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David McKinnon: Diophantine approximation and filtrations

Banff

<u>Question</u>: How close can rational pts get to an algebraic point?

<u>Liouville</u> (1844): Let α be an algebraic number $\in \bar{\mathbb{Q}}$ and $\frac{p}{q} \in \mathbb{Q}$

If $d = [\mathbb{Q}(\alpha) : \mathbb{Q}]$ then $|\alpha - \frac{p}{q}| > \frac{c}{q^d}$

accept for fin. many $\frac{p}{q}$ ($c>0$ doesn't dep. on $\frac{p}{q}$)

↳ this is sharp for $d = 1, 2$ & not else

<u>Sketch of proof</u>:

<u>Step 1</u>: Find $_{\wedge}$ polynomial $P(x)$ s.t. $P(\alpha) = 0$

nonzero

<u>Step 2</u>: If q big and $|\alpha - \frac{p}{q}|$ small, then

$$P\left(\frac{p}{q}\right) = 0, \quad P(X) = \sum_{n=1}^{d} \frac{1}{n!} P^{(n)}(\alpha)[x-\alpha]^n$$

$$\Rightarrow \left|P\left(\frac{p}{q}\right)\right| \leq \left|\alpha - \frac{p}{q}\right| \cdot B$$

B: const. indep of $\frac{p}{q}$

$\left|P\left(\frac{p}{q}\right)\right| = \frac{\text{Integer}}{q^d}$

$$\Rightarrow |\text{Integer}| \leq \left|\alpha - \frac{p}{q}\right| \cdot B \cdot q^d \Rightarrow P\left(\frac{p}{q}\right) = 0$$

Step 3: Since P has fin. many roots, $|\frac{p}{q} - \alpha|$ can't be small if q is big.

Step 4: In any infinite set of $\frac{p}{q}$, $q \to \infty$ □

Theorem (Roth, 1955)

Given $\alpha \in \overline{\mathbb{Q}}$ there are only fin. many $\frac{p}{q} \in \mathbb{Q}$ s.t. $|\alpha - \frac{p}{q}| < \frac{c}{q^{2+\varepsilon}}$ $c > 0$ const. dep. on $\frac{p}{q}$

To generalize these theorems to arbitrary alg. var. X we replace

$\alpha \longleftrightarrow$ alg. pt on X

$\frac{p}{q} \longleftrightarrow$ rat. pt. on X

$|\alpha - \frac{p}{q}| \longleftrightarrow$ distance on X

$q \longleftrightarrow$ height of approximating pt

$[x_0 : x_1 : \cdots : x_n] \in \mathbb{P}^n(\mathbb{Q})$, $x_i \in \mathbb{Z}$, $\gcd(x_i) = 1$

$$H(P) = \max_i \{|x_i|\}$$

If L is a very ample line bundle on X, define $H_L(x) = H(\phi(x))$, where $\phi : X \to \mathbb{P}^n$ corr. to L

In general, $H_{L_1 + L_2}(x) = H_{L_1}(x) H_{L_2}(x)$

So $H_M(x) = \dfrac{H_{L_1}(x)}{H_{L_2}(x)}$ where $M = L_1 - L_2$, L_1, L_2 very ample

Joint with Mike Roth

<u>Theorem</u> (McKinnon-Roth)

X alg. var. def. over $\mathbb{Q}$, $x \in X(\bar{\mathbb{Q}})$, L ample line bundle on X, $n = \dim X$, $\varepsilon > 0$

There is a proper closed (Zariski) subset $Z \subset X$ s.t.

$$\operatorname{dist}(x,y) \geq \frac{1}{H_L(y)^{\frac{1}{\beta}+\varepsilon}} \qquad \beta = \int_0^\infty f(v)\,dv$$

for all $y \in X(\mathbb{Q}) - Z(\mathbb{Q})$, where

$$f(v) = \frac{\operatorname{vol}(mL - vE)}{\operatorname{vol}(L)}$$

<u>Faltings-Wüstholz</u>: Given conditions, there is a proper subvariety $Z \subset X$ s.t. except for fin. many $y \in X(\mathbb{Q}) - Z(\mathbb{Q})$ we have

$$|s_{ij}(y)| \geq H_L(y)^{-c_i} = \frac{1}{H_L(y)^{c_i}} \qquad \forall i.$$

Let $K = k(X)$. $V = \Gamma(X, mL)$, $m \gg 0$

$$V_K = V \otimes K \cong \Gamma(\tilde{X}, \pi^* mL)$$

$$V_i = \Gamma(\tilde{X}, \pi^* mL - iE)$$

$$V_K \supseteq V_1 \supseteq V_2 \supseteq \cdots \supseteq V_\gamma = (0)$$

for each i, choose basis $\{s_{ij}\}$ for V_i

<u>idea</u>: filtration by order of vanishing

Given a $\mathbb{Q}$-subspace $W \subset V$, def. the slope of W

$$\mu(W) = \sum \frac{c_i(\dim((W\otimes k)\cap V_i) - \dim((W\otimes k)\cap V_{i-1})}{\dim W}$$

Condition: $\mu(W) > 1$ for some W

How to define the c_i's ?

$$m\beta = \int_0^\infty f(v)\,dv \approx \sum_{i=1}^{r} (v_i - v_{i-1}) f(v_i)$$

$f(v) = \frac{vol(mL - vE)}{vol(mL)}$ for some v_i

$$\Rightarrow \sum_{i=1}^{r} v_i(f(v_i) - f(v_{i+1})) \approx m\beta$$

Define $c_i = \frac{1}{m\beta} v_i + \delta$ δ small

$$\Rightarrow \sum_{i=1}^{r} c_i(f(v_i) - f(v_{i+1})) > 1$$

$s_{ij} \in V_i \Rightarrow s_{ij} \in \Gamma(m\pi^*L - iE)$ vanishes to order at least $\frac{i}{m}$ at x

$$\Rightarrow dist(x,y)^{\frac{v_i}{m} - \varepsilon} \geq |s_{v_i j}(y)| \geq H_L^{-\frac{1}{m\beta}v_i - \varepsilon'}(y)$$

$$\Rightarrow dist(x,y) \geq \frac{1}{H^{\frac{1}{\beta}+\varepsilon}(y)}$$ away from Z + fin. many other y