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Finiteness of global dimension of endomorphism algebras

Λ ring, $M \in \operatorname{mod}\Lambda$, $\operatorname{End}_\Lambda(M)$

$\operatorname{egd} M := \operatorname{gl.dim} \operatorname{End}_\Lambda(M)$ <u>end-global dimension</u>

Study modules with $\operatorname{egd} M < \infty$

<u>History</u>:

- Auslander: "Representation dimension of artin algebras." 1971
- Quasi-hereditary [Cline-Parshall-Scott 1988] [Dlab-Ringel 1989]
- Finitistic conjecture
- dim of derived category [Rouquier]

  $\dim \mathcal{D}^b(\operatorname{mod}\Lambda) \leq \operatorname{egd} M$

  $\forall M$: generator of Λ
- n-cluster tilting module
- Non-commutative crepant resolution (NCCR) [van den Bergh]

Example:

M semisimple $\rightsquigarrow$ $\mathrm{End}_\Lambda M$ semisimple

$\Rightarrow$ egd $M = 0$

Impose M $\begin{cases} \text{faithful} & \rightsquigarrow \text{excludes trivial case} \\ \text{generator } (-\text{cogenerator}) \end{cases}$

Definition: [Dao-Takahashi-Vial-I.]

Call $M \in \mathrm{mod}\,\Lambda$ a non-commut. resolution (NCR) if egd $M < \infty$ and M is faithful.

Theorem A

Λ fin dim alg / field k

[A] 1 If Λ is representation-finite then

$$\mathrm{egd}\Big(\bigoplus_{\substack{X \text{ indec.} \\ \Lambda\text{-mod.}}} X \Big) \leq 2$$

[A] 2 Λ has an NCR egd $\bigoplus \Lambda/\mathrm{rad}^i \Lambda < \infty$
$\leq ll(\Lambda)$

[I] 3 M: n-cluster tilting module

$\Rightarrow$ egd $M \leq n+1$

(gen. of 1 : $n = 1$)

[I] 4 $\forall\, M \in \mathrm{Mod}\,\Lambda$ $\exists$ $N \in \mathrm{Mod}\,\Lambda$:

$$\mathrm{egd}(M \oplus N) < \infty$$

Construction of N in 4

$M_0 := M$ $\quad M_i$ is Λ-$\mathrm{End}_\Lambda M_i$-bimodule

set $\quad M_{i+1} = \mathrm{rad}(M_i)_{\mathrm{End}_\Lambda(M_i)} \subset M_i$

$$\Rightarrow \quad M_0 \supsetneq \underbrace{M_1 \supsetneq M_2 \supsetneq \cdots \supsetneq M_n}_{n} = 0$$

then $\quad N := \bigoplus_{i=1}^{n} M_i$

Ingredient:

Lemma [A-Platzeck-Todorov]

Γ ring, $\Gamma \ni e$ idempotent then

$$\mathrm{gldim}\,\Gamma \leq \mathrm{gldim}(e\Gamma e) + \mathrm{gldim}(\Gamma/(e)) + \mathrm{pd}_\Gamma(\Gamma/(e)) + 1.$$

Proposition:

Λ ring, $N \in \mathrm{mod}\,\Lambda$. Assume

$\exists$ $\quad \mathrm{add}\,N = \mathcal{C}_0 \supset \mathcal{C}_1 \supset \cdots \supset \mathcal{C}_n = 0$ chain of full subcats with $\mathcal{C}_i = \mathrm{add}\,N_i$

right rejective chain

satisfies

1. $\forall i \;\exists\; 0 \to \underbrace{L_\ell \to \cdots \to L_0}_{\in \mathcal{C}_{i+1}} \to N_i$ complex

s.t. $\quad 0 \to (-, L_\ell) \to \cdots (-, L_0) \to (-, N_i) \to 0$ is exact in $\mathcal{C}_{i+1}$.

② $\operatorname{gldim} \operatorname{End}_{\mathcal{C}_i/[\mathcal{C}_{i+1}]}(N_i) < \infty \quad \forall i$

$\quad\llcorner$ factor cat. of $\mathcal{C}_i$

Then $\quad \operatorname{egd} N < \infty$.

<u>Proof of Thm A ④:</u>

$\mathcal{C}_i := \operatorname{add}\left(\bigoplus_{j \geq i} M_j\right)$

$\operatorname{add}(M \oplus N) = \mathcal{C}_0 \supset \mathcal{C}_1 \supset \cdots \supset \mathcal{C}_n = 0$

right rejective chain. $\square$

<u>Remark</u>: $\operatorname{End}_\Lambda(M \oplus N)$ is $\overbrace{\text{quasi-hereditary}}^{\text{strongly}}$ algebra in the sense of Ringel

<u>Theorem B</u>: [I., Leuschke $M = \Lambda$] $\dim R = 1$

R complete DVR, Λ R-order

K quotient field of R

Assume $\Lambda \otimes_R K$ is semisimple. Then

$\forall M \in CM(\Lambda) \; \exists N \in CM(\Lambda)$ s.t. $\operatorname{egd}(M \oplus N) < \infty$

$\therefore \quad \Lambda$ has an NCR.

Construction same as in Thm A, but sequence is infinite. $\exists n \gg 0$: $\operatorname{add} M_n = \operatorname{add} M_{n+1}$

Question: What about $\Lambda \otimes_R k$ not semisimple?

Theorem C [DITV, cf D-Faber-Ingalls]

R complete local normal domain, $\dim R = 2$ with $R/\mathrm{rad} R$ alg. closed. Then

R has an NCR $\iff$ R is a rational singularity.

Sketch:

$\exists$ M: NCR of R i.e. $K_0(\mathrm{mod}\,\Gamma)$ fin. gen.

$\Gamma = \mathrm{End}_R M$

$\exists\ e\Gamma e = R$

$\Downarrow$

M gen. [Wunram, I-Wemyss]

$K_0(\mathrm{mod}\,R)$ fin. gen.

$\Downarrow$

R: rational singularity $\iff$ $Cl(R)$ fin. gen.
[Lipman 69]