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Noncommutative resolutions of discriminants

jt Buchweitz, Ingalls

Goal: NCR of non-normal hypersurfaces

$\rightsquigarrow$ McKay correspondence for reflection groups.

I Reflection groups: Δ

Assume $k = \mathbb{C}$, $G_{\text{finite}} \subseteq GL_n(k)$ gen. by order 2 reflections

$G \circlearrowright V$ also on $S = \mathrm{Sym}_k V = k[x_1, \dots, x_n]$

$T = S^G$ invariant ring

Thm (Chevalley-Shephard-Todd)
$G \subseteq GL_n(k)$ finite. Then
$T = S^G$ is a polynomial ring
$\Leftrightarrow$ G gen. by pseudoreflections

$T \cong k[f_1, \dots, f_n] \hookrightarrow S = k[x_1, \dots, x_n]$

Geometrically:

$$\begin{array}{cccc} \pi : & \mathrm{Spec}(S) & \longrightarrow & \mathrm{Spec}(T) \\ & \cong & & \cong \\ & V & & V/G \end{array}$$

$$\pi : V \longrightarrow V/G$$

$\cup\!\!/$

mirrors of the refl. grp $\longrightarrow$ $\mathrm{Im}(V(z)) =: \Delta$ <u>Discriminant of G</u>

$$z = \prod_{g \in G} \ell_g$$

hyperplane arrangement

$$z = \mathrm{Jac}(f_1, \dots, f_n) = \det\left(\frac{\partial f_i}{\partial x_j}\right) \qquad \Delta = z^2$$

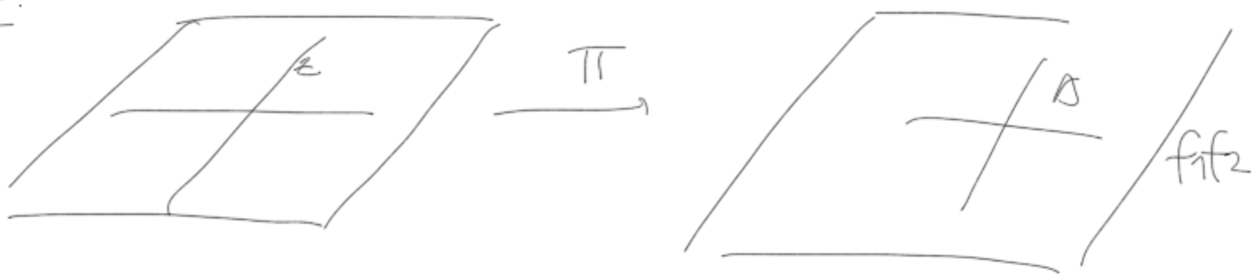
<u>Example</u> $G = \mu_2 \times \mu_2 \circlearrowright k[x,y]$

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad T = k[\underset{f_1}{x^2}, \underset{f_2}{y^2}]$$

$$z = 4xy$$

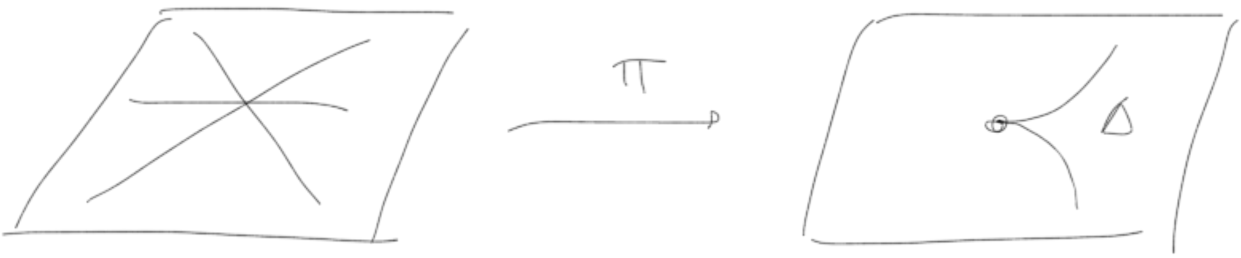
$$z^2 = \Delta = x^2y^2 = f_1 f_2$$

<u>Picture</u>:



$\rightsquigarrow$ same for $(\mu_2)^n$ $\rightsquigarrow$ Δ = normal crossing divisor

Example: $G = S_3$



Those Δ's are singular in codim 1

Saito free divisors

Q: construct NCR's for Δ?

$\boxed{\text{II NC(C)Rs}}$

Def: Let R be commut. Noeth. ring (R reduced)

An R-algebra A is called NCR if

$A = \operatorname{End}_R M$ for M fg faithful R-module

with $\operatorname{gldim} A < \infty$.

(Dao-Iyama-Takahashi-Vial)

(van den Bergh): A is NCCR (NC crepant R)

if A is nonsingular order, i.e.

$\operatorname{gldim} A_p = \dim R_p$ for $p \in \operatorname{Spec} R$

and A is maximal CM.

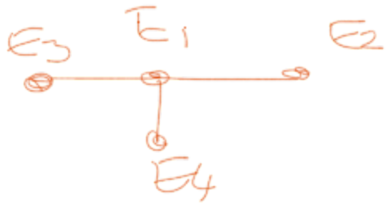
Example McKay correspondence

Let $\Gamma \subseteq SL_2(k)$ finite acting on V, $\dim V = 2$

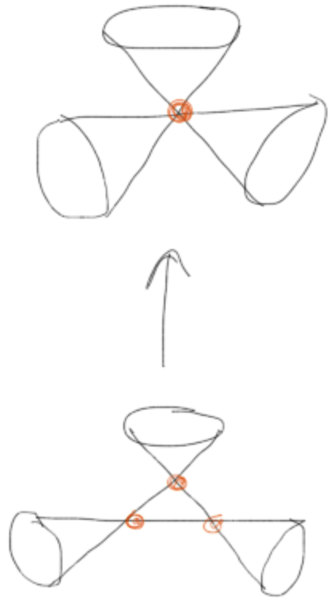
$S = \mathrm{Sym}_k V$. $\quad R = S^\Gamma$

then $X = \mathrm{Spec}(R) = V/\Gamma$ is a Kleinian singularity $\rightsquigarrow$ classified by ADE-Dynkin diagrams.

E.g. D_4 : $z^2 + x(x^2+y^2)$



Seq. of blowups



McKay: 1979

considered McKay graph $\rightsquigarrow$ 1-1 corresp.

$$\{\text{irr reps of } \Gamma\}/_\sim \overset{1:1}{\longleftrightarrow} \{\text{indec. of ex. div. of } \tilde{X}\}$$

$$1:1 \updownarrow \text{Auslander}$$

$$\left\{\begin{matrix}\text{indec. max. CM} \\ \text{over } R = S^\Gamma\end{matrix}\right\} \overset{1:1}{\longleftrightarrow} \{\text{Proj. } S * \Gamma\text{-modules}\} \leftrightarrow \mathrm{add}_R S$$

Theorem (Auslander 1986)

Let $\Gamma \subseteq GL_n(k)$ small (no pseudo refl.)

Let $A = S * \Gamma$ skew group ring :

elts $\sum_{\gamma \in \Gamma} s_\gamma \gamma$ with mult. $(s\gamma)(s'\gamma') = s\gamma(s')\cdot\gamma\gamma'$

Then

$$S * \Gamma \longrightarrow \mathrm{End}_{S^\Gamma}(S) \quad \text{isom.}$$

Also $S * \Gamma$ is max. CM over $S^\Gamma = R$

gldim $= n$ and $Z(S*\Gamma) = R$

Remark ① $A = S * \Gamma$ is NCCR of R.

② $S * \Gamma \underset{\text{Morita}}{\sim} \Pi(\tilde{\Delta})$

extended Dynkin diag

(Reiten - Van den Bergh)

$$\boxed{\text{III NCR of } \Delta}$$

$G \subseteq Gl_n(k)$ refl. grp. , $T = S^G$

still have $S * G$ but not isom to $\mathrm{End}_T S$

<u>Theorem</u> (BFI)

let $G \subseteq Gl_n(k)$ be true refl grp, $\Gamma = G \cap Sl_n(k)$

$R = S^\Gamma$, $T = S^G$. Then

$$1 \longrightarrow \Gamma \longrightarrow G \longrightarrow \underset{\substack{\cong \\ \mu_2 = \langle \sigma \rangle}}{H} \longrightarrow 1$$

and $A = S * G \cong \mathrm{End}_{R*H}(S * H)$

<u>with</u> $Z(A) = T$.

Denote by $e_\pm = \frac{1}{2}(1 \pm \sigma) \in R * H$ idempotents

Recollment: $B = R * H$

$$\mathrm{mod}\ \underbrace{B/Be_-B}_{\substack{\cong \\ T/\Delta}} \xleftarrow{\ i^* = - \otimes_B B/Be_-B\ } \mathrm{mod}\ B$$

Theorem: (BFI)

$$CM(B)/(e_B) \simeq CM(T/\Delta)$$

Knörrer dim 2

Compute $i^*(A) \cong End_{T/\Delta}(i^*(S*H))$

$CM(B)/e_B \rightsquigarrow A \rightsquigarrow \bar{A} \cong A/AeA$

where $e = \frac{1}{|G|}\sum_{g\in G} g$

Theorem (BFI)

$\bar{A} \cong End_{T/\Delta}(i^*(S*H))$ is NCR of T/Δ

of gldim n. with

$i^*(S*H) \cong S/z$

hyperplane arr.

$\top$ gldim n also follows from Auslander – Platzeck Todorov

$$Proj(\bar{A}) \overset{1:1}{\longleftrightarrow} add_{T/\Delta}(S/z) \overset{1:1}{\longleftrightarrow} \text{nontriv. } G \text{ irreps.}$$

+ (slides)