth
- $\tilde{C} \subset \tilde{X}$ is a negative curve, $0 \in \tilde{C}$

$\tilde{D}_0 \equiv \pi^* B - tE$

$$0 > (\tilde{C}^2) = (\pi^* C - (\mathrm{mult}_0 C) E)^2$$
$$= (C^2) - 1 \geq -1$$
$$\Rightarrow (C^2) = 0 \rightsquigarrow C \text{ elliptic curve}$$

$B - C = (1-c)B + cB - C \equiv (1-c)B +$ effective $\mathbb{Q}$-div.

$\hookrightarrow$ ample

$(B - C \cdot C) = (B.C) > 1$

$\longrightarrow$ $\exists$ $D' \equiv B - C$ s.t. $\mathrm{mult}_0(D'|_C) > 1$

$\rightsquigarrow$ $C + sD'$ satisfies $\otimes$ for $0 < s < 1$

Further $B - (C + \varepsilon D') = (1-\varepsilon)(B-C)$ ample

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§ Naive Question

Conj. (Fujita)

X smooth proj. var., L ample, $\dim X = n$

(1) $K_X + mL$ is bpf if $m \geq n+1$

(2) $K_X + mL$ is very ample if $m \geq n+2$

$(1)'$ $x \in X$ If $(L^n) > n^n$ & $(L^{\dim Z} \cdot Z) \geq \boxed{n}^{\dim Z}$

$\forall\ Z \subsetneq X,\ x \in Z$

then $K_X + L$ is bpf at x.

naive Question: X abelian var., L ample, $Z \subseteq X$ ab.

If $\left(\left(\frac{1}{p+2} L\right)^{\dim Z} \cdot Z\right) > \boxed{\dim Z}^{\dim Z}$ does (N_p) hold then for L?

$\left(\frac{1}{p+2} L\right)^2 > 2^2 = 4$

$\left(\frac{1}{p+2} L \cdot Z\right) > 1$ $\quad Z$ elliptic

$\Big\}$ case of Lemma

$cD + \varepsilon' D'$ Fujita

$C + \varepsilon' D'$ N_p