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Kiumars Kaveh: Khovanskii bases, Newton-Okounkov polytopes and tropical geometry

jt. Chris Manon

arXiv

A (graded) algebra , ν valuation full-rank

$S(A,\nu)$ fin. gen. $\rightsquigarrow$ Eff. computation in A

$\rightsquigarrow$ Toric degen.

$\downarrow$

Application in Kähler/Sympl. geometry

k field

$x = (x_1, \ldots, x_n)$

$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ monomial

$>$ term order on $\mathbb{N}^n$

0 unique maximum

$f = \sum c_\alpha x^\alpha$, $in_>(f) = c_\beta x^\beta$

$\beta = \min\{\alpha \mid c_\alpha \neq 0\}$

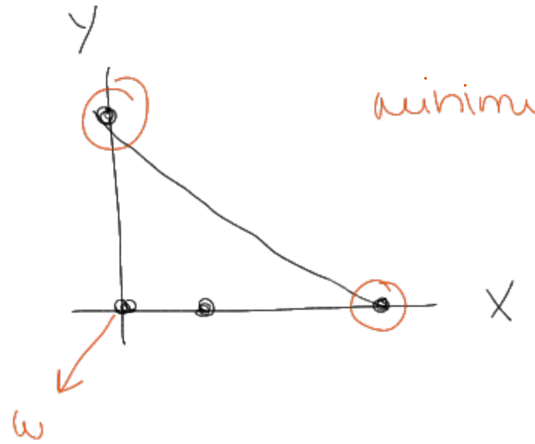
$I \subset k[x]$ $\quad in_>(I) = \langle in_>(f) \mid f \in I \rangle$

$\omega \in \mathbb{Q}^n$

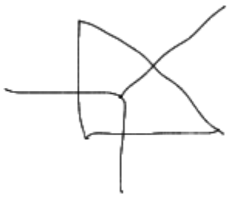
$in_\omega(f) = \sum_\beta c_\beta x^\beta \rightsquigarrow \min\{\omega \cdot \alpha \mid c_\alpha \neq 0\}$

attained at β

__Ex__: $f(x,y) = y^2 + x^3 - ax - b$



Newton Polytope



Gröbner fan

__Def__: $in_\omega(I) = \langle in_\omega(f) \mid f \in I \rangle$

__Def__: $\mathcal{G} \overset{\text{fin.}}{\subset} I$ is a __Gröbner basis__ wrt $<$ if $in_<(\mathcal{G})$ generate (as ideal) $in_<(I)$

- Given Gröbner basis for I $\Longrightarrow$ effective comp.
  e.g. given $f \in k[X]$ can test if $f \in I$

__Remark__: $in_<(I)$ is fin. gen.

replace $I \rightsquigarrow A \underset{\text{subalg.}}{\subset} k[X]$

<u>Def</u>: (SAGBI basis)

fin. $\mathcal{B} \subset A$ is a <u>SAGBI basis</u> if $in_<(\mathcal{B})$

gen. $\underbrace{in_<(A)}_{\text{semigroup algebra}}$ as k-algebra.

<u>Remark</u>: $in_<(A)$ may not be fin. gen. algebra

<u>Example</u> (Göbel) ☹

$A = k[x_1, x_2, x_3]^{A_3}$ $\quad >$ lex order (same, e.g. opp.)

$in_<(A)$ is not fin. gen.

<u>Example</u> ☺ ♡

$Gr(2,n)$ 2-planes in k^n

$$\begin{bmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{bmatrix} \longmapsto (p_{ij})$$

$$p_{ij} := \det\begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix} \qquad i<j$$

$Gr(2,n) \hookrightarrow \mathbb{P}^{\binom{n}{2}-1}$ Plücker embedding

Plücker ideal = ideal of rel's among p_{ij}'s

(quadratic relations)

$A = k[P_{ij}] \subset k[x_1, \dots, x_n, y_1, \dots, y_n]$

$\quad\llcorner$ Plücker algebra

<u>Theorem</u> (Sturmfels)

$\{P_{ij}\}$ SAGBI basis for A

wrt (some natural) term order $>$

<u>Fact</u>: SAGBI basis

$\Rightarrow \quad \mathrm{Spec}(A) \overset{\text{deg.}}{\rightsquigarrow} \mathrm{Spec}(\mathrm{in}_\alpha(A))$

toric variety (ev. non normal)

<u>Extend the setup to arbitrary algebra</u>:

A fin. gen. k-algebra (domain), $d = \dim_{Kr}(A)$

$v : A \setminus \{0\} \longrightarrow \mathbb{Z}^d$ (or $\mathbb{Q}^d$)

($v|_k$ trivial) $\quad >$

$(\star)$ $k_v = k$ i.e. $v(f) = v(g) \Rightarrow v(f - cg) > v(f)$

$\exists\, 0 \neq c \in k$

<u>Proposition</u>: $k = \bar{k}$ & $\mathrm{rank}(v) = d$ then

$(\star)$ is satisfied.

<u>Example</u>: $v : k[x] \setminus \{0\} \longrightarrow \mathbb{N}^n$

$$v(f) = \min\{\alpha \mid c_\alpha \neq 0\}$$

$f = \sum c_\alpha x^\alpha$

Def: $\mathcal{B} \subset A$ fin. subset is a Khovanskii basis, if $\nu(\mathcal{B})$ generates $S(A,\nu)$ value semigp

$$S(A,\nu) = \{ \nu(f) \mid 0 \neq f \in A \}$$

Example: $A = k[X]_{homog.}$

$$\{ ZY^2 - X^3 - aXZ^2 - bZ^3 = 0 \} \subset \mathbb{P}^2$$

$\nu(f) = (\deg(f), \mathrm{ord}_p(f))$ ⇝ see e.g. Lazarfeld–Mustata, Anderson

p generic pt $\Rightarrow$ $S(A,\nu)$ not fin. gen.

$p = \infty \Rightarrow S(A,\nu)$ fin. gen.

Remark: (not serious) issue of $S(A,\nu)$ not (max.) well-ordered. If $A = \bigoplus_{i \geq 0} A_i$ OK ✓

- **Subduction algorithm** $\mathcal{B} = \{ b_1, \ldots, b_n \}$

Input: (A,ν), $\mathcal{B}$, $f \in A \setminus \{0\}$

Output: f as polynomial in elts of $\mathcal{B}$.

$$\nu(f) = \sum_i \nu(b_i) k_i$$

replace f by $f - \varepsilon \prod b_i^{k_i}$ & repeat

↳ reason for effective computations

Remark: $(A, \nu) \rightsquigarrow \{A_{\nu \geq a}\}$ filtration

$$gr_\nu(A) = \bigoplus_a A_{\nu \geq a} / A_{\nu > a}$$

Proposition: $S(A, \nu)$ is fin. gen $\iff$ $gr_\nu(A)$ fin. gen

Prop. $gr_\nu(A) \cong k[S(A, \nu)]$

Remark: If ν is not of max. rank, still things work (but not last Prop.)

Examples: homog. coord. rings of flag varieties & more general spherical when there ex. natural valuation with fin. Khovanskii basis

Def: ν "subductive valuation" if it has finite Khovanskii basis for which subduction algo always terminates.

Question: When do we have a subductive valuation ν on A?

Tropical variety

$I \subset k[x^{\pm}]$ (or $k[x]$)

$\mathrm{Trop}(I) \subset \mathbb{Q}^n$

$\quad \overset{\cdot\cdot}{\|}$

$\{ \omega \in \mathbb{Q}^n \mid in_\omega(I) \text{ is monomial-free} \}$

- $\mathrm{Trop}(I)$ encodes "behaviour at ∞" of $V(I) \subset (k^*)^n$
- $\mathrm{Trop}(I)$ $\underset{\text{rational}}{\wedge}$ polyhedral fan

$w_1, w_2 \underset{\text{rel. int.}}{\in} C \subset \mathrm{Trop}(I)$ cone C

then $in_{w_1}(I) = in_{w_2}(I)$

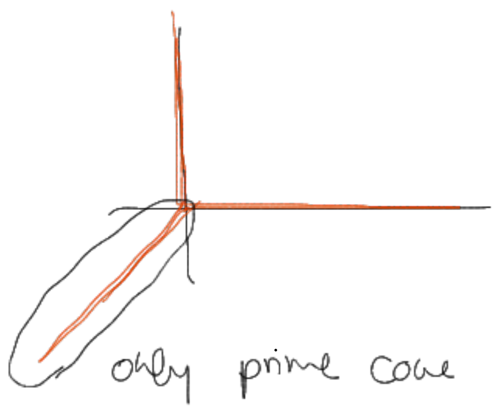
<u>Def</u>: A cone $C \subset \mathrm{Trop}(I)$ is <u>prime</u> if $in_C(I)$ is a prime ideal.

$\quad \| $

$in_w(I)$

$\forall\, w \in C^\circ$

<u>Example</u>: $y^2 - x^3 - ax - b$



only prime cone

Theorem (K-Manon)

A k-algebra, $A = \bigoplus_{i \geq 0} A_i$

$\mathcal{B} = \{b_1, \dots, b_n\}$ algebra generators for A

I ideal of rel's among b_i's

$k[x]/_I \cong A$

Then $\mathcal{B}$ is a Khovanskii basis for subductive valuation v

$\iff$ Trop(I) contains a "prime" cone. maximal dim

"$\Leftarrow$"

$\{u_1, \dots, u_d\} \subset C$ prime (max) cone gen.

v_M weight valuation

$$M = \begin{pmatrix} — u_1 — \\ — u_2 — \\ \vdots \\ — u_d — \end{pmatrix}$$

$x_1, \dots, x_n$

weight of x_i is ith column of M.

Theorem: • v_M is subductive valuation

• $S(A, v)$ = semigrp gen. by columns of M

Example: $zy^2 - x^3 - axz^2 - bz^3$

$$M = \begin{bmatrix} 1 & 1 & 1 \\ -3 & -2 & 0 \end{bmatrix}$$

$\uparrow \quad \uparrow \quad \uparrow$

generators of $S(A, v_M) = S(A, v)$

same as $v(f) = (\deg f, \operatorname{ord}_\infty f)$

$\rightsquigarrow$ wonderful compactification fits into this setting

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why prime cone?

$f = \sum c_\alpha x^\alpha$

$\tilde{v}_M(f) := \min\{ M\alpha \mid c_\alpha \neq 0 \}$ usual val. ass. to matrix

$k[X] \longrightarrow k[X]/I = A$

in gen. only quasi-val. on A

$\hookrightarrow$ need prime to be valuation

$gr_{\tilde{v}_M}(A) = k[X]/\operatorname{in}_C(I)$