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Brauer configuration algebras & multiserial algebras

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$k = \bar{k}$, $\Lambda = kQ/I$, I admissable

we say Λ is biserial if each left & right indec. proj Λ-mod P satisfies

$$\mathrm{rad}(P) = U + U' \quad U, U' \text{ uniserial}$$

and $U \cap U' = 0$ or simple

↳ Skowronski-Waschbüsch 1983

we say Λ is multiserial if each left & right indec. proj. P satisfies

$$\mathrm{rad}(P) = \sum U_i \quad , U_i \text{ uniserial}$$

if $i \neq j$ $U_i \cap U_j = 0$ or simple

↳ Von Höhne-Waschbüsch 1984

We say Λ is special biserial if

1. for each $a \in Q_1$ there is at most one arrow b s.t. $ab \notin I$ and at most one arrow c s.t. $ca \in I$.
2. for each vertex v, there are at most two arrows ending at v and at most two arrows starting at v

↳ Wald-Waschbüsch 1985

We say that Λ is <u>special multiserial</u> if

1. holds.

- special biserial $\Rightarrow$ biserial
  └ Wald-Waschbüsch
- biserial $\Rightarrow$ tame
  indec. mod's are string or band mod's

- special multiserial $\Rightarrow$ multiserial
  └ G.S.
- wild (usually)

? ? ?

<u>Λ symmetric</u>: $\Lambda \cong D(\Lambda) = \mathrm{Hom}_k(\Lambda, k)$

a cycle in Q is <u>simple</u> if it has no repeated arrows.

$\mathcal{S}$ = set of simple cycles

$\mu : \mathcal{S} \longrightarrow \mathbb{Z}_{\geq 1}$

we say $(\mathcal{S}, \mu)$ is a <u>defining pair</u> if

1) $\mathcal{S}$ closed under cyclic permutation

$$(a_1 \cdots a_n) \in \mathcal{S} \Rightarrow (a_2 a_3 \cdots a_n a_1) \in \mathcal{S}$$

2) μ is constant on cyclic permutation classes

3) every arrow is in a unique cyclic permutation class.

we call the cycles in $\mathcal{S}$ <u>special cycles</u>

Define an <u>ideal</u> for a defining pair $(\mathcal{S}, \mu)$

I is gen. by

1) ab s.t. ab does not lie on a special cycle

2) If $C, C' \in \mathcal{S}$ cycles at vertex v then

$$C^{\mu(C)} - (C')^{\mu(C')}$$

3) If $C \in \mathcal{S}$ then $C^{\mu(C)} a_1 \in I$

where $C = (a_1 \cdots a_n)$

$\Lambda = kQ/I$ (for a defining pair) is called an $\boxed{\text{algebra generated by cycles}}$

Brauer Configuration:

$(\Gamma_0, \Gamma_1, \mu, \Theta)$ where

$\Gamma_0 = \{\alpha_1, \dots, \alpha_m\}$

$\Gamma_1 = \{V_1, \dots, V_n\}$, V_i multiset of elts of Γ_0
↳ can have repeated elt.

$\mu : \Gamma_0 \longrightarrow \mathbb{Z}_{\geq 1}$

Θ orientation

Idea: $\Gamma_0 \longleftrightarrow$ & cycles

for each $\alpha \in \Gamma_0$ consider $V_{i_1}, \dots, V_{i_s}$
s.t. $\alpha \in V_{i_j}$ $1 \leq j \leq s$

Θ gives an order

wlog $V_{i_1} < V_{i_2} < \dots < V_{i_s}$ $\qquad$ V_i vertices
$\qquad\qquad$ arrows

$$\updownarrow$$

$$V_{i_1} \to V_{i_2} \to \dots \to V_{i_s} \to V_{i_1}$$

corresp. algebra is called Brauer conf. alg. BCA

(If Γ_0, Γ_1 graph Brauer graph alg.)

Theorem Λ symm. TFAE

1) Λ sym. special multiserial alg.
2) Λ algebra def. by cycles
3) Λ is BCA.

Def: Λ is gentle if it is special biserial with quadratic monomial ideal I.

Def: Λ is almost gentle if special multi-serial with I quadr. monomial.

let ${}_\Lambda M_\Lambda$ bimodule, take $\Lambda \ltimes M$ with mult. $(a,m)(b,n) = (ab, an+mb)$

Theorem: If A is almost gentle then $A \ltimes D(A)$ is BCA.

partial converse:
Suppose Λ is BCA with def. pair (δ, μ)
An edge cut $\mathcal{E} = \left\{ a \,\middle|\, \begin{array}{l} \exists!\ C \text{ upto cyclic perm.} \\ \text{with } a \in C \end{array} \right\}$
set of arrows.
$\rightarrow$ For each perm. class choose one arrow.

Proposition
Λ BCA with edge cut $\mathcal{E}$ then
$A := k(Q \setminus \mathcal{E}) / I \cap k(Q\setminus\mathcal{E})$ is almost gentle.
Moreover, $A \ltimes D(A) \cong \Lambda$.

Theorem:

Let Λ be special multiserial algebra. Then $\exists$ a BCA Λ^* and a ring surjection

$$\Lambda^* \longrightarrow \Lambda$$

Def: We say a Λ-mod M is *multiserial* if

$$\operatorname{rad}(M) = \sum_i U_i, \quad U_i \text{ uniserial}$$

with $i \neq j$ then $U_i \cap U_j = 0$ or simple.

Theorem (*):

If Λ multiserial alg. then every fg Λ-mod. is multiserial.

rad M 

$r^3 = 0$:

Theorem: Let Λ be sym. indec. alg. with $r^3 = 0$ but $r^2 \neq 0$. Then Λ is BCA.

Open: biserial $\Rightarrow$ special biserial?

multiserial $\Rightarrow$ special multiserial?

<u>Open:</u>

A fg alg, if $A \ltimes D(A)$ is BCA
$\Rightarrow$ A almost gentle?

"Well known"

M $n \times n$ sym. matrix with entries in $\mathbb{Z}_{\geq 0}$

$M \longleftrightarrow G$ graph with n vertices

given $G \longleftrightarrow$ covered sym. $\mathrm{rad}^3 = 0$ alg.

$$\circ\!\!-\!\!\circ \longleftrightarrow \circ \underset{a^*}{\overset{a}{\rightleftarrows}} \circ$$

kQ/I where $\left.\begin{array}{l} aa^* - a^*a \\ ab, a^*b, ab^* \end{array}\right\} \in I$

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \qquad M' = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$\hookrightarrow$ isom.

$M = (1) \rightsquigarrow k[x,y]/(xy - yx, x^2, y^2)$

don't get $k[x,y]/(xy, x^2 - y^2)$

$\Big\}$ isom if $\sqrt{-1} \in k$ else not