10/02/2017

(How K-(semi)stability should have been defined)

- G - reductive lin. alg. grp over $\mathbb{C}$ ($SL_{N+1}(\mathbb{C})$)
- $\mathbb{E}, \mathbb{W}$ fin. dim. rat. G-modules
- $0 \neq u \in \mathbb{E}$, $0 \neq w \in \mathbb{W}$
- $\mathbb{P}(\mathbb{E})$ = proj. sp. of <u>lines</u>
- $[u] \in \mathbb{P}(\mathbb{E})$

<u>Assume</u>: $\langle G \cdot u \rangle = \mathbb{E}$

$\langle G \cdot w \rangle = \mathbb{W}$

<u>Def</u> (Vinberg)

$(\mathbb{E}, u) \gtrsim (\mathbb{W}, w) \iff \exists$ G map $\pi \in \mathrm{Hom}(\mathbb{E}, \mathbb{W})$

st. $\pi(u) = w$ & the rational map

$\pi : \mathbb{P}(\mathbb{E}) \dashrightarrow \mathbb{P}(\mathbb{W})$ restricts to a

reg. map $\overline{G.[u]} \longrightarrow \overline{G.[w]}$

$\iff \overline{G.[u]} \cap \mathbb{P}(\ker \pi) = \emptyset$.

<u>Def</u> (P., 2012) $\mathbb{V}, \mathbb{W}$ reps, $0 \neq v \in \mathbb{V}$, $0 \neq w \in \mathbb{W}$

(v, w) is a <u>semi-stable pair</u> $\iff$

$\overline{G.[v,w]} \cap \overline{G.[v,0]} = \emptyset$ inside $\mathbb{P}(\mathbb{V} \oplus \mathbb{W})$

Amusing: study the function

$$\sigma \in G \longrightarrow \log\|\sigma\cdot w\|^2 - \log\|\sigma\cdot v\|^2 \overset{?}{\geqslant} c$$

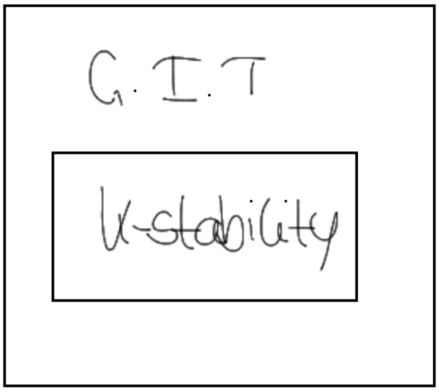
boundedness holds $\iff$ (v,w) semi-stable pair

Rmk: "$-$" is important

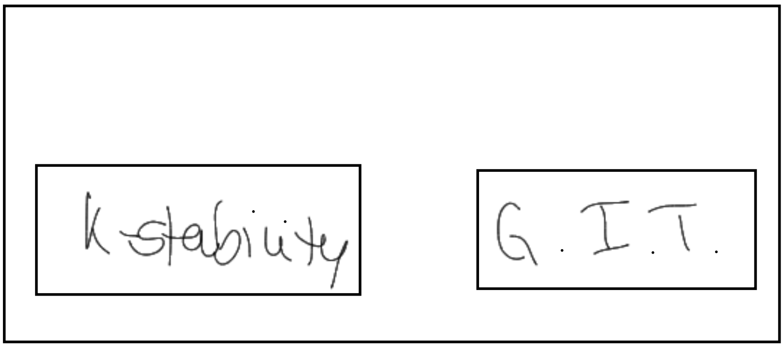
if "$+$" instead get $\mathbb{V}\otimes\mathbb{W} \rightsquigarrow$ Mumford

Exp: $\mathbb{V}=\mathbb{C}, v=1$, w anything $\neq 0$ in $\mathbb{W}$

Then $(1,w)$ is ss-pair $\iff 0 \notin \overline{G\cdot w} \subset \mathbb{W}$



not true, rather:



Exp: $G = SL_2(\mathbb{C})$, $\mathbb{V} = S^e(\mathbb{C}^{2\vee}) \ni v = f \neq 0$

$\mathbb{W} = S^d(\mathbb{C}^{2\vee}) \ni w = g \neq 0$

Q: $\overline{SL_2(\mathbb{C})\cdot[f,g]} \cap \overline{SL_2(\mathbb{C})\cdot[f,0]} = \emptyset$?

Answer: (f,g) is ss-pair $\iff \forall\, p \in \mathbb{P}^1$

$$\mathrm{ord}_p(g) - \mathrm{ord}_p(f) \leq \frac{d-e}{2}$$

- implies $e \leq d$
- $e=0$, $\mathbb{V}=\mathbb{C}$, $f=1$

  $(1,g)$ ss-pair $\iff$ $\mathrm{ord}_p(g) \leq \frac{d}{2}$

- $e=d-1$

  $\Rightarrow \mathrm{ord}_p(g) - \mathrm{ord}_p(f) \leq \frac{1}{2}$

  $\Rightarrow$ $\nexists$ no ss-pairs $\therefore$ $(\mathbb{V}\oplus\mathbb{W})^{ss} = \emptyset$

  prehomog. vec. sp.

  $\dim(G) \geq \dim(\mathbb{W})$

  $\rightsquigarrow$ When does $(\mathbb{V}\oplus\mathbb{W})^{ss} = \emptyset$ hold?

  $\hookrightarrow$ which $\mathbb{V}$ & $\mathbb{W}$ & G

- $e=d$ (f,g) ss-pair $\iff$ $[f]=[g]$

  $\rightsquigarrow$ set of ss-pairs is not open

<u>In general</u>:

$\mathbb{V} = \mathbb{E}_\lambda$, $\mathbb{W} = \mathbb{E}_\mu$ irred. $SL_2(\mathbb{C})$-rep.

If $\exists$ a ss-pair (v,w) $\Rightarrow$ $\lambda \nleq \mu$.

$\overline{G.[v,w]} \cap \overline{G.[v,0]} = \emptyset$

T alg. torus

$\Rightarrow \overline{T.[v,w]} \cap \overline{T.[v,0]} = \emptyset \iff N(v) \subset N(w)$

weight polytope

Theorem: (P.)

These are the same

$\overline{G.[v,w]} \cap \overline{G.[v,0]} = \emptyset \iff \mathcal{N}(v) \subset \mathcal{N}(w)$
$\forall T$

Q: Is there some "worst torus" s.t. $\mathcal{N}(v)$ is as far away from $\mathcal{N}(w)$ as possible

- Popov-Vinberg "Method of supports"

Example: $\mathbb{V} = \mathbb{E}_{(2,2,0)}$ for $SL_3(\mathbb{C}) = G$

$\mathbb{W} = \mathbb{E}_{(3,1,0)}$ $v := \xi_{(2,2,0)}$ h. wt

$w := E_{2,1} \cdot \xi_{(3,1,0)}$

neg. root vect $-\alpha_1$

claim: $(\xi_{220}, E_{2,1}\xi_{310})$ ss pair

$$B_\Delta(\mathbb{P}^2 \times \mathbb{P}^2) = \overline{SL_3(\mathbb{C})[v,w]} \subset \mathbb{P}(\mathbb{V} \oplus \mathbb{W})$$

Any two-orbit variety $\overline{G.[u]} \subset \mathbb{P}(\mathbb{E})$ gives a semi-stable pair. (stephanie's example)

Cupit-Foutou

Exp: (difficult)

$$p = a_m z^m + \cdots$$

$$q = b_n z^n + \cdots$$

$\mathrm{Res}_{m,n}(p,q) = R_{m,n}(a_\cdot, b_\cdot)$ vanishes

$\Leftrightarrow$ P & Q have common root

$m = n = d \geq 2$ $\quad R_d \in \mathbb{C}_{2d}[M_{2\times(d-1)}]^{SL_2}$ — poly with coeff. from P,Q

$\Delta_d(P) = R_{d,d-1}(P, \partial P) \quad \Delta_d \in \mathbb{C}_{2d-2}[M_{1\times(d+1)}]$

$(R_d^{2d-2}, \Delta_d^{2d})$ is semistable for action of $SL_{d+1}(\mathbb{C})$

Reason: $\mathbb{P}^1$ is Kähler-Einstein

$(2d-2)\,N(R_d) \subset 2d\,N(\Delta_d)$ special torus $(\mathbb{C}^*)^{d+1}$

$\Uparrow$

Gelfand-Kapranov-Zelevinsky

TABLE

Mumford-Hilbert	Pairs
$\forall\, T \leq G\ \exists\, d \in \mathbb{Z}_{>0}$ & $f \in \mathbb{C}_{\leq d}[W]^T$ s.t. $f(w) \neq 0$ & $f(0) = 0$	$\forall\, T \leq G$ & $\chi \in \mathfrak{X}_T(V)$ $\exists\, d \in \mathbb{Z}_{>0}$ & $f \in \mathbb{C}_d[V \oplus W]^T_{d\chi}$ s.t. $f(v,w) \neq 0$ & $f\vert_V = 0$
$0 \notin \overline{G w}$	$\overline{G.[v,w]} \cap \overline{G.[v,0]} = \emptyset$
$w_\lambda(w) \leq 0\ \forall \lambda$	$w_\lambda(w) - w_\lambda(v) \leq 0 \quad \forall \lambda$

$[\lambda(\alpha)v, \lambda(\alpha)w] \in \mathbb{P}(V\oplus W)$

$[\alpha^{w_\lambda(v)}\{\alpha^{-w_\lambda(v)}\lambda(\alpha).v\}, \alpha^{w_\lambda(w)}\{\alpha^{-w_\lambda(w)}\lambda(\alpha).w\}]$

$\rightsquigarrow [\{\alpha^{-w_\lambda(v)}\lambda(\alpha).v\}, \alpha^{w_\lambda(w)-w_\lambda(v)}\{\alpha^{-w_\lambda(w)}\lambda(\alpha).w\}]$

TABLE:

$0 \in N(w) \quad \forall T$	$N(v) \subset N(w) \quad \forall T$

Kähler-Problem:

$\nu_\omega : \mathcal{H}_\omega = \{\varphi \in C^\infty(X^n) \mid \omega_\varphi := \omega + \frac{\sqrt{-1}}{2\pi}\partial\bar\partial\varphi > 0\}$

$X^n \hookrightarrow \mathbb{P}^N$

$\omega_{FS}|_X = \omega$

ν_ω K-energy map $\nu_\omega(\varphi) = -\frac{1}{V}\int_X \log\left(\frac{\omega_\varphi^n}{\omega^n}\right)\omega_\varphi^n + \dots$

Core-Problem: $\nu_\omega \geq C$? $(*)$

$SL_{N+1}(\mathbb{C}) \longrightarrow \mathcal{H}_\omega$

$\sigma \longmapsto \sigma^*(\omega_{FS}) = \omega_{FS} + \partial\bar\partial\varphi_\sigma|_X$

Instead: $\boxed{\nu_\omega|_{SL_{N+1}(\mathbb{C})} \geq C\,?}$

Theorem:

Given any smooth $X^n \hookrightarrow \mathbb{P}^N$ (lin. normal)

$\exists$ reps $\mathbb{V}, \mathbb{W}$ irred. of $G = SL_{N+1}(\mathbb{C})$

& "encodings"

$$\begin{tikzcd}
 & \nu(X^n) \in \mathbb{V} \\
X^n \arrow[ur] \arrow[dr] & \\
 & W(X^n) \in \mathbb{W}
\end{tikzcd}$$

$\nu(\sigma X^n) = \sigma\, \nu(X^n)$

Then: $\nu_\omega|_{SL_{N+1}(\mathbb{C})} \geq -C$ iff $(\nu(X), w(X))$

is a semistable pair, where

$$C = \log \tan^2 d_{FS}\left(\overline{G[v,w]}, \overline{G[v,0]}\right)$$

• $\nu(X) = R_X^{\circ}$ Cayley-Chow form of X to some power

$$\begin{array}{c} R_X^{\circ} \\ \cap \\ \mathbb{C}_\bullet[M_{N+1 \times N+1}]^{SL_{N+1}(\mathbb{C})} = E_\lambda . \\ \parallel \\ \mathbb{V} \end{array}$$

• $X^n \times \mathbb{P}^{n-1} \xrightarrow{\text{Segre}} \mathbb{P}(M_{n \times (N+1)})$

$$\begin{array}{c} \mathbb{P}(M_{n \times (N+1)}) \\ \cup \\ Y \end{array} \quad \text{image}$$

consider $Y^\vee$: $\operatorname{codim}(Y^\vee) = 1$

$\deg(Y^\vee) = n(n+1)d - d\mu > 0$

polynomial $\Delta(X)$

Ricci curvature of X

so $(v(x), w(x)) = \left(R_X^{\deg \Delta}, \Delta^{\deg(R)}\right)$ ö the pair

K-energy is <u>coercive</u>.