15/08/16

Hugh Thomas:
Semistable subcategories for preprojective algebras

or Preprojective algebra & Stability
jt David Speyer
also drawing on work Iyama, Reiten, Reading

1. Semistable subcategories

A fd alg. simples $S_1, -, S_n$

$$K_0(A) = \bigoplus_{i=1}^{n} \mathbb{Z}[S_i]$$

$M \in A\text{-mod} \rightsquigarrow [M] \in K_0(A)$

Q: how many times each simple appears in a composition series?

dual $K_0^*(A) = \mathrm{Hom}_{\mathbb{Z}}(K_0(A), \mathbb{R})$
$\quad \ni \varphi$

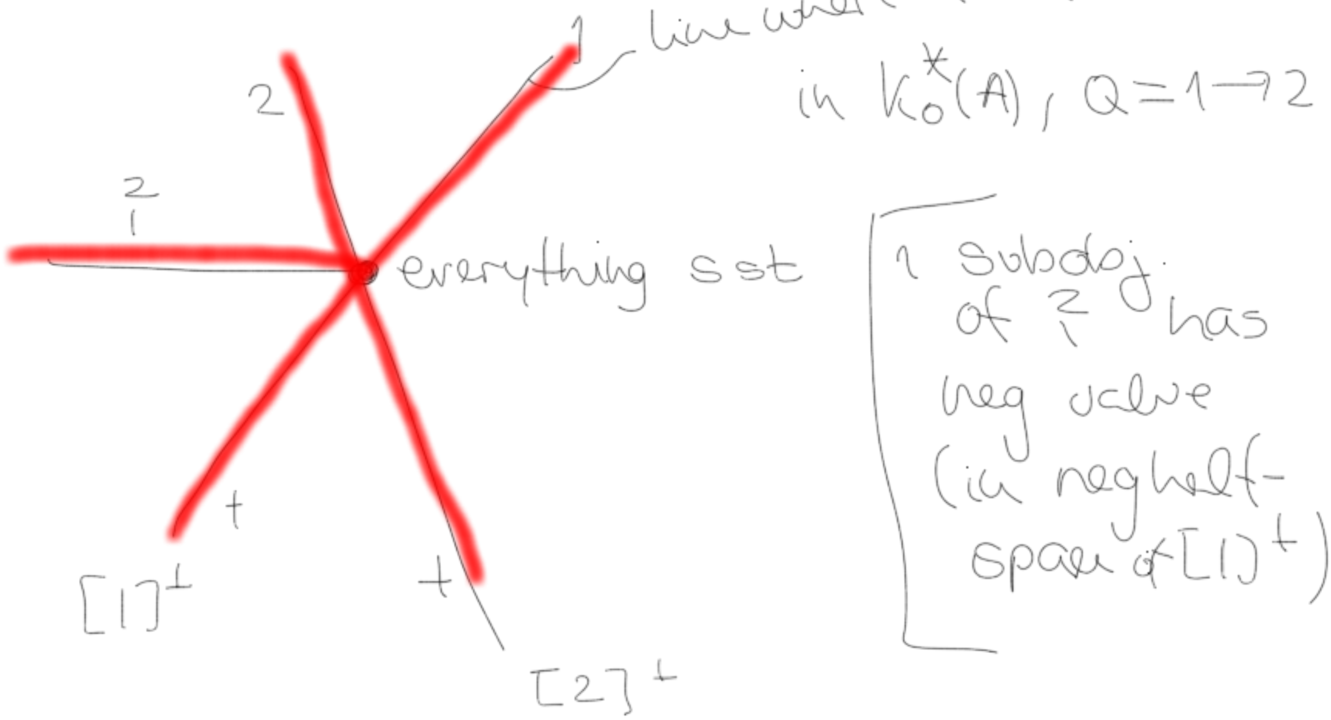
$M \in A\,\mathrm{mod}$ is semistable wrt φ if

$$\varphi([M]) = 0 \qquad \varphi([N]) \leq 0 \quad \forall N \hookrightarrow M$$

$$(A\text{-mod})_\varphi = \varphi \text{ sstable } A\text{-mod}$$

[King] abelian and extension closed.

Ex:



Bricks of A are A-modules M st.

$\mathrm{End}_A(M)$ is division algebra

$\hookrightarrow$ $Br(A)$

Abelian, ext clsd subcat is def. by its bricks

Q simply laced Dynkin

for each $i \xrightarrow{a} j$ add $j \xrightarrow{a^*} i$ $\rightsquigarrow$ $\overline{Q}$

def. $\Pi = k\overline{Q} / (\sum_a aa^* - a^*a)$ preproj. alg

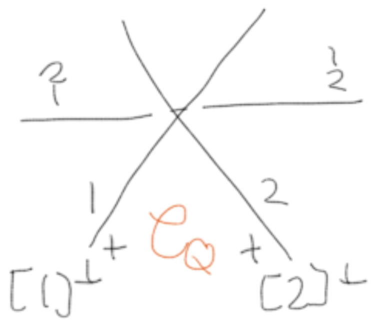
dimensions of bricks of Π form pos. roots of root sys. of type Q inside $K_0(\Pi)$

$\mathcal{H}_Q$ = collection of dual hyperplanes inside $K_0^*(\Pi)$

Theorem (Crawley-Brevey)

$$(\Pi\text{-mod}) \ \varphi \neq 0 \iff \varphi \in \bigcup_{H \in \mathcal{H}_Q} H$$

Exp:



$$\overline{Q} = 1 \rightleftarrows 2$$

$$\operatorname{Br}\Pi = \operatorname{ind}\Pi = \{1, 2, \begin{smallmatrix}2\\1\end{smallmatrix}, \begin{smallmatrix}1\\2\end{smallmatrix}\}$$

$$\varphi \in \mathcal{C}_Q \iff \varphi([S_i]) > 0$$

$\mathcal{H}$ finite collection of hyperplanes in $\mathbb{R}^n$

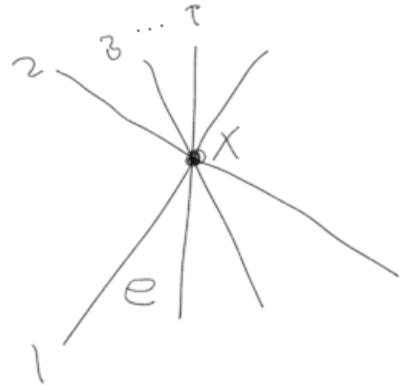
chambers: comp. of $\mathbb{R}^n \setminus \bigcup_{H \in \mathcal{H}} H$

technical assump.: chambers are simplicial
 i.e. n extremal rays

Choose base chamber $\mathcal{C}$ and def

$$\mathcal{H}^2 = \{ H \cap K \mid H, K \in \mathcal{H} \text{ distinct} \}$$

$X \in \mathcal{H}^2$



number hyperplanes cont. X $1,\dots,r$ s.t.
e btw 1 and r.

def $\mathrm{Split}(X) = \{H_2, \dots, H_{r-1}\}$

$$\mathcal{W}_H(\mathcal{H}) = \text{components of } \mathcal{H} \setminus \bigcup_{X : H \in \mathrm{Split}(X)} X$$

$$\mathcal{W}(\mathcal{H}) = \bigcup_{H \in \mathcal{H}} \mathcal{W}_H(\mathcal{H})$$

Def. by Nathan Reading

$\mathcal{P}(\mathcal{H}, e)$ lattice, then $\mathcal{W}(\mathcal{H})$
poset str.

$\updownarrow$ corr.

join irred of $\mathcal{P}(\mathcal{H}, e)$

consider $\mathcal{H} = \mathcal{H}_Q$ $e = e_Q$

Theorem (Iyama-Reading-Reiten-T.)

$$\mathrm{Br}(\Pi) \longleftrightarrow \mathcal{W}(\mathcal{H}_Q)$$

Theorem (Speyer-T.)

$$M \longleftrightarrow E_M \qquad M \in (\Pi\text{-mod})^{\varphi} \iff \varphi \in \overline{E}_M$$

Def $\mathcal{W}\mathrm{Int} = \{\text{intersections of } \overline{E} \text{ for } E \in \mathcal{W}(\mathcal{H})\}$

then $(\mathcal{W}\mathrm{Int}, \overset{\text{order}}{``\subseteq"}) \longleftrightarrow$ (semistable subcat of Π, "$\subseteq$")

$\wr\wr$ $\qquad$ " [Mizuno]

coxeter grp $(W, ``\leq_{\text{shard}}")$ $\qquad$ abelian, ext-closed

$A = \Pi / I$

$\operatorname{Br} A = \{ M \mid M \in \operatorname{Br}(\Pi),\ MI = 0 \}$

$\rightsquigarrow$ semistability picture for Π/I is obtained from that of Π by erasing some strands.

non cross. part.

$$
\begin{tikzcd}
\text{abelian ext-closed subcat of } kQ \arrow[r, leftrightarrow] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowright"] & NC_Q(W) \arrow[d, hook] \\
\text{abelian ext-closed subcats of } \Pi \arrow[r, leftrightarrow] & W
\end{tikzcd}
$$