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Toric degen. of Grassmannians & plabic graphs

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Speyer-Sturmfels 2003	Rietsch-Williams 2015
$\underline{w} \in$ maximal prime cone of $\mathrm{Trop}(\mathrm{Gr}(k,n))$	(certain) plabic graph
$\rightsquigarrow$	$\rightsquigarrow$
flat toric degenerations	NO body
	$\rightsquigarrow$
	flat toric degeneration

compare!

① The (tropical) Grassmannian

$$\mathrm{Gr}(k,n) = \{ V \subset \mathbb{C}^n \mid \dim V = k \} \hookrightarrow \mathbb{P}(\Lambda^k \mathbb{C}^n)$$

$$(k \times n \text{ matrix} \longleftrightarrow \overset{\Psi}{M} = [v_1 \cdots v_n]$$

$$\underset{\{i_1, \ldots, i_k\}}{\overset{\shortparallel}{I}} \subset \binom{[n]}{k} \rightsquigarrow P_I = \det [v_{i_1} \cdots v_{i_k}]$$

$I_{k,n}$ def. ideal gen. by Plücker relations

$\boxed{k=2}$

$(p_{ij})_{1\le i,j\le n}$

$I_{2,n}$ gen. by $p_{ij}p_{kl} - p_{ik}p_{jl} + p_{il}p_{jk}$

for $1 \le i < j < k < l \le n$

$\underline{w} \in \mathbb{R}^N$, $N = k(n-k)$, $f = \sum_{\underline{u}\in\mathbb{N}^N} a_{\underline{u}} x^{\underline{u}}$

$$\rightsquigarrow \quad \mathrm{in}_{\underline{w}}(f) = \sum_{\substack{\underline{u} \\ \underline{w}\cdot\underline{u} \text{ min.}}} a_{\underline{u}} x^{\underline{u}}$$

$I \subset \mathbb{C}[x_1, \dots, x_N]$ then $\mathrm{in}_{\underline{w}}(I) = \langle \mathrm{in}_{\underline{w}}(f) \mid f \in I \rangle$

$\underline{w} \in \mathbb{R}^N \rightsquigarrow \exists$ flat family over $\mathbb{A}^1$ s.t.

fibre $t \neq 0$ isom $V(I)$

fibre $t = 0$ isom $V(\mathrm{in}_{\underline{w}}(I))$

Def: The tropical Grassmannian
[SS]

$$\mathrm{Trop}(\mathrm{Gr}(k,n)) = \{ w \in \mathbb{R}^N \mid \mathrm{in}_{\underline{w}}(I_{k,n}) \text{ monomial free} \}$$

support of polyhedral fan in $\mathbb{R}^{\binom{n}{k}}$

$\underline{w}$ in rel. int. of max. prime cone of $\mathrm{Trop}(\mathrm{Gr}(k,n))$

$\rightsquigarrow$ flat toric degen.

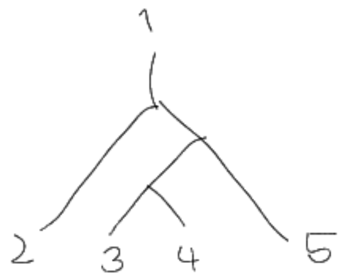
$\boxed{k=2}$

[SS] classify all max. prime cones of $\mathrm{Trop}(Gr(2,n))$

$$\left\{ \begin{array}{c} \text{max prime} \\ \text{cones of } \mathrm{Trop}(Gr(2,n)) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{trivalent trees} \\ \text{with } n \text{ leaves} \end{array} \right\}$$

$$(\deg_\tau(p_{ij}))_{ij} = \omega_\tau \longleftarrow\!\shortmid \ \tau$$

Example: $\boxed{n=5}$



$\deg_\tau(p_{ij}) := -\#$ int. edges btw i & j

here

	12	13	14	15	23	24	25	34	35	45
$\deg(p_{ij})$	0	−2	−2	−1	−2	−2	−1	0	−1	−1

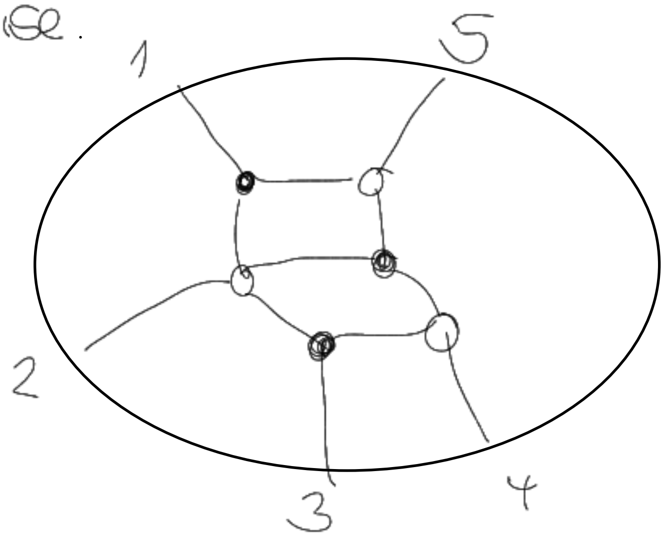
$$\overset{0\ \ 0}{p_{12}p_{34}} - \underbrace{\overset{-4}{p_{13}p_{24}} + \overset{-4}{p_{14}p_{23}}}_{= \ \mathrm{in}_{\omega_\tau}}$$

② Plabic graphs

I will need
- plabic graph
- reduced
- trip permutation
- perfect orientation
- I-flow $I \subset \binom{[n]}{k}$

Def: [Postnikov]
A plabic graph is planar graph in disk with n boundary vertices and internal vertices colored black or white (bicolored).
Each boundary vertex is adjacent to exactly one internal vertex, they are numberd $1, \dots, n$ counterclockwise.

e.g.



Moves on plabic graphs:

(M1)

(M2) 

(M3)  (same with $\circ$)

<u>Def</u> plabic graph $\mathcal{G}$ is reduced if $\nexists\ \mathcal{G}' \underset{\text{move}}{\sim} \mathcal{G}$

containing 

$\mathcal{G}$ reduced plabic graph with n boundary vert.

$\rightsquigarrow \pi_{\mathcal{G}} \in S_n$

A trip on $\mathcal{G}$ starting in $i \in \{1, \dots, n\}$ is a walk on $\mathcal{G}$ starting in i which follows

1) go maximally right at $\bullet$
2) go maximally left at $\circ$

$\pi_{\mathcal{G}}(i) = j$ boundary vertex where trip ends

<u>Def</u>: A perfect orient. of a plabic graph $\mathcal{G}$ is an orientation of its edges s.t.

1) every $\bullet$ has $\exists!$ outgoing $\bullet\!\!\longrightarrow$
2) every $\circ$ has $\exists!$ incoming $\longrightarrow\!\!\circ$

<u>Def</u> $\deg_{\mathcal{G}}(P_I)$

$\deg_{\mathcal{G}}(\text{oriented path})$

$\deg_{\mathcal{G}}(I\text{-flow})$

From now on $\mathcal{G}$ reduced plabic graph with trip perm.

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & k & \cdots & n \\ n-k+1 & \cdots & & & 1 & & n-k \end{pmatrix}$$

perfectly oriented s.t. $\{1, 2, \ldots, k\}$ are sources (the only)

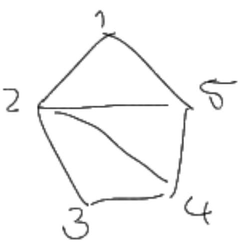
$\deg_{\mathcal{G}}$(oriented path) = # internal faces to the left of the path

$I \subset \binom{[n]}{k}$, <u>I-flow</u> is a collection of paths from sources to I : selfavoiding & vertex disjoint

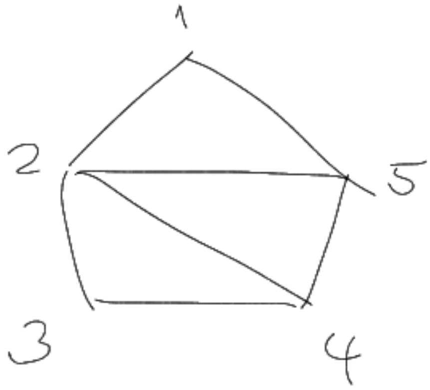
$$\deg_{\mathcal{G}}(I\text{-flow}) = \sum_{\substack{\text{paths} \\ \text{in } I\text{-flow}}} \deg_{\mathcal{G}}(\text{path})$$

$$\deg_{\mathcal{G}}(P_I) = \min_{I\text{-flow}} \{ \deg_{\mathcal{G}}(I\text{-flow}) \}$$

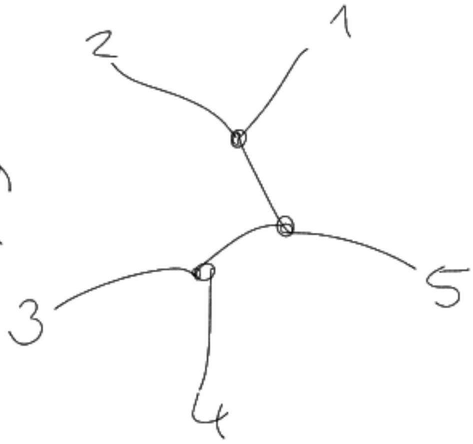
③ <u>What has all this to do with [SS]?</u>

Consider n-gon 

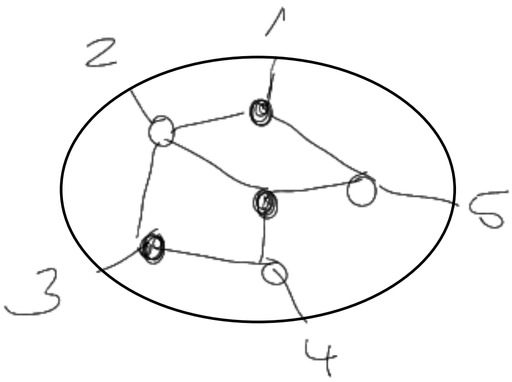
$$\mathcal{G} \longleftarrow\!\shortmid \ \Delta \ \longmapsto \tau$$



dual graph



Kodama-Williams



<u>Theorem</u>: [BFFHL]

$$\mathrm{in}_{\mathcal{T}_\Delta}(I_{2,n}) = \mathrm{in}_{\mathcal{G}_\Delta}(I_{2,n})$$

<u>Q</u>: Is $w_{\mathcal{G}_\Delta}$ in general in $\mathrm{Trop}(Gr(k,n))$?

↳ Yes for $Gr(3,6)$