TROPICAL GEOMETRY AND REPRESENTATION THEORY OF REDUCTIVE GROUPS

CHRISTOPHER MANON

Lecture 1

We start by recalling that the tropical variety of an affine variety equipped with an embedding can be constructed as the image of the set of all valuations on its coordinate ring. This means that if we are given a source of valuations we can create and study portions of the tropicalization. For G-varieties and other related spaces, the representation theory of a connected reductive group G provides a mechanism to create portions of tropical varieties and study them with an established combinatorial language.

1. Valuations and tropical geometry

Let V be an algebraic variety over $\mathbf{k}$ an algebraically closed, trivially valued field, and suppose that V is the zero locus of an ideal $I \subset \mathbf{k}[\mathbf{x}]$ for $\mathbf{x} = \{x_1, \ldots, x_n\}$. One indicator that the tropical variety of V is capturing useful information is that it can be constructed in several apparently different ways. For example, from [MS15, Theorem 3.2.3] we see the Gröbner theoretic point of view (the initial ideal $in_w(I)$ $w \in \mathrm{Trop}(I)$ contains no monomials) connected with valuations on the coordinate ring $\mathbf{k}[V]$ associated to the $\mathbb{K}$ points of V for $\mathbf{k} \subset \mathbb{K}$ a valued field extension. The following (see [Pay09]) provides another perspective on the valuative description of a tropical variety.

Remark 1.1. For these lectures we use the convention that valuations are sub-additive: $v(f+g) \leq MAX\{v(f), v(g)\}$ to conform with conventions of the dominant weight ordering in the dominant weights of a reductive group.

Proposition 1.2. *Let $I \subset \mathbf{k}[\mathbf{x}]$ be a prime ideal which cuts out a variety $V \subset \mathbb{A}^n$, and let V^{an} be the set of valuations $v : \mathbf{k}[V] \setminus \{0\} \to \mathbb{R}$ which restrict to the trivial valuation on $\mathbf{k}$. Then the map $ev_{\mathbf{x}} : V^{an} \to \mathbb{R}^n$, $ev_{\mathbf{x}}(v) = (v(x_1), \ldots, v(x_n))$ surjects onto the tropical variety $\mathrm{Trop}(I) \subset \mathbb{R}^n$.*

The notation V^{an} for the set of valuations is a reference to the fact that this is the underlying set of the Berkovich Analytification of V (see [Pay09]). The topology on V^{an} is the coarsest topology which makes the evaluation functions $ev_f : V^{an} \to \mathbb{R}$, $ev_f(v) = v(f); f \in \mathbf{k}[V]$ continuous.

The space V^{an} defies description outside of restricted cases (ie $dim(V) = 1$). However, when the variety V comes equipped with a distinguished class of valuations, 1.2 implies that every tropicalization of V sees a part of such a class. This is the case with varieties equipped with a reductive group action, and other varieties closely related to the representation theory of G.

2. Notation

(1) G- a connected, reductive group over $\mathbf{k}$,

(2) T- a maximal torus of G,

(3) U- a maximal unipotent subgroup of G.

(4) B- a Borel subgroup of G, recall that for compatible choices $B = TU$,

(5) $\Lambda = Hom(T, \mathbb{G}_m)$- the lattice of weights (associated to the choice of T),

(6) Λ_+- the monoid of dominant weights (associated to the choice of B),

(7) $V(\lambda)$, $\lambda \in \Lambda_+$- the irreducible representation associated to λ,

(8) $\mathfrak{g}$- the Lie algebra of G,

(9) $\mathfrak{h}$- the Lie algebra of T,

(10) $\mathfrak{n}$- the (nilpotent) Lie algebra of U,

(11) R- the roots (weights which appear in adjoint representation on $\mathfrak{g}$),

(12) $\mathcal{R}$- the root lattice (generated by R),

(13) R_+- positive roots (associated to the choice B),

(14) Δ- Weyl chamber, the convex hull of $\Lambda_+ \subset Hom(T, \mathbb{G}_m)$,

(15) $\Delta^\vee \subset Hom(\mathbb{G}_m, T) \otimes \mathbb{R}$- dual Weyl chamber (coweights which pair to non-negative real numbers with positive roots).

3. A motivating example

The Grassmannian $Gr_2(n)$ of 2-planes in an n-dimensional space, it's projective coordinate ring $R_{2,n}$, and the tropical variety $\mathcal{T}(n) = \mathrm{Trop}(I_{2,n})$ of the ideal which vanishes on the Plücker generators $p_{ij} \in R_{2,n}, 1 \leq i < j \leq n$ are very well understood (see [SS04]). Nevertheless, there is something to be gained by revisiting what we know about these objects from the point of view of representation theory. The Grassmannian case will provide a useful example of how representation theory of a reductive group G can influence the structure of the tropical varieties of spaces related to G. Some of what follows will appear in joint work with Jessie Yang [YM], see also [Man11].

The Grassmannian $Gr_2(n)$ is a flag variety for GL_n, and the associated Pĺucker algebra $R_{2,n}$ is the projective coordinate ring associated to the Pĺucker line bundle $\mathcal{L}_{\omega_2^*}$, so it has a natural homogeneous grading $R_{2,n} = \bigoplus_{m\geq 0} V(m\omega_2)$, where $V(m\omega_2)$ is the irreducible GL_n representation associated to the dominant weight $m\omega_2 = (m, m, 0, \dots, 0)$. However, there is another interpretation of $R_{2,n}$ in terms of the representation theory of SL_2. A classical result of representation theory states that $R_{2,n}$ is the algebra of SL_2 invariants inside the coordinate ring $\mathbf{k}[M_{2\times n}]$ of the space of $2 \times n$ matrices. In particular $R_{2,n} \subset \mathbf{k}[M_{2\times n}]$ is generated by the 2×2 minors of a $2 \times n$ matrix of indeterminants: $\{x_{1i}, x_{2,i}, 1 \leq i \leq n\}$, $p_i j = x_{1,i}x_{2,j} - x_{1,j}x_{2,i}$.

Changing perspective slightly, we may view $M_{2\times n}$ as the $n-$fold product $\mathbb{A}^2 \times \mathbb{A}^2$. The i-th copy of $\mathbb{A}^2$ has coordinate ring a polynomial ring on two variables: $\mathbf{k}[x_{1,i}, x_{2,i}]$. The group SL_2 naturally acts on $\mathbf{k}[x_{1,i}, x_{2,i}]$, so its coordinate ring has an *isotypical* decomposition into the irreducible representations of SL_2. Happily, for $\mathbf{k}[x_{1,i}, x_{2,i}]$, this decomposition is both multiplicity-free, and contains each irreducible representation of SL_2 exactly once.

$$\mathbf{k}[x_{1,i}, x_{2,i}] \cong \bigoplus_{m\geq 0} Sym^m(\mathbf{k}^2). \tag{1}$$

Here we may think of $Sym^m(\mathbf{k}^2)$ as the monomials of total degree m in $x_{1,i}, x_{2,i}$. We will write $V(m) = Sym^m(\mathbf{k}^2)$ for the m-th irreducible. As consequence, we obtain the following descriptions of $\mathbf{k}[M_{2\times n}]$ and $R_{2,n}$ in terms of the representation theory of SL_2:

$$\mathbf{k}[M_{2\times n}] = \bigoplus_{\mathbf{r}\in\mathbb{Z}^n_{\geq 0}} V(r_1) \otimes \dots \otimes V(r_n), \tag{2}$$

$$R_{2,n} = \mathbf{k}[M_{2\times n}]^{SL_2} = \bigoplus_{\mathbf{r}\in\mathbb{Z}^n_{\geq 0}} [V(r_1) \otimes \dots \otimes V(r_n)]^{SL_2}. \tag{3}$$

Immediately we see some structure. For one, $R_{2,n}$ is multigraded by $\mathbb{Z}^n$, and the component associated to $\mathbf{r} \in \mathbb{Z}^n_{\geq 0}$ has representation theoretic meaning: it is the space of invariant vectors $[V(r_1) \otimes \dots \otimes V(r_n)]^{SL_2}$. This grading coincides with the homogeneous grading on $R_{2,n}$ induced by the action of the diagonal matrices $T \subset GL_n$.

Example 3.1. *Suppose $n = 3$, then we are dealing with $3-$fold tensor products $[V(i) \otimes V(j) \otimes V(k)]^{SL_2}$. The Clebsch-Gordon rule states that this space is either* $\mathbf{k}$

or 0*, and the former occurs precisely when* $i+j+k \in 2\mathbb{Z}$ *and* i, j, k *form the sides of a triangle. It follows that* $R_{2,3}$ *is a multiplicity-free under the grading by* $\mathbb{Z}^3$*, so it is an affine semigroup algebra. The affine semigroup of integral* (i, j, k) *which satisfy the Clebsch-Gordon condition is generated freely by* $(1, 1, 0), (0, 1, 1), (1, 0, 1)$ *[Pop Quiz: which tensor products of representations are these?]. This is the coordinate ring of* $\bigwedge^2(\mathbf{k}^3)$*, the affine cone over* $Gr_2(3)$*.*

Before we go further with $R_{2,n}$, let's review two rules for representations of a reductive group. Any irreducible $V(n)$ has a dual $V(n)^*$, which is canonically isomorphic to $V(n)$ itself in the SL_2 case (this does not always happen). For any pair of representations $Hom_G(V, W) = Hom_G(V(0), V^* \otimes W)$, where $V(0) \cong \mathbf{k}$ is the trivial representation; this is an instance of $Hom - \otimes$ adjunction. The space $Hom_G(V(0), V^* \otimes W)$ is the space of invariants $[V^* \otimes W]^G$. For any pair of irreducibles, $Hom_G(V, W)$ is 0 unless $V = W$, so we see that there is an invariant in a $2-$fold tensor product of irreducible representations is if and only if we are considering $V^* \otimes V$.

Now we will use properties of the category $Rep(SL_2)$ of finite dimensional SL_2 representations to tease out some more structure from $R_{2,n}$. For now let's pick one graded component $[V(r_1) \otimes \ldots \otimes V(r_n)]^{SL_2} \subset R_{2,n}$. We choose one additional piece of information, a trivalent tree $\mathcal{T}$ with n leaves labelled $1, \ldots, n$.

We will play a game with $\mathcal{T}$ and the space $[V(r_1) \otimes \ldots \otimes V(r_n)]^{SL_2}$ which begins by labelling the edge i-th connected to the i-th leaf of $\mathcal{T}$ with the integer r_i. Next we pick any two labelled edges which share a common vertex, say r_i and r_j, and we look at the tensor product $V(r_i) \otimes V(r_j)$. The Clebsch-Gordon rule in Example 3.1 says that this tensor product has a multiplicity-free decomposition:

$$V(r_i) \otimes V(r_j) = \bigoplus_{s \geq 0} [V(s) \otimes V(r_i) \otimes V(r_j)]^{SL_2} \otimes V(s). \tag{4}$$

You should be sure to understand this expression, and do an example (say $V(4) \otimes V(7)$); note that the triangle inequality ensures that only a finite number of summands contribute. Also, let's double check that indeed $[V(s) \otimes V(r_i) \otimes V(r_j)]^{SL_2} = Hom_{SL_2}(V(s), V(r_i) \otimes V(r_j))$, so each summand corresponds to a map (unique up to scalar!) $V(s) \to V(r_i) \otimes V(r_j)$. The product $V(r_i) \otimes V(r_j)$ appears in the tensor product $V(r_1) \otimes \ldots \otimes V(r_n)$, so we get the following expression:

$$[V(r_1) \otimes \ldots \otimes V(r_n)]^{SL_2} = \tag{5}$$

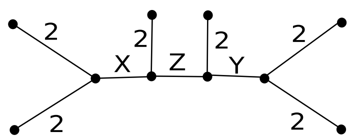
$$\bigoplus_{s \geq 0} [V(s) \otimes V(r_i) \otimes V(r_j)]^{SL_2} \otimes [V(s) \otimes \ldots \bar{V(r_i)} \ldots \bar{V(r_j)} \ldots \otimes V(r_n)]^{SL_2}.$$

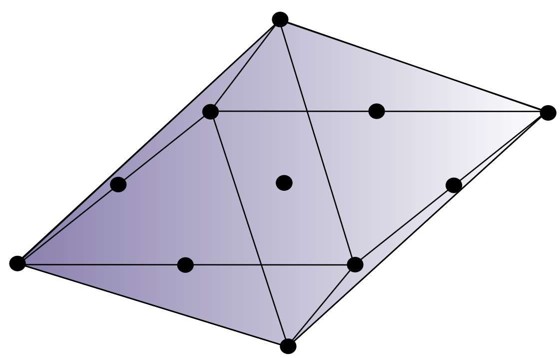
We can imagine that we have replaced $\mathcal{T}$ with labels $r_1, \ldots, r_n$ with a finite collection of labellings of $\mathcal{T}$, where we put an appropriate s on the edge connected to the edges labelled r_i, r_j whenever $[V(s) \otimes V(r_i) \otimes V(r_j)]^{SL_2} \neq 0$. Note that for each of these, if we "forget" i, j we have a tree with $n - 1$ labelled leaves, and we can continue with this game. Let's pretend we did, here's the result:

Proposition 3.2. *There is direct sum decomposition* $[V(r_1) \otimes \ldots \otimes V(r_n)]^{SL_2} = \bigoplus_{\mathbf{s}} W_{\mathcal{T}}(\mathbf{s}, \mathbf{r})$*, where each* $W_{\mathcal{T}}(\mathbf{s}, \mathbf{r})$ *is* $1-$*dimensional space, and corresponds to a labelling of the edges* $E(\mathcal{T})$ *of* $\mathcal{T}$ *by* $\mathbf{s}$ *in a way such that any time three edges* e, f, g

meet at a vertex, their labels $\mathbf{s}(e), \mathbf{s}(f), \mathbf{s}(g)$ *satisfy the Clebsch-Gordon condition. In particular,* $W_{\mathcal{T}}(\mathbf{s}, \mathbf{r}) = \bigotimes_{v \in V(\mathcal{T})} [V(\mathbf{s}(e)) \otimes V(\mathbf{s}(f)) \otimes V(\mathbf{s}(g))]^{SL_2}$.

Remark 3.3. This proposition allows us to compute the dimension of $[V(r_1) \otimes \ldots \otimes V(r_n)]^{SL_2}$ as the set of labellings $\mathbf{s}$ satisfying the Clebsch-Gordon conditions at every vertex of $\mathcal{T}$; these are the lattice points in a polytope $P_{\mathcal{T}}(\mathbf{r})$, see Figure 3.





A notable feature of this decomposition is that the product $W_{\mathcal{T}}(\mathbf{s}, \mathbf{r}) W_{\mathcal{T}}(\mathbf{s}', \mathbf{r}')$ is a subspace of the sum $\bigoplus_{\mathbf{s}'' \prec \mathbf{s} + \mathbf{s}'} W_{\mathcal{T}}(\mathbf{s}'', \mathbf{r} + \mathbf{r}')$, where $\prec$ indicates that *each entry* in $\mathbf{s}''$ is smaller than the sum of entries in $\mathbf{s} + \mathbf{s}'$. This is a hint that the tensor product decomposition for SL_2 has uncovered a useful algebraic, and ultimately tropical structure in $R_{2,n}$. To uncover this structure, we build $R_{2,n}$ in a different way which depends on $\mathcal{T}$ from the start.

Place a direction on each edge in $\mathcal{T}$ (this choice won't matter). We're going to build $R_{2,n}$ as the coordinate ring of a kind of quiver variety. let $L(\mathcal{T})$ be edges of $\mathcal{T}$ which are connected to leaves, and $F(\mathcal{T})$ be the non-leaf edges, and let

$$M_{\mathcal{T}} = \prod_{e \in L(\mathcal{T})} \mathbb{A}^2 \times \prod_{f \in F(\mathcal{T})} SL_2. \tag{6}$$

We're going to act on this space with a group defined by the non-leaf vertices $V(\mathcal{T})$:

$$G(\mathcal{T}) = \prod_{v \in V(\mathcal{T})} SL_2 \tag{7}$$

We use the direction to define the action of the group $G(\mathcal{T})$ on $M_{\mathcal{T}}$. The copy of SL_2 corresponding to a vertex v acts on the left of the space corresponding to any outgoing edge, and on the right of the space corresponding to an incoming edge. The proof of the following is not difficult; it makes repeated use of the identity $G\backslash\backslash[G \times X] \cong X$ for any G-space X.

Proposition 3.4.

$$R_{2,n} \cong \mathbf{k}[M_{\mathcal{T}}]^{G(\mathcal{T})}. \tag{8}$$

By the Peter-Weyl theorem, the coordinate ring of SL_2 has a multiplicity-free decomposition (as an $SL_2 \times SL_2$ space):

$$\mathbf{k}[SL_2] = \bigoplus_{m \geq 0} V(m) \otimes V(m). \tag{9}$$

Pick an edge $f \in F(\mathcal{T})$, and think of each of these representations as sitting at f, with the left $V(m)$ sitting at the tail edge of the edge, and the right $V(m)$ sitting at the head. Recall also that $\mathbf{k}[\mathbb{A}^2] = \bigoplus_{r \geq 0} V(r)$; think of each of these representations sitting at a leaf-edge $e \in L(\mathcal{T})$. Make a choice of $\mathbf{s}, \mathbf{r}$ from all of these direct sums; this is an isotypical summand of the coordinate ring of $M_{\mathcal{T}}$; by taking invariants by $G(\mathcal{T})$ we exactly get the space $W_{\mathcal{T}}(\mathbf{s}, \mathbf{r})$.

Finally, we see how this extravagant construction of $R_{2,n}$ helps us understand the tropical geometry of this algebra. Both coordinate rings $\mathbf{k}[SL_2]$ and $\mathbf{k}[\mathbb{A}^2]$ have a distinguished valuation, and therefore a distinguished ray of valuations by taking $\mathbb{R}_{\geq}0$ multiples. On $\mathbf{k}[\mathbb{A}^2]$ this valuation is given by homogeneous degree $deg : \mathbf{k}[\mathbb{A}^2] \setminus \{0\} \to \mathbb{Z}$, ie $g \in V(n)$ is sent to n. To compute the valuation $v : \mathbf{k}[SL_2] \setminus \{0\} \to \mathbb{Z}$ we do almost the same thing; $f \in V(n) \otimes V(n)$ is sent to n. Any multiple of deg is in fact still a valuation on $\mathbf{k}[\mathbb{A}^2]$, while only positive multiplies are allowed for v; this is because the decomposition of $\mathbf{k}[SL_2]$ is not a homogeneous grading. We let $v_e, e \in E(\mathcal{T})$ be the appropriate valuation for the space at e.

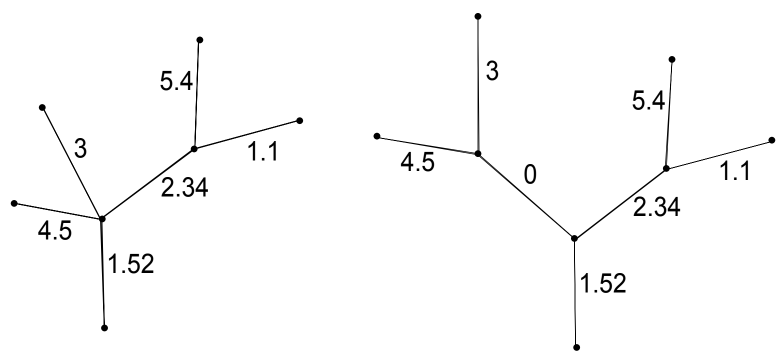
Let $C_{\mathcal{T}} \subset \mathbb{R}^{E(\mathcal{T})}$ be the set of functions $w : E(\mathcal{T}) \to \mathbb{R}$ which are non-negative on $F(\mathcal{T})$. For any $w \in C_{\mathcal{T}}$ we get a valuation v_w on $\mathbf{k}[M_{\mathcal{T}}]$ defined on a tensor $\otimes_{e \in E(\mathcal{T})} f_e$ by the rule $\sum w(e) v_e(f_e)$. Since $R_{2,n}$ is a subalgebra of $\mathbf{k}[M_{\mathcal{T}}]$, each valuation v_w passes to a $R_{2,n}$. The following proposition tells us how to evaluate v_w on any member of $W_{\mathcal{T}}(\mathbf{s}, \mathbf{r})$.

Proposition 3.5. *Let $f \in W_{\mathcal{T}}(\mathbf{s}, \mathbf{r})$, then*

$$v_w(f) = \sum_{e \in L(\mathcal{T}} w(e)\mathbf{r}(e) + \sum_{f \in F(\mathcal{T})} w(f)\mathbf{s}(f). \tag{10}$$

Remark 3.6. Any choice of basis member for each $W_{\mathcal{T}}(\mathbf{s}, \mathbf{r})$ defines an *adapted basis* for every valuation in $C_{\mathcal{T}}$. We will see what this means in the next lecture.

In other words, we "dot" the labelling $w \in C_{\mathcal{T}}$ with the labelling data $\mathbf{s}, \mathbf{r}$. If $\mathcal{T}'$ is obtained from $\mathcal{T}$ by a projection $\pi : \mathcal{T} \to \mathcal{T}'$ which collapses one or more edges in $F(\mathcal{T})$, we have a canonical inclusion $C_{\mathcal{T}'} \subset C_{\mathcal{T}}$ as those weightings which are 0 on the collapsed edges, see Figure 3. By gluing the $C_{\mathcal{T}}$ together over common faces we obtain a polyhedral complex $\mathcal{T}_n$ called the space of phylogenetic trees. We have shown that there is a realization of each point $w \in \mathcal{T}_n$ as a valuation $v_w : R_{2,n} \setminus \{0\} \to \mathbb{R}$; it follows that this complex maps into every tropical variety of $R_{2,n}$.



We finish this example by seeing what v_w for $w \in C_{\mathcal{T}}$ must do to a Pl$\acute{}$ucker generator p_{ij}. The space $[V(r_1) \otimes \ldots \otimes V(r_n)]^{SL_2}$ containing p_{ij} is easy to describe; it is the invariants in the product $V(0) \otimes \ldots V(1) \ldots V(1) \ldots \otimes V(0)$, where $V(1)$ only appears in the i-th and j-th places. Playing our game with $\mathcal{T}$, we see that since this space is 1-dimension, only one summand $W_{\mathcal{T}}(\mathbf{s}, \mathbf{r})$ can be non-zero; this is precisely the $\mathbf{s}$ which assigns a 1 to every edge on the unique path in $\mathcal{T}$ betwee i and j, and a 0 elsewhere. As a consequence $v_w(p_{ij})$ is the sum of the weights on this path. If we think of w as a metric on $\mathcal{T}$, this sum is the "distance" between the leaves labelled i and j.

4. The coordinate algebra of a G-variety

The SL_2 and $SL_2 \times SL_2$-invariant valuations of $\mathbb{A}^2$ and SL_2 played an important role in constructing the space $\mathcal{T}_n$ of valuations on $R_{2,n}$. Now we'll see what kind of valuations more general reductive groups can offer. In what follows X will be an affine G-variety (if you like, take X to be the affine cone over a projective G variety). In particular, the coordinate ring $\mathbf{k}[X]$ is an integral domain over $\mathbf{k}$ and comes equipped with a rational G action. By collecting the irreducible sub-representations with the same highest weight, we can form the isotypical decomposition of $\mathbf{k}[X]$:

$$\mathbf{k}[X] = \bigoplus_{\lambda \in \Lambda_X^+} W_\lambda. \tag{11}$$

The space $W_\lambda \cong Hom_G(V(\lambda), \mathbf{k}[X]) \otimes V(\lambda)$ is called the isotypical space of weight λ. The set $\Lambda_X^+ \subset \Lambda_+$ is composed of those weights for which $W_\lambda \neq 0$. If $b_\lambda \in W_\lambda, b_\eta \in W_\eta$ are highest weight vectors, must have that $b_\lambda b_\eta \neq 0$ is a highest weight vector of weight $\lambda + \eta$. It follows that Λ_X^+ is closed under addition; it's also easy to see that $0 \in \Lambda_X^+$ as $\mathbf{k} \subset W_0$. The lattice generated by Λ_X^+ is denoted Λ_X.

Example 4.1 (Affine Toric Varieties). *In the fashion of toric geometry we let N be a lattice and $M = Hom(N, \mathbb{Z})$ be the dual lattice. Let $\sigma \subset N \otimes \mathbb{Q}$ be a rational polyhedral cone with dual cone $\sigma^\vee = \{u \mid \langle u, v\rangle \geq 0 \ \forall v \in \sigma\} \subset M \otimes \mathbb{Q}$. The algebraic torus $T_N = N \otimes \mathbb{G}_m$ has group of characters $Hom(N \otimes \mathbb{G}_m, \mathbb{G}_m)$ $= Hom(N, Hom(\mathbb{G}_m, \mathbb{G}_m)) = Hom(N, \mathbb{Z}) = M$, each of which corresponds to an irreducible T_N representation: $\chi_u : T_N \to \mathbb{G}_m \curvearrowright \mathbf{k}$ (in fact every irreducible representation of T_N is of this form). The cone σ defines the affine semigroup algebra $\mathbf{k}[S_\sigma] = \bigoplus_{u \in \sigma^\vee \cap M} \mathbf{k}\chi_u$, this is the free vector space over the set of lattice points in $\sigma^\vee$ equipped with multiplication $\chi_u \chi_{u'} = \chi_{u+u'}$. An element $t \in T_N$ acts on $f = \sum c_i \chi_{u_i} \in \mathbf{k}[S_\sigma]$ by $t \circ f = \sum \chi_{u_i}(t) c_i \chi_{u_i}$; this makes $\mathbf{k}[S_\sigma]$ into a T_N-algebra. The variety $U_\sigma = Spec(\mathbf{k}[S_\sigma])$ is then the (normal) affine toric variety associated to the cone σ.*

Example 4.2 (The Plücker algebra). *Let $R_{m,n}$ be global section ring of the Plücker line bundle $\mathcal{L}(\omega_{n-m})$ on the Grassmannian variety $Gr_m(n)$. This algebra can be realized as the SL_m invariant subalgebra of the polynomial ring $\mathbf{k}[x_{ij}, 1 \leq i \leq m, 1 \leq n \leq n]$, where the x_{ij} are thought of as the entries of an $m \times n$ matrix X of parameters, and SL_m acts by left transformations. This algebra is generated by the $m \times m$ determinant minors p_I of X, where $I \subset [n]$ is a subset of size m. The p_I form a basis of the representation $V(\omega_m) = \bigwedge^m(\mathbf{k}^n)$, note that this is the irreducible representation of GL_n associated to the dominant weight $\omega_m = (1, \ldots, 1, 0, \ldots 0)$ (here there are exactly m 1's). Likewise, the degree k monomials in the p_I span the irreducible representation $V(k\omega_m)$ of GL_n with highest weight $k\omega_m$. This gives us the following isotypical decomposition of $R_{m,n}$ as a GL_n algebra:*

$$R_{m,n} = \bigoplus_{k \geq 0} V(k\omega_m). \tag{12}$$

Example 4.3 (SL_2). *The coordinate ring $\mathbf{k}[SL_2]$ can be presented as the quotient of $\mathbf{k}[x_{11}, x_{21}, x_{12}, x_{22}]$ by the principal ideal $< x_{11}x_{22} - x_{21}x_{12} - 1 >$. The x_{ij} are specializations of the coordinate functions obtained from the inclusion of varieties $SL_2 \to M_{2,2}$. This map can viewed as an inclusion $M^*_{2,2} = End(V(1)) \subset \mathbf{k}[SL_2]$, where the vectors in $End(V(1))$ is identified with the space spanned by the entry functions on $M_{2,2}$. In particular, $f = Ax_{11} + Bx_{21} + Cx_{12} + Dx_{22} \in End(V(1))$ has value $Aa + Bb + Cc + Dd$ on the matrix with entries a, b, c, d.*

In the same way, we have inclusions $End(V(n)) \subset \mathbf{k}[SL_2]$ for every irreducible representation $V(n) = Sym^n(\mathbf{k}^2)$ of SL_2. These inclusions end up being enough to describe the whole coordinate ring, in particular we get the following decomposition:

$$\mathbf{k}[SL_2] = \bigoplus_{n \geq 0} End(Sym^n(\mathbf{k}^2)). \tag{13}$$

Example 4.4 (G). *In general, the Peter-Weyl theorem tells us that the coordinate ring of $\mathbf{k}[G]$ is the sum of the "matrix entry" functions as a $G \times G$ variety.*

Theorem 4.5 (Peter-Weyl). *The coordinate ring $\mathbb{K}[G]$ decomposes as a multiplicty-free sum over all dominant weights Λ:*

$$\mathbf{k}[G] = \bigoplus_{\lambda \in \Lambda_+} End(V(\lambda)). \tag{14}$$

Since $V(\lambda^*)$ is the dual vector space of $V(\lambda)$, we can identify the space of endomorphisms $End(V(\lambda))$ with $V(\lambda^*) \otimes V(\lambda)$ with by sending $f \otimes v$ to the endomorphism $w \to f(w) \otimes v$. For this reason, elements of $\mathbf{k}[G]$ can be thought of as (sums of) entries in matrices obtained by having G act on the representations $V(\lambda)$.

Example 4.6 ($\mathbb{A}^2$)**.** *The space $\mathbb{A}^2$ comes equipped with an action of SL_2. The isotypical components of $\mathbf{k}[\mathbb{A}^2] = \mathbf{k}[x, y]$ with respect to this action are exactly the irreducible representations $V(n)$. In particular $V(n) = Sym^n(\mathbf{k}^2)$ can be taken to be the space of monomials of total degree n in the x, y. Let $U \subset SL_2$ be the set of 2×2 upper triangular matrices with 1's along the diagonal. Up to right multiplication by elements of U, the equivalence class of $A \in SL_2$ is determined by its first column, both of whose entries cannot be simultaneously 0. In this way, $SL_2/U \cong \mathbb{A}^2 \setminus \{0\}$. Since $\mathbb{A}^2 \setminus \{0\} \subset \mathbb{A}^2$ differs only in codimension 2, we conclude that $\mathbf{k}[SL_2/U] = \bigoplus_{n\geq 0} V(n)$.*

Example 4.7 (G/U)**.** *Let $U \subset G$ be a maximal unipotent subgroup (take for example the upper triangular matrices in $G = GL_n$ with 1's along the diagonal). The quotient G/U is a quasi affine variety with coordinate ring $\mathbf{k}[G]^U = \bigoplus_{\lambda\in\Lambda_+} V(\lambda) \otimes V(\lambda^*)^U$. For any irreducible representation $V(\lambda)$, the space of invariants $V(\lambda)^U$ is just the line through any highest weight vector kkb_λ. It follows that $kk[G/U] = \bigoplus_{\lambda\in\Lambda_+} V(\lambda)$; ie the coordinate ring of G/U is the multiplicity-free sum of the irreducible representations of G. We let $G//U$ denote the spectrum $Spec(\mathbf{k}[G/U])$. The quotient space G/U includes into $G//U$ as a dense open subset, and these spaces agree in codimension 2.*

The algebra $\mathbf{k}[G/U]$ retains a right action by a maximal torus $T \subset G$ (indeed, $V(\lambda)^U$ is the weight space $\mathbf{k}b_\lambda$ for T). As a consequence, $\mathbf{k}[G/U]$ has a grading by $\Lambda^+ \subset \Lambda$, so that $V(\lambda) \otimes V(\eta)$ is mapped onto $V(\lambda + \eta)$ by the multiplication operation in $\mathbf{k}[G/U]$. This operation is called Cartan multiplication.

Example 4.8 (G/B)**.** *Let $\mathcal{L}_{\lambda^*}$ be the line bundle on G/B associated to the B-character $\lambda^* \in \Lambda_+$. The global section space $H^0(G/N, \mathcal{L}_{\lambda^*})$ is isomorphic to $V(\lambda)$ as a G-representation. It follows that the graded section ring $R_{\mathcal{L}_{\lambda^*}}$ is the multiplicity free sum $\bigoplus_{N\geq 0} V(N\lambda)$. In particular the multiplication operation in this ring maps $V(N\lambda) \otimes V(M\lambda)$ onto $V(N + M\lambda)$; this must be Cartan multiplication. As a consequence, $R_{mathcalL_{\lambda^*}}$ is realized as a Λ-graded subalgebra of $\mathbf{k}[G/U]$ for any λ.*

Example 4.9 (Matrices)**.** *Let $M_{m\times n}$ be the space of $m \times n$ matrices with $m \leq n$. This space has an action by the group $GL_m \times GL_n$, where GL_m acts by row transformations on the left and GL_n acts by column transformations on the right. Let $\Pi_+(m) \subset \Lambda_+(m)$ be the subset of those dominant GL_m weights with all positive components. In particular $\lambda \in \Pi_+(m)$ is a tuple $(\lambda_1, \ldots, \lambda_m)$ where $0 \leq \lambda_m \leq \ldots \leq \lambda_1$. Notice that each weight in $\Pi_+(m)$ also defines a dominant weight of GL_n. With these definitions, we have:*

$$\mathbf{k}[M_{m\times n}] = \bigoplus_{\lambda\in\Pi_+(m)} V_m(\lambda)^* \otimes V_n(\lambda), \tag{15}$$

where $V_n(\lambda)$ and $V_m(\lambda)$ are the GL_n, resp. GL_m representations associated to λ. In the transposed situation we have $\mathbf{k}[M_{n\times m}] = \bigoplus_{\lambda\in\Pi_+(m)} V_n(\lambda)^* \otimes V_m(\lambda)$.

Recall the dominant weight ordering on Λ_+, where $\lambda \prec \eta$ if and only if $\eta - \lambda$ is a positive sum of positive roots. If we take two irreducible representations $V(\lambda)$, $V(\eta)$, we can consider the irreducible decomposition of the tensor product $V(\lambda)\otimes V(\eta) = \bigoplus T_\mu^{\lambda,\eta}\otimes V(\mu)$. For any weight in one of the $V(\mu)$ in this decomposition, we must have $\mu \prec \lambda+\eta$.

Now fix two isotypical components $W_\lambda, W_\eta \subset \mathbf{k}[X]$ and consider the product $W_\lambda W_\eta = m(W_\lambda \otimes W_\eta) \subset \mathbf{k}[X]$. The multiplication map $m : \mathbf{k}[X]\otimes \mathbf{k}[X] \to \mathbf{k}[X]$ must be a map of G-representations, so $W_\lambda W_\eta$ must be isomorphic to a subrepresentation of $W_\lambda \otimes W_\eta$; this implies that $W_\lambda W_\eta \subset \bigoplus_{\mu\prec\lambda+\eta} W_\mu$. When W_μ appears in $W_\lambda W_\eta$ for $\lambda,\eta \in \Lambda_X^+$ we say that the weight $\lambda+\eta-\mu \in \mathcal{R}$ (here $\mathcal{R}$ is the root lattice) is a *tail* of X. Likewise, the cone C_X generated by the tails in $\Lambda_X \otimes \mathbb{Q}$ is called the *tail cone* of X.

A normal affine G-variety is said to be a *spherical variety* if it has a dense, open B-orbit, where $B \subset G$ is a Borel subgroup. A torus T is its own Borel subgroup, so we can conclude that any toric variety is a spherical variety for T. The isotypical components of the coordinate ring $\mathbf{k}[X]$ of an affine spherical variety are always multiplicity-free, that is, each $W_\lambda \subset \mathbf{k}[X]$ is a single copy of $V(\lambda)$. All of the examples we have discussed are spherical varieties.

5. G-valuations

A G-valuation on an affine G-variety X is a valuation $v : \mathbf{k}(X)\setminus\{0\} \to \mathbb{R}$ that is invariant under the action of G. The set of G-valuations on $\mathbf{k}(X)$ is denoted V_X; we will see that V_X comes with a polyhedral structure. For the basics of G-valuations and spherical varieties we refer to the book of Timashev, [Tim11, Chapter 4]. In what follows $\mathbf{k}(X)^{(B)}$ denotes the monoid of B-eigenfunctions in $\mathbf{k}(X)$ (caution: this is not the field of B-invariant rational functions). The following proposition is due to Knop [Kno]:

Proposition 5.1. *Any G-valuation $v \in V_X$ is determined by its restriction to $\mathbf{k}(X)^{(B)}$.*

For any G-variety, the dual Weyl chamber $\Delta^\vee$ defines a cone of G-valuations on $\mathbf{k}[X]$. Taking $h \in \Delta^\vee$, the valuation $v_h : \mathbf{k}[X]\setminus\{0\} \to \mathbb{R}$ satisfies $v_h(f_\lambda) = h(\lambda)$ for any $f_\lambda \in W_\lambda$, and $v_h(\sum f_\lambda) = MAX\{h(\lambda) \mid f_\lambda \neq 0\}$. The restriction of v_h to the subfield $\mathbf{k}(X)^B \subset \mathbf{k}(X)$ of B-invariant rational functions is the trivial valuation.

For the remainder of this section we will make the (extreme) assumption that X is an affine spherical variety, as this both simplifies matters and handles all of the classes of tropical variety that we will discuss. Note in this case $\mathbf{k}(X)^B = \mathbf{k}$. Any $f \in \mathbf{k}(X)^{(B)}$ has an associated weight $\lambda(f) \in \Lambda_X$, and the function $\lambda : \mathbf{k}(X)^{(B)} \to \Lambda_X$ is a map of groups. We've assumed that X is spherical, so it has an open B-orbit, this implies that the kernel of λ is $\mathbf{k}^*$; this gives an identification $\Lambda_X \cong \mathbf{k}(X)^{(B)}/\mathbf{k}^*$.

Any G-valuation $v : \mathbf{k}[X]\setminus\{0\} \to \mathbb{Q}$ over $\mathbf{k}$ defines a linear map on $\mathbf{k}(X)(B)$ by restriction, and therefore a linear function $\rho(v) : \Lambda_X \to \mathbb{Q}$. This leads to the following description of G-valuations on an affine spherical variety due to Luna and Vust, [LV].

Theorem 5.2 (Luna, Vust)**.** *The map $\rho : V_X \to Hom(\Lambda_X, \mathbb{Q})$ is $1-1$, and the image is a convex polyhedral cone which contains $\Delta^\vee$.*

In particular, for $f \in \mathbf{k}(X)^{(B)}$ and $v \in V_X$, the value $v(f)$ is computed by the pairing $\langle \rho(v), \lambda(f)\rangle$. For $f \in \mathbf{k}[X]$ with isotypical decomposition $f = \sum f_\eta$ we then have:

$$v(f) = MAX\{\langle \rho(v), \eta\rangle \mid f_\eta \neq 0\}. \tag{16}$$

Brion [Bri] and Knop [Kno] tell us more about the geometry of the cone V_X.

Theorem 5.3 (Brion, Knop). *The set V_X is a simplicial cone in $Hom(\Lambda_X, \mathbb{Q})$. Furthermore, there is a finite set $\beta_1, \dots, \beta_\ell \in \Lambda_X \otimes \mathbb{Q}$ such that:*

(1) $V_X = \{v \in Hom(\Lambda_X, \mathbb{Q}) \mid \langle v, \beta_i\rangle \leq 0 \ \forall i = 1, \dots, \ell\}$,
(2) $\beta_1, \dots, \beta_\ell$ define a root system, and V_X is the fundamental domain for the action of the associated Weyl group $\mathcal{W}_X$.

The set $\beta_1, \dots, \beta_\ell$ is call the spherical root system of X. It can be shown that the tail cone C_X is generated over $\mathbb{R}_{\geq}0$ by the β_i, and moreover that $C_X = (V_X)^\vee$. The Weyl group $\mathcal{W}_X$ is called the *little Weyl group* of X.

Remark 5.4. The cone V_X is a generalization of the vector space $N \otimes \mathbb{R}$, which indexes T_N-valuations on a toric variety $Y(\Sigma)$ with fan $\Sigma \subset N \otimes \mathbb{Q}$, and contains the tropicalizations of very affine varieties $X \subset Y(\Sigma)$. This analogy leads to *spherical tropicalization*, see [Vog], [KMa].

Example 5.5 (G). *The weights Λ_G^+ can be identified with Λ_+, and the corresponding description of the tail cone C_G contains $\lambda - \eta$ for all $\eta \prec \lambda$. It follows that V_G can be identified with the dual Weyl chamber $\Delta^\vee$.*

Remark 5.6. A version of the description of V_X for spherical varieties remains true for general G-varieties. If we fix $v_0 : \mathbf{k}(X)^B \setminus \{0\} \to \mathbb{Q}$ a valuation on the subfield of B-invariant rational functions, and consider the collection V_0 of G-invariant valuations whose restriction to $\mathbf{k}(X)^B$ coincides with v_0. Pick $v_1 \in V_0$ and define $\rho(v) : V_0 \to Hom(\Lambda, \mathbb{Q})$ as $\rho(v)(\lambda) = v(f_\lambda) - v_1(f_\lambda)$, where $f_\lambda \in \mathbf{k}(X)^{(B)}$. This is independent of our choice of eigenfunction, and identifies V_0 with a simplicial cone in the $\mathbb{Q}$-vector space $Hom(\Lambda_X, \mathbb{Q})$. In particular, any $v_h : \mathbf{k}(X) \setminus \{0\} \to \mathbb{Q}$ for $h \in \Delta^\vee$ restricts to the trivial valuation on $\mathbf{k}(X)^B$. For general X, the set V_X can be realized as a polyhedral subcomplex of an infinite union of half-spaces (over all non-trivial v_0) along a common hyperplane (containing valuations whose restriction to $\mathbf{k}(X)^B$ is trivial), see [Tim11, Chapter 20].

6. Branching Problems, branching varieties, branching cones

Let $\phi : H \to G$ be a map of connected, reductive groups over $\mathbf{k}$. The *branching problem* associated to ϕ amounts to describing the functor $\phi^* : Rep(G) \to Rep(H)$ given by pulling back G-representations along ϕ. Due to the semi-simplicity of $Rep(H)$ and $Rep(G)$ the functor ϕ^* is determined by what it does to irreducible representations of G:

$$\phi^*(V(\lambda)) = \bigoplus_{\eta \in \Lambda_+(H)} Hom_H(V(\eta), \phi^*(V(\lambda))) \otimes V(\eta). \tag{17}$$

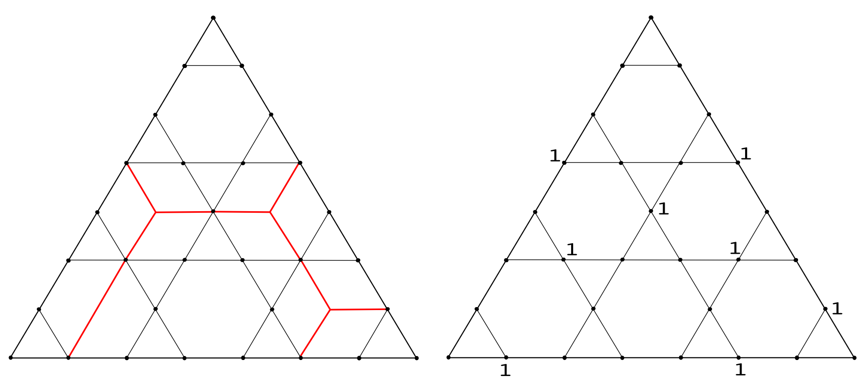
In this way, ϕ^* is computed from the branching spaces $Hom_H(V(\eta), \phi^*(V(\lambda)))$ composed of the H representation maps from irreducibles of H into irreducibles of G. From now on we will drop the ϕ^* and write $Hom_H(V(\eta), V(\lambda))$ when the pullback functor is clear from context.

Example 6.1. *[$GL_{n-1} \subset GL_n$] Let $GL_{n-1} \subset GL_n$ be the upper diagonal inclusion. The branching space $Hom_{GL_{n-1}}(V(\eta), V(\lambda))$ is always either $\mathbf{k}$ or 0. It has dimension 1 if and only if the weights η, λ* interlace*:*

$$\lambda_n \geq \eta_{n-1} \geq \lambda_{n-1} \geq \ldots \geq \eta_1 \geq \lambda_1. \tag{18}$$

Example 6.2 ($SL_2 \subset SL_2 \times SL_2$)**.** *The branching problem for the diagonal inclusion $SL_2 \subset SL_2 \times SL_2$ requires us to find $Hom_{SL_2}(V(i), V(j) \otimes V(k))$ for all i, j, k. Recall that $Hom_{SL_2}(V(i), V(j) \otimes V(k)) = [V(i) \otimes V(j) \otimes V(k)]^{SL_2}$, and that the latter spaces are described by the polyhedral Clebsch-Gordon rule.*

Example 6.3. *[$G \subset G \times G$] Describing $Hom_G(V(\lambda), V(\eta) \otimes V(\mu))$ requires us to understand $[V(\lambda^*) \otimes V(\eta) \otimes V(\mu)]^G$. In the case SL_n, this space has a combinatorial description given by the Littlewood-Richardson rule. In fact,* [BZ01a] *give a way to construct a number of polyhedral rules for every G. One such rule, equivalent to the Littlewood-Richardson rule, is given by the Berenstein-Zelevinsky triangles:*



This is a picture of a Berenstein-Zelevinsky pattern. Each vertex is assigned a positive integer in a way that pairs of pairs of vertices across a common hexagon from each other have equal sum. By choosing the clockwise orientation, we can read the list of sums of pairs: $a_1, \ldots, a_r$, $b_1, \ldots, b_r$, $c_1, \ldots, c_r$ along each of the three edges. Let $\alpha = \sum a_i\omega_i, \beta = \sum b_i\omega_i, \gamma = \sum c_i\omega_i$, where ω_i is the i-th fundamental weight of SL_{r+1}. Each triangle with boundary α, β, γ labels a unique invariant in $[V(\alpha) \otimes V(\beta) \otimes V(\gamma)]^{SL_{r+1}}$.

As is the case with many important vector spaces in representation theory, branching spaces $Hom_H(V(\eta), V(\lambda))$ can be studied by considering the appropriate algebra. Consider the coordinate algebra of the variety $H/U_H \times G/U_G$, where U_H, U_G are choices of maximal unipotent subgroups. This space is spherical as a

$H \times G$ space, and in fact carries a right action by a maximal torus $T_H \times T_G \subset H \times G$. Accordingly, the coordinate ring $\mathbf{k}[H/U_H \times G/U_H]$ is graded by $\Lambda_+(H) \times \Lambda_+(G)$ with isotypical components $V(\eta) \otimes V(\lambda)$ for all dominant η, λ.

The map $\phi : H \to G$ enables us to define a diagonal H action on $H/U_H \times G/U_G$ by acting on the left of H and the left of G through ϕ. We let $R(\phi) = \mathbf{k}[H/U_H \times G/U_G]^H$ be the invariants with respect to this action; this is the *branching algebra* associated to ϕ. The branching algebra retains the right action by $T_H \times T_G$, defining the following decomposition as a $\Lambda_+(H) \times \Lambda_+(G)$-graded algebra:

$$R(\phi) = \bigoplus_{\eta \in \Lambda_+(H), \lambda \in \Lambda_+(G)} Hom_H(V(\eta), V(\lambda)) \tag{19}$$

Here we have identified $Hom_H(V(\eta), V(\lambda))$ with the invariant space $[V(\eta^*) \otimes V(\lambda)]^H$. The GIT quotient $\mathrm{Spec}[R(\phi)] = H\backslash\backslash[H//U_H \times G//U_G] = B(\phi)$ is called the *branching variety* associated to ϕ. See [Man16], [HMM], [HJL$^+$09] for more on the structure of branching varieties.

Remark 6.4. Instead of taking unipotent quotients in the definition of branching variety we could have taken Borel quotients resulting in a GIT quotient of the product of flag varieties $H/B_H \times G/B_G$. Such a construction requires the choice of an $H-$linearized line bundle on $H/B_H \times G/B_G$, for example $\mathcal{L}_\eta \boxtimes \mathcal{L}_\lambda$. The resulting *projective branching variety* $B_{\eta,\lambda}(\phi)$ is also an interesting object. For generic η, λ it is in fact a Mori dream space with Cox ring $R(\phi)$.

Example 6.5 ($B(n-1,n)$)**.** *Let $B(n-1,n)$ be the branching variety of the upper diagonal inclusion $GL_{n-1} \subset GL_n$. The branching rule in example 6.1 can be used to show that the associated branching algebra is a polynomial ring on $2n-1$ variables, where one of the variables is allowed to be inverted. As a consequence, $B(n-1,n) \cong \mathbb{G}_m \times \mathbb{A}^{2n-2}$.*

Example 6.6 ($P_n(SL_2)$)**.** *We have in fact already met the branching algebra of the diagonal inclusion $SL_2 \subset SL_2^{n-1}$, it is the Plücker algebra $R_{2,n}$. We let $P_n(SL_2)$ denote the associated branching variety.*

Example 6.7 ($P_n(G)$)**.** *This branching problem amounts to describing the n-fold tensor product invariants for irreducible representations of G. In Example 6.3 we saw that the case $n = 3$ has a polyhedral counting rule, this hints at some beautiful tropical geometry.*

The space $B(n, n+1)$ is a toric variety, accordingly the branching problem associated to the inclusion $GL_n \subset GLn+1$ is essentially combinatorial. This is part of a broader theme, where the more combinatorial structures (e.g. tropical, toric) we can attach to $R(\phi)$, the more equipment we have to understand the branching problem associated to ϕ.

Now consider a factorization $\phi = \psi \circ \pi$ of the map $\phi : H \to G$ in the category of reductive groups, where $\psi : H \to L$, $\pi : L \to G$ for L a connected, reductive group. Using the identity $X \cong G\backslash\backslash[G \times X]$ for the diagonal action of G on $G \times X$ for any G-variety X, we can conclude:

$$B(\phi) = H \times L\backslash\backslash[H/U_H \times L \times G/U_G], \tag{20}$$

$$R(\phi) = [\mathbf{k}[H]^{U_H} \otimes \mathbf{k}[L] \otimes \mathbf{k}[G]^{U_G}]^{H\times L}. \tag{21}$$

The description of $R(\phi)$ as an $H \times L$ invariant algebra corresponds to an iterated decomposition of branching spaces, where first we branch $V(\lambda)$ over π and then we branching the resulting irreducibles over ψ:

$$Hom_H(V(\eta), V(\lambda)) = \bigoplus_{\mu\in\Lambda_+(L)} Hom_H(V(\eta), V(\mu)) \otimes Hom_L(V(\mu), V(\lambda)). \tag{22}$$

Through the factorization $\phi = \psi \circ \pi$ each of the cones $\Delta_H^\vee, \Delta_L^\vee, \Delta_G^\vee$ define valuations on $R(\phi)$. In fact, the vector spaces $\Lambda_H^\vee \otimes \mathbb{R} \supset \Delta_H^\vee$ and $\Lambda_G^\vee \otimes \mathbb{R} \supset \Delta_G^\vee$ define valuations, since $\mathbf{k}[H/U_H]$ and $\mathbf{k}[G/U_G]$ are in fact homogeneously graded by the actions of T_H and T_G. For the following see [HMM].

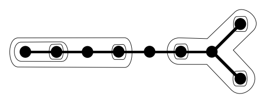
Proposition 6.8. *Each point in the cone $\Delta(\psi, \pi) = [\Lambda_H^\vee \otimes \mathbb{R}] \times \Delta_L^\vee \times [\Lambda_H^\vee \otimes \mathbb{R}]$ defines a valuation on the branching algebra $R(\phi)$. If $(h_1, h_2, h_3) \in \Delta(\psi, \pi)$ with associated valuation v_{h_1,h_2,h_3} and $f \in Hom_H(V(\eta), V(\mu)) \otimes Hom_L(V(\mu), V(\lambda))$ then $v_{h_1,h_2,h_3}(f) = h_1(\eta) + h_2(\mu) + h_3(\lambda)$.*

Proposition 6.8 can be extended to any finite factorization $\phi = \psi_1 \circ \ldots \circ \psi_k$ in the expected way; this should remind us of the construction of the cone $C_\mathcal{T}$ of valuations on the Plücker algebra $R_{2,n}$. In particular, for any such factorization there is a corresponding cone $\Delta(\bar{\psi})$ of valuations, and these can be computed on elements of $R(\phi)$ by evaluating weights on coweights. Furthermore, the cone $\Delta(\psi_1, \ldots, \psi_j, \psi_{j+1}, \ldots, \psi_k)$ naturally contains $\Delta(\psi_1, \ldots, \psi_j \circ \psi_{j+1}, \ldots, \psi_k)$ as the face where the $\Delta_{L_i}^\vee$ coweight is chosen to be 0. In this way, a small diagram of factorizations of ϕ corresponds to a polyhedral complex of valuations on $R(\phi)$.

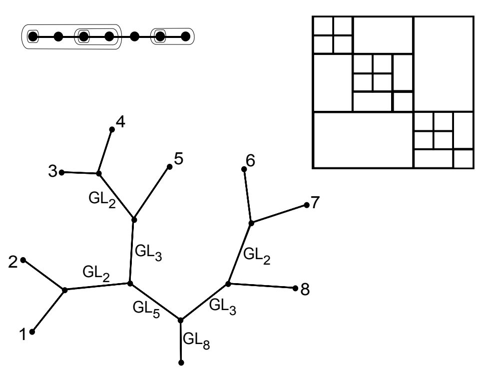
Example 6.9. *Go back to Section 3 to see an instance of one of these complexes: the space of metric trees $\mathcal{T}_n$ built from the cones $C_\mathcal{T}$. A tree $\mathcal{T}$ gives a factorization of the inclusion $SL_2 \subset SL_2^{n-1}$ by the following rule. Place a copy of SL_2 on each edge of $\mathcal{T}$, and start at the edge connected to the vertex labelled 1. We'll put the unique orientation on $\mathcal{T}$ which makes 1 a source and all other leaves sinks. Now each trivalent vertex has an incoming copy of SL_2 and two or more outgoing copies of SL_2; each time we see this we make the associated diagonal inclusion. The result can be turned into a factorization of $SL_2 \subset SL_2^{n-1}$. For SL_2, $\Delta^\vee = \mathbb{R}_{\geq} 0$ and $\Lambda^\vee \otimes \mathbb{R} \cong \mathbb{R}$, as a consequence the associated cone is $C_\mathcal{T}$.*

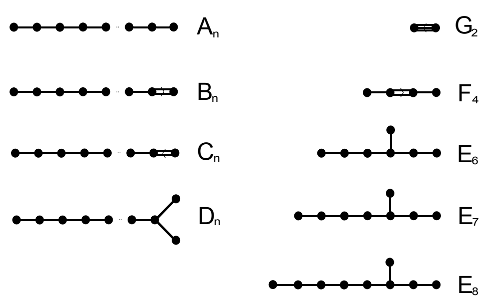
Example 6.10 (Tubings of Dynkin diagrams)**.** *Let G be simple and simply-connected. It is well-known that each such G corresponds to a Dynkin diagram. Dynkin diagrams represent the simple roots of the Lie algebra $Lie(G) = \mathfrak{g}$ along with the angles between pairs. Collections of simple roots in turn correspond to distinguished subgroups of G. For example, for a collection S of simple roots there is an associated Levi subgroup $L_S \subset G$ of semi-simple type given by the corresponding sub-Dynkin diagram. In this way, a chain of Levi subgroups $L_{S_1} \subset \ldots \subset L_{S_k} \subset G$ can be represented by a* tubing *of the Dynkin diagram of G, see Figure 6.10.*

We can let $1 \subset G$ be the bottom of this inclusion. We know there must then be a corresponding cone of valuations in the branching algebra $\mathbf{k}[G/U]$, and likewise in any subalgebra, e.g. the projective coordinate ring of a flag variety. When we carry out this operation for type A_n, we can recover the space $\mathcal{T}_n^+$ of metric trees



whose leaves are cyclically ordered $1, \ldots, n$*, see Figure 6.10. This applies to* any *flag variety of type* A*, not just the Grassmannian* $Gr_2(n)$*; and we recover the full space* $\mathcal{T}_n$ *by moving* $\mathcal{T}_n^+$ *around by the action of the Weyl group* S_n*. Similarly, we see that the space* $\mathcal{T}_n^+$ *can be realized as a set of valuations on any flag variety of types* B, C, G*, and* F*, and the underlying graph of these Dynkin diagrams are all chains. However, we do not recover* $\mathcal{T}_n$ *as a set of valuations on the flag varieties of these groups as their Weyl groups are different (ie we get different spaces of metric trees).*





For more on this example see [Man12].

Lecture 2

In this lecture we continue with the theme of relating portions of tropical varieties to combinatorial objects derived from representation theory. We will need another tool to construct these sets: the connection between higher rank valuations and cones of rank 1 valuations. Varieties equipped with an action by a reductive group G come with a distinguished class of higher rank valuations come from the associated action of the Lie Algebra. I will describe this construction and several associated connections between tropical geometry and branching problems in representation theory.

7. A motivating Example II

First we return to the Plücker algebra $R_{2,n}$ for some more inspiration. Recall that if we are given a trivalent tree $\mathcal{T}$ we obtain a cone of valuations $C_{\mathcal{T}}$ on $R_{2,n}$ along with a direct sum decomposition of $R_{2,n}$ as a $\mathbf{k}$ vector space:

$$R_{2,n} = \bigoplus_{\mathbf{s},\mathbf{r} \in S_{\mathcal{T}}} W_{\mathcal{T}}(\mathbf{s},\mathbf{r}), \tag{23}$$

where $\mathbf{s},\mathbf{r} \in \mathbb{Z}^{E(\mathcal{T})}$ are an integral labelling of the edges of $\mathcal{T}$ (recall that $\mathbf{r} = (r_1,\dots,r_n)$ labels the leaf-edges $L(\mathcal{T})$). The set $S_{\mathcal{T}}$ is collection of all labellings $(\mathbf{s},\mathbf{r})$ of the edges of $\mathcal{T}$ which satisfy the Clebsch-Gordon conditions at every non-leaf vertex of $\mathcal{T}$. It is straightforward to check that $S_{\mathcal{T}}$ is an affine semigroup; in fact it is the set of lattice points in a polyhedron $P_{\mathcal{T}} \subset \mathbb{R}^{E(\mathcal{T})}$ defined by the imposing the triangle inequalities at each non-leaf vertex of $\mathcal{T}$. Furthermore, $S_{\mathcal{T}}$ is saturated (ie if $m\omega \in S_{\mathcal{T}}$ for some $m \in \mathbb{Z}_{\geq 0}$, then $\omega \in S_{\mathcal{T}}$) with respect to the lattice $\mathcal{L}_{\mathcal{T}} \subset \mathbb{Z}^{E(\mathcal{T})}$ defined by imposing the even-sum portion of the Clebsch-Gordon conditions at each trivalent vertex of $\mathcal{T}$. It follows that the $\mathbf{s}$ which make this work for a given $\mathbf{r}$ are the lattice points in a polytope $P_{\mathcal{T}}(\mathbf{r})$.

All of this is interesting because $R_{2,n}$ is not quite graded by $S_{\mathcal{T}}$, but it is very close to being so: indeed $W_{\mathcal{T}}(\mathbf{s},\mathbf{r})W_{\mathcal{T}}(\mathbf{s}',\mathbf{r}') \subset \bigoplus_{\mathbf{s}'' \prec \mathbf{s}+\mathbf{s}'} W_{\mathcal{T}}(\mathbf{s}'',\mathbf{r}+\mathbf{r}')$. Here $\prec$ indicates that each entry of $\mathbf{s}''$ is smaller than the sum of the corresponding entries in $\mathbf{s}+\mathbf{s}'$. Furthermore, it is possible to show that the component $W_{\mathcal{T}}(\mathbf{s}+\mathbf{s}',\mathbf{r}+\mathbf{r}')$ is never 0. When $(\mathbf{s},\mathbf{r}) \in S_{\mathcal{T}}$, the $1-$dimension space $W_{\mathcal{T}}(\mathbf{s},\mathbf{r})$ defines an SL_2 invariant in the tensor product $V(r_1) \otimes \dots \otimes V(r_n)$. Each tree $\mathcal{T}$ provides a different tool to count the dimensions of the tensor product invariant spaces of SL_2, and these trees are encoded into the tropical objects $C_{\mathcal{T}} \subset \mathcal{T}_n$. Tropical geometry provides an organizing tool for combinatorial results in representation theory.

Returning to the multiplication operation in $R_{2,n}$, we consider a valuation $v_p : R_{2,n} \setminus \{0\} \to \mathbb{R}$ for p an interior point of $C_{\mathcal{T}}$; ie p labels every edge $e \in E(\mathcal{T})$ with a non-zero number $p(e)$. The valuation v_p defines a filtration F^p on $R_{2,n}$, where $F^p_{\leq d}$ is the $\mathbf{k}$ vector space of all $f \in R_{2,n}$ with $v_p(f) \leq d$. A key observation is that each space F^p_d is a direct sum of the spaces $W_{\mathcal{T}}(\mathbf{s},\mathbf{r})$. For now we pick a basis member $b_{\mathcal{T},\mathbf{s},\mathbf{r}} \in W_{\mathcal{T}}(\mathbf{s},\mathbf{r})$ for each space.

$$F^p_{\leq d} = \bigoplus_{v_p(b_{\mathcal{T},\mathbf{s},\mathbf{r}}) \leq d} W_{\mathcal{T}}(\mathbf{s},\mathbf{r}) \tag{24}$$

We say that the basis $\mathbb{B}_{\mathcal{T}} = \{b_{\mathcal{T},\mathbf{s},\mathbf{r}} \mid (\mathbf{s},\mathbf{r}) \in S_{\mathcal{T}}\}$ is *adapted* to v_p for all $p \in C_{\mathcal{T}}$. The *associated graded* algebra $gr_p(R_{2,n})$ is the algebra $\bigoplus F^p_{\leq d}/F^p_{<d}$. The following results are now possible from the fact $W_{\mathcal{T}}(\mathbf{s},\mathbf{r})W_{\mathcal{T}}(\mathbf{s}',\mathbf{r}') \subset \bigoplus_{\mathbf{s}'' \prec \mathbf{s}+\mathbf{s}'} W_{\mathcal{T}}(\mathbf{s}'',\mathbf{r}+\mathbf{r}')$.

Proposition 7.1. *For any p in the interior of $C_{\mathcal{T}}$ we have:*

1. *the associated graded algebra $gr_p(R_{2,n})$ is isomorphic to the affine semigroup algebra $\mathbf{k}[S_{\mathcal{T}}]$,*
2. *the equivalence classes of the $\mathbb{B}_{\mathcal{T}}$ in $gr_p(R_{2,n})$ form a basis and certain subsets of $\mathbb{B}_{\mathcal{T}}$ solve our tensor product counting problem,*
3. *the equivalence classes of the Plücker generators $\bar{p}_{ij} \in \mathbf{k}[S_{\mathcal{T}}]$ are an algebraic generating set.*

We say that the Plücker generators $p_{ij}, 1 \leq i < j \leq n$ form a *Khovanskii basis* of $R_{2,n}$ with respect to every point $p \in C_{\mathcal{T}}$. As a consequence, the ideal $J_{\mathcal{T}}$ which presents the affine semigroup algebra $\mathbf{k}[S_{\mathcal{T}}]$ as a quotient by the generators $\bar{p}_{ij}$ is the initial ideal $in_{d(p)}(I_{2,n})$ of the Plücker ideal $I_{2,n}$:

$$J_{\mathcal{T}} = in_{d(p)}(I_{2,n}). \tag{25}$$

Here $d(p)$ is the vector $(\ldots v_p(p_{ij}), \ldots)$; ie the list of distances between pairs of leaves of $\mathcal{T}$.

8. Khovanskii bases

Now we will learn more about the elements of tropical geometry and combinatorial commutative algebra which made the theory of the Plücker algebra work. We refer to [KMb] for much of what follows. Let A be a $\mathbf{k}$-domain, let $v : A \setminus \{0\} \to \mathbb{Q}$ be a valuation which is trivial over $\mathbf{k}$, and let the $S(A, v) = \{q \mid \exists f, v(f) = q\}$ denote the *value semigroup* of v. For any $q \in \mathbb{Q}$ we can define the subspace $F_q = \{f \mid v(f) \leq q\} \subset A$. This is a vector space over $\mathbf{k}$. Similarly we can define $F_{<q}$ to be the subspace of regular functions with value strictly less than q. Clearly if $q < r$ then $F_q \subset F_{<r}$ is a (possibly not proper) subspace. The *associated graded algebra* is defined to be the graded algebra $gr_v(A) = \bigoplus_{q \in \mathbb{Q}} F_q/F_{<q}$, notice that this is also a $\mathbf{k}$ algebra.

If q is *not* a value of v then $F_q/F_{<q} \cong 0$, so we can take this sum to be over the value semigroup of v: $gr_v(A) = \bigoplus_{q \in S(A,v)} F_q/F_{<q}$. For any $f \in A$ we can make the *initial form* $\bar{f} \in gr_v(A)$ with respect to v by taking the equivalence class $\bar{f} \in F_{v(f)}/F_{<v(f)}$. The map which assigns elements in A to their equivalence classes respects multiplication: $\bar{fg} = \bar{f}\bar{g}$.

Definition 8.1. A subset $\mathcal{B} \subset A$ is called a Khovanskii basis with respect to v if the equivalence classes $\bar{\mathcal{B}} \subset gr_v(A)$ generate $gr_v(A)$ as a $\mathbf{k}$-algebra.

Suppose that A is positively graded, ie $A = \bigoplus_{n \geq 0} A_n$; this is a simplifying assumption which holds for many of the algebras we consider. Suppose additionally that A possesses a *finite* Khovanskii basis $\{b_1, \ldots, b_n\} = \mathcal{B} \subset A$ with respect to v. We let $\pi : \mathbf{k}[\mathbf{x}] \to A$ be the associated map from a polynomial ring so that $\pi(x_i) = b_i$, and $I = ker(\pi)$. Finally, let $ev_{\mathcal{B}}(v) = (v(b_1), \ldots, v(b_n)) \in \mathbb{Q}^n$ be the vector of weights defined by v. The following summarizes some of the results in [KMb] which relates v and $gr_v(A)$ to the tropical geometry of the ideal I.

Proposition 8.2 (classification of valuations with Khovanskii basis $\mathcal{B}$)**.** *Let v, $\mathcal{B}$ be as above, then:*

(1) There is a cone C_v in the tropical variety $\mathrm{Trop}(I)$ *so that $ev_{\mathcal{B}}(v) \in C_v$ and $gr_v(A) \cong \mathbf{k}[\mathbf{x}]/in_{\mathbf{u}}(I)$ for any $\mathbf{u}$ in the interior of C_v*

(2) In particular, $in_{\mathbf{u}}(I)$ is a prime ideal, for any $\mathbf{u} \in C_v$.

(3) For any $\mathbf{u} \in C_v$ there is a valuation $v_{\mathbf{u}} : A \setminus \{0\} \to \mathbb{Q}$ with Khovanskii basis $\mathcal{B}$ so that $ev_{\mathcal{B}}(v_{\mathbf{u}}) = \mathbf{u}$.

(4) A valuation v' has Khovanskii basis $\mathcal{B}$ if and only if $v' = v_{\mathbf{u}}$ for some $\mathbf{u} \in \mathrm{Trop}(I)$ with $in_{\mathbf{u}}(I)$ a prime ideal.

When $C \subset \mathrm{Trop}(I)$ is C_v for some valuation with Khovanskii basis $\mathcal{B}$, we say that C is a *prime cone* of $\mathrm{Trop}(I)$ and we let $in_C(I)$ denote the initial ideal $in_{\mathbf{u}}(I)$. The valuation $v_{\mathbf{u}} : A \setminus \{0\} \to \mathbb{Q}$ for $\mathbf{u} \in C$ is called the *weight valuation* associated to $\mathbf{u}$; it can be computed by the following formula:

$$v_{\mathbf{u}}(f) = MAX\{\langle \alpha, \mathbf{u}\rangle \mid \exists p(\mathbf{x}) = \sum C_\beta \mathbf{x}^\beta \in \mathbf{k}[\mathbf{x}], C_\alpha \neq 0, \pi(p) = f\}. \tag{26}$$

In a moment we will see how this formula can be simplified to a finite expression for any $f \in A$.

Definition 8.3. We say that $\mathbb{B} \subset A$ is an *adapted basis* with respect to v if it is a $\mathbf{k}$ vector space basis of A with the additional property that the set $\mathbb{B} \cap F_q$ is a $\mathbf{k}$ vector space basis for any $q \in S(A, v)$. See [KMb,].

The lineality space $Lin(J)$ of an ideal $J \subset \mathbf{k}[\mathbf{x}]$ is the set of weights $\mathbf{w} \in \mathbb{Q}^n$ such that $in_{\mathbf{w}}(J) = J$. It can be shown that $Lin(J)$ is a $\mathbb{Q}$-vector subspace of $\mathbb{Q}^n$. We let $N_C = Lin(in_C(I))$.

Proposition 8.4. *Let $C \subset$* $\mathrm{Trop}(I)$ *be a prime cone, with corresponding lineality space N_C. We have:*

(1) there is a basis $\mathbb{B} \subset A$ composed of certain monomials in the $\mathcal{B}$ which is adapted to each $v_{\mathbf{u}}$ for $\mathbf{u} \in C$,

(2) for any $v_{\mathbf{u}}$ for $\mathbf{u} \in C$, if $f = \sum C_\alpha b_\alpha$ is the linear expression in the basis $\mathcal{B}$, $v_{\mathbf{u}}(f) = MAX\{\langle \alpha, \mathbf{u}\rangle \mid C^\alpha \neq 0\}$,

(3) for any $\mathbf{u}, \mathbf{w} \in C$ and $b^\alpha \in \mathbb{B}$, $v_{\mathbf{u}+\mathbf{w}}(b^\alpha) = v_{\mathbf{u}}(b^\alpha) + v_{\mathbf{w}}(b^\alpha)$, ie any element of $\mathbb{B}$ defines a linear function on C.

Definition 8.5. For a $\mathbb{Q}$-vector space N let $PL(N, \mathbb{Q})$ denote the semialgebra of piecewise linear functions on N with values in $\mathbb{Q}$ under the operations $F \otimes G = F + G$ and $F \oplus G = MAX\{f, g\}$.

For a cone $\sigma \subset N \otimes \mathbb{Q}$, a function $\mathfrak{v} : A \setminus \{0\} \to PL(N, \mathbb{Q})$ is σ-valuation if it satisfies $\mathfrak{v}(fg) = \mathfrak{v}(f) + \mathfrak{v}(g)$, $\mathfrak{v}(f + g) \geq MAX\{\mathfrak{v}(f), \mathfrak{v}(g)\}$ (computed pointwise) when restricted to σ.

We let M_C be the dual vector space of N_C above. As we have seen, any $b^\alpha \in \mathbb{B}$ defines a linear function $\ell_\alpha : C \to \mathbb{Q}$; we extend this function to all of M_C. Now we let $\mathfrak{v}_C(f) \in PL(N_C, \mathbb{Q})$ be the function $v_C(f) = MAX\{\ell_\alpha \mid f = \sum C_\alpha b_\alpha\}$. The function $\mathfrak{v}_C(f)$ is piecewise-linear, by construction $\mathfrak{v}_C$ is a C-valuation, and $v_C(f)[\mathbf{u}] = v_{\mathbf{u}}(f)$ for any $\mathbf{u} \in C$ and $f \in A$.

Definition 8.6. The value semigroup $S(A, \mathfrak{v}_C) \subset M$ is taken to be the image of $\mathbb{B}$ under the map $\mathfrak{v}_C$ defined above.

Note that $\mathfrak{v}_C(b^\alpha b^\beta)[\mathbf{u}] = v_{\mathbf{u}}(b^\alpha b^\beta) = v_{\mathbf{u}}(b^\alpha) + v_{\mathbf{u}}(b^\beta) = \langle \mathfrak{v}_C(b^\alpha), \mathbf{u}\rangle + \langle \mathfrak{v}_C(b^\alpha), \mathbf{u}\rangle$, so $S(A, \mathfrak{v}_C)$ is actually a semigroup. [Caution: $b^{\alpha+\beta}$ is not necessarily in $\mathcal{B}$, however $\langle \alpha + \beta, \mathbf{u}\rangle$ equals the the top term of $\mathfrak{v}_C(b^\alpha b^\beta)[\mathbf{u}]$ for every $\mathbf{u} \in C$].

Remark 8.7. This construction can be repeated with $\mathbb{R}$ in place of $\mathbb{Q}$. With this modification, any $f \in A$ defines a continuous map $\mathfrak{v}_C(f) : C \to \mathbb{R}$. It follows that for any prime cone C the construction above defines a continuous map $\Phi : C \to U^{an}$, where U^{an} is the analytification of $U = Spec(A)$ from the previous lecture.

Remark 8.8. If $dim(C) = dim_{\mathbb{Q}}(N_C)$ is the Krull dimension of A, the assignment $\mathfrak{v}_C : \mathbb{B} \to S(A, \mathfrak{v}_C)$ is $1-1$. This will be related to constructions involving higher rank valuations shortly. If A has representation theoretic meaning, $S(A, \mathfrak{v}_C)$ can be a useful discrete object.

Remark 8.9. For any affine spherical variety X over $\mathbf{k}$, the valuation cone V_X can be mapped into an appropriate prime cone C. One observes that for any G-valuation $w : \mathbf{k}[X] \setminus \{0\} \to \mathbb{Q}$, $gr_w(\mathbf{k}[X])$ is finitely generated, and all of the associated graded algebras have $gr_v(\mathbf{k}[X]))$ for v in the interior of V_X as a common associated graded algebra. It follows that a Khovanskii basis of an interior v is a Khovanskii basis $\mathcal{B} \subset \mathbf{k}[X]$ for any valuation in V_X. Let I be the defining ideal of this Khovanskii basis. The assignment $v \to ev_{\mathcal{B}}(v) \in \mathrm{Trop}(I)$ then maps V_X linearly into some prime cone C.

Remark 8.10. Following [KMb,], various *partial compactifications* of $U \subset X$ can be constructed from choices of integral points in a prime cone C and the dual lattice inside M. This construction recovers the wonderful compactification when G is a simple of adjoint form.

Remark 8.11. In [KMb] a flat family $E \to Y(C)$ is defined over a toric variety associated to C with general fiber U and special fibers the spectra of the various associated graded algebras defined by the valuations in C. When $U = G$ a reductive group, and C is the cone $\Delta^\vee$, this family recovers the *Vinberg enveloping monoid* $S(G)$ of G, see [HMM].

9. Revisiting branching algebras

A number of the features we saw in the case of the Plücker algebra $R_{2,n}$ carry over to the cones $\Delta(\bar{\psi})$ of valuations on the branching algebra $R(\phi)$ corresponding to a factorization $\phi : \psi_k \circ \cdots \circ \psi_0$.

$$H \xrightarrow{\psi_0} L_1 \xrightarrow{\psi_2} \dots \xrightarrow{\psi_{k-1}} L_k \xrightarrow{\psi_k} G$$

In analogy with the spaces $W_{\mathcal{T}}(\mathbf{s}, \mathbf{r})$, let $W(\bar{\mu}) = Hom_H(V(\mu_0), V(\mu_1)) \otimes \cdots \otimes Hom_{L_k}(V(\mu_k), V(\mu_{k+1}))$ for a tuple of dominant weights $\bar{\mu}$ with $\mu_i \in \Lambda_+(L_i)$.

Similarly to the semigroup $S_{\mathcal{T}}$, we let $S(\bar{\psi}) \subset \prod \Lambda_+(L_i)$ be the set of $\bar{\mu}$ with $W(\bar{\mu}) \neq 0$. We summarize the properties of these objects in the following proposition.

Proposition 9.1. *Let $\phi = \psi_1 \circ \cdots \circ \psi_k$ be a factorization of $\phi : H \to G$, and let $R(\phi)$ be the corresponding branching algebra, then the following must hold:*

1. *There is a direct sum decomposition $R(\phi) = \bigoplus_{\bar{\mu} \in S(\bar{\psi})} W(\bar{\mu})$ as a $\mathbf{k}$ vector space.*
2. *For any $W(\bar{\lambda}), W(\bar{\eta})$ we have $W(\bar{\lambda})W(\bar{\eta}) \subset \bigoplus_{\bar{\mu} \prec \bar{\lambda} + \bar{\eta}} W(\bar{\mu})$, where $\prec$ is the dominant weight ordering in $\prod \Lambda_+(L_i)$. Moreover, the top component in $W(\bar{\eta} + \bar{\lambda})$ is always nonzero.*
3. *For any point $h \in \Delta(\bar{\psi})$, the associated graded algebra $gr_h(R(\phi))$ is finitely generated, and the decomposition $gr_h(R(\phi)) = \bigoplus_{\bar{\mu} \in S(\bar{\psi})} W(\bar{\mu})$ is a grading. In particular, any choice of basis for each space $W(\bar{\mu})$ defines a basis of $R(\phi)$ adapted to each valuation v_h.*

For a general point $h \in \Delta(\bar{\psi})$, the associated graded algebra $gr_h(R(\phi))$ has a description in terms of the *horosherical contractions* (see [Pop87]) of the groups $L_1, \ldots, L_k$. For a reductive group G, we can consider a general point $h \in \Delta^\vee$. The spectrum G^c of the graded algebra $gr_h(\mathbf{k}[G])$ of the $G \times G$ valuation v_h has an explicit description as a GIT quotient:

$$G^c = T \backslash\backslash [G//U \times U \backslash\backslash G], \tag{27}$$

where T acts on the right of $G//U$ and the left of $U\backslash\backslash G$ by a certain action, see [HMM]. The coordinate ring $\mathbf{k}[G^c]$ has an identical isotypical decomposition with that of G, but it inherits an extra T action.

Proposition 9.2. *For a general point $h \in \Delta(\bar{\psi})$, the associated graded algebra $gr_j(R(\phi))$ is isomorphic to the following invariant ring:*

$$gr_h(R(\phi)) = [\mathbf{k}[H/U_H] \otimes \ldots \otimes \mathbf{k}[L_i^c] \otimes \ldots \otimes \mathbf{k}[G/U_G]]^{H \times \ldots \times L_k}. \tag{28}$$

Roughly, $gr_h(R(\phi))$ is formed in the same way build $R(\phi)$ from the maps $\bar{\psi}$, only with L_i^c in place of L_i.

It follows that every valuation corresponding to a point in $\Delta(\bar{\psi})$ has a finite Khovanskii basis. This means that there exists some way to identify $S(\bar{\psi})$ with one of the value semigroups $S(A, \mathfrak{v}_C)$ from the previous section.

Remark 9.3. Following the previous section, there is a compactification $X(\bar{\psi}) \supset B(\phi)$ by a combinatorial normal crossings divisor associated to every choice of factorization of ϕ. Presumably the boundary of this compactification is interesting and contains some representation theoretic data.

Remark 9.4. We can think of any chain of inclusions $H_1 \subset \ldots \subset H_k \subset G$ as a factorization of $1 \to G$. The branching algebra of the latter is $\mathbf{k}[G/U]$; so each such inclusion defines a cone of valuations on $\mathbf{k}[G/U]$. These valuations are all T-homogeneous, so they pass to the projective coordinate rings of the flag varieties G/B.

Example 9.5. *We consider the factorization* $1 \to GL_1 \to GL_2 \to \ldots \to GL_{n-1} \to GL_n$ *of the inclusion of the identity into* GL_n *by upper diagonal inclusions. We've seen that each step of this factorization* $GL_k \to GL_{k-1}$ *has a simple branching algebra composed of interlacing patterns. Putting the pieces together from above yields a cone of valuations isomorphic to* $\Delta_1^\vee \times \ldots \times \Delta_n^\vee$ *on* $\mathbf{k}[GL_n/U]$*, where* $\Delta_i^\vee$ *is the dual Weyl chamber of* GL_i*. The value semigroup* S_n *attached to this construction is then the semigroup of integral* Gel'Fand-Zetlin patterns*. A general choice from the valuation cone realizes* $\mathbf{k}[S_n]$ *as an associated graded algebra of* $\mathbf{k}[GL_n/U]$*. This recovers the Gel'fand-Zetlin toric degenerations of flag varieties of type A.*

$$\begin{tikzcd}
\mu_{1,4} & & \mu_{2,4} \arrow[dl] & & \mu_{3,4} \arrow[dl] & & \mu_{4,4} \arrow[dl] \\
& \mu_{1,3} \arrow[ul] & & \mu_{2,3} \arrow[ul] \arrow[dl] & & \mu_{3,3} \arrow[ul] \arrow[dl] & \\
& & \mu_{1,2} \arrow[ul] & & \mu_{2,2} \arrow[ul] \arrow[dl] & & \\
& & & \mu_{1,1} \arrow[ul] & & &
\end{tikzcd}$$

10. Higher rank valuations and cones of valuations

Now we shift our attention to valuations of higher rank. Instead of transmitting information from a variety into another space, higher rank valuations can capture important combinatorial structures in their images. Higher rank valuations also help with asymptotics of counting problems related to algebraic geometry. This application is most prominantly on display in the developing theory of Newton-Okounkov bodies, [LM09], [KK12].

For simplicity we will consider valuations which take values in $\mathbb{Z}^r$ equipped with the lexicographic ordering. As in the rank 1 case, a valuation $\mathfrak{v} : A \setminus \{0\} \to \mathbb{Z}^r$ is a function which satisfies $\mathfrak{v}(fg) = \mathfrak{v}(f) + \mathfrak{v}(g)$, $\mathfrak{v}(f+g) \leq MAX\{\mathfrak{v}(f), \mathfrak{v}(g)\}$, and $\mathfrak{v}(C) = 0 \ \ \forall C \in \mathbf{k} \setminus \{0\}$. The definitions for Khovanskii basis $\mathcal{B} \subset A$ and adapted basis $\mathbb{B} \subset A$ can be taken verbatim from above. We let $gr_{\mathfrak{v}}(A)$ denote the associated graded algebra and $S(A, \mathfrak{v}) = \{\mathfrak{v}(f) \mid f \in A \setminus \{0\}\}$ denote the value semigroup of a valuation $\mathfrak{v}$. For a graded algebra $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$ we say that $\mathfrak{v}$ is homogeneous if $\mathfrak{v}(f) = MAX\{\mathfrak{v}(f_i) \mid f_i \in A_i\}$; ie if the value of any $f \in A$ is achieved on one of its homogeneous components.

Definition 10.1. The Newton-Okounkov cone $P(A, \mathfrak{v})$ is the convex hull of $S(A, \mathfrak{v})$ in $\mathbb{R}^r$. If A is positively graded, the Newton-Okounkov body $\Delta(A, \mathfrak{v})$ is the convex hull of the set $\{\frac{\mathfrak{v}(f)}{deg(f)} \mid f \in A \setminus \{0\}\}$. [Caution: $\Delta(A, \mathfrak{v})$ is usually not polyhedral!]

Remark 10.2. The value semigroup $S(A, \mathfrak{v})$ is comparable to the affine semigroup $S(A, \mathfrak{v}_C)$ from Section 2. In fact, we will see that these two constructions always coincide in a certain sense.

We let $rank(\mathfrak{v})$ be the rank of the lattice generated by $S(A, \mathfrak{v})$, this is bounded above by the Krull dimension of A.

Remark 10.3. If $rank(\mathfrak{v})$ is equal to the Krull dimension of A then the volume of $\Delta(A, \mathfrak{v})$ recovers the *degree* of A; ie the first coefficient of its Hilbert polynomial, see [KK12,].

In general, a branching problem takes the following form: we have a domain A (the branching algebra) which is graded by a group Γ (a lattice of dominant weights). In this case a homogeneous maximal rank valuation $\mathfrak{v} : A \to \mathbb{Z}^n$ can help with a solution. Over an algebraically closed field, any maximal rank valuation must have *one dimensional leaves*, this means that the vector space $F_q/F_{<q} \subset gr_{\mathfrak{v}}(A)$ is always 1-dimensional. If $\mathfrak{v}$ is homogeneous and if the grading support of A in Γ generates Γ as a group, then there must be a homomorphism $\pi : \mathbb{Z}^n \to \Gamma$ so that $\pi(\mathfrak{v}(f)) = \gamma$ for all $f \in A_\gamma$.

To see this, note that we can simply send $\mathfrak{v}(f)$ to γ; the homogeneity of $\mathfrak{v}$ and the valuation axioms then ensure that this map is well-defined and a group homomorphism. Now we can recover $dim(A_\gamma)$ as the number of points in the set $S(A, \mathfrak{v})_\gamma = \{\mathfrak{v}(f) \mid f \in A_\gamma\}$. If additionally $S(A, \mathfrak{v})$ is a saturated semigroup, ie it satisfies $S(A, \mathfrak{v}) = P(A, \mathfrak{v}) \cap \mathbb{Z}^n$, then $S(A, \mathfrak{v})_\gamma$ is the number of lattice points in a convex set $P(A, \mathfrak{v})_\gamma \subset P(A, \mathfrak{v})$. Finally, if $\mathfrak{v}$ has a finite Khovanskii basis, $P(A, \mathfrak{v})_\gamma$ is a convex polytope, and the solution to our counting problem is reduced to an issue of linear programming.

Following this description, each valuation $\mathfrak{v}$ with finite Khovanskii basis is potentially a distinct "polyhedral counting rule" for the counting problem represented by A. It is then useful to compare and relate these rules; this is where tropical geometry can step in as an organizing tool.

Theorem 10.4. *[*[KMb]*] Let A be graded by a group Γ, and suppose that A_γ is always finite dimensional. Let $\mathfrak{v} : A \setminus \{0\} \to \mathbb{Z}^r$ be a homogeneous valuation of rank r with finite Khovanskii basis $\mathcal{B} \subset A$ and associated ideal I, then we have the following:*

(1) there is a prime cone $C_{\mathfrak{v}} \subset \mathrm{Trop}(I)$ of dimension $\geq rank(\mathfrak{v})$ such that $gr_{\mathfrak{v}}(A) \cong \mathbf{k}[\mathbf{x}]/in_{\mathbf{u}}(I)$ for all $\mathbf{u}$ in the interior of $C_{\mathfrak{v}}$,

(2) for any collection $M = \{bu_1, \dots, \mathbf{u}_r\}$ of $rank(\mathfrak{v})$ integral linearly independent points from the interior of $C_{\mathfrak{v}}$ there is a valuation $\mathfrak{v}_M : A \setminus \{0\} \to \mathbb{Z}^r$ with $gr_{\mathfrak{v}}(A) \cong gr_{\mathfrak{v}_M}(A)$ and $S(A, \mathfrak{v}) \cong S(A, \mathfrak{v}_M)$,

(3) there is a basis $\mathbb{B} \subset A$ of $\mathcal{B}$ monomials which is simultaneously adapted to $\mathfrak{v}$ and every $\mathfrak{v}_M$ constructed from $C_{\mathfrak{v}}$.

In particular, Theorem 10.4 implies that all of the counting rules given by valuations with Khovanskii basis $\mathcal{B} \subset A$ are represented by distinct prime cones $C_{\mathfrak{v}} \subset \mathrm{Trop}(I)$. Furthermore, a prime cone in $\mathrm{Trop}(I)$ of dimension r gives a way to produce a valuation of rank r with prescribed associated graded algebra. In this case $S(A, \mathfrak{v}) \cong S(A, \mathfrak{v}_C)$ from Section 2. Also notice that if $\mathfrak{v}$ has maximal rank then $\mathfrak{v}(\mathbb{B}) = S(A, \mathfrak{v})$, and each space $F_q/F_{<q} \subset gr_{\mathfrak{v}}(A)$ is spanned by exactly one member of $\mathbb{B}$.

11. String valuations

We have seen that higher rank valuations can potentially resolve counting problems which manifest on multigraded algebras. Now we'll see a way to produce higher rank valuations from the equipment of a G action.

Let $e : A \to A$ be a nilpotent $\mathbf{k}-$linear operator. This means that for any $f \in A$, $e^k(f)$ for $k = 1, 2, \dots$ is eventually 0. Any such operator defines a discrete valuation by the formula $v_e(f) = MAX\{\ell \mid e^\ell(f) \neq 0\}$. There is a similar formula which assigns a higher rank valuation $\mathfrak{v}_{\mathbf{e}}$ to a sequence $e_1, \dots, e_r$ of nilpotent operators. Starting with the previous formula, we define ℓ_i to be $\{\ell \mid e_i^\ell(f_{i-1}) \neq 0\}$, where f_{i-1} is $e^{\ell_{i-1}}(f_{i-2})$. Conceptulally, using e_1, we "raise" f until we "hit a wall" (ie $e_1^\ell(f) = 0$), then we do the same with e_2 and the last non-zero function obtained from the previous sequence. We continue this way until we are out of operators e_i.

Proposition 11.1. *For a sequence $\mathbf{e} = e_1, \dots, e_r$, the function $\mathfrak{v}_{\mathbf{e}} : A \setminus \{0\} \to \mathbb{Z}^r$ defined by $\mathfrak{v}_{\mathbf{e}}(f) = (\ell_1, \dots, \ell_r)$ defines a valuation on A when $\mathbb{Z}^r$ is given the lexicographic ordering.*

We call $\mathfrak{v}_{\mathbf{e}}$ the *string valuation* associated to the sequence $e_1, \dots, e_k$ (note: this valuation depends on the ordering of the e_i).

Question 11.2. When does the valuation $\mathfrak{v}_{\mathbf{e}}$ have a finite Khovanskii basis?

We use this construction with a particular set of nilpotent operators. Let $\mathfrak{g}$ be the Lie algebra of G with its decomposition $\mathfrak{g}^{ss} = \mathfrak{z} \oplus \mathfrak{g}^{ss}$, where $\mathfrak{z}$ is the Lie algebra of the center and $\mathfrak{g}^{ss}$ is the semisimple part. We fix a system of Chevallay generators; $H_1, \dots, H_r; f_1, \dots, f_r; e_1, \dots, e_r \in \mathfrak{g}^{ss}$, where r is the rank of $\mathfrak{g}$. The $H_1, \dots, H_r$ span a Cartan subalgebra $\mathfrak{h}^{ss}$ of $\mathfrak{g}$, and together with $\mathfrak{z}$ they generate the Lie algebra $\mathfrak{h}$ of a maximal torus T. The elements $f_1, \dots, f_r$ and $e_1, \dots, e_r$ span the Lie subalgebras $\mathfrak{n}_-, \mathfrak{n}_+$ of a corresponding pair of maximal unipotent subgroups $U_-, U_+ \subset G$. Each triple $\{f_i, H_i, e_i\}$ generates a copy of the Lie algebra sl_2. The essential relations of this Lie algebra are $[H, f_i] = -\alpha_i(H) f_i, [H, e_i] = \alpha_i(H) e_i$ for any $H \in \mathfrak{h}^{ss}$, where $\alpha_i \in R$ is the corresponding root. This information is stored in the Cartan matrix $A = [a_{ij}]$, where $a_{ij} = \alpha_j(H_i)$.

Recall that the Weyl group $\mathcal{W}$ of $\mathfrak{g}$ is generated by elements called simple reflections $s_1, \dots, s_r$ which are in bijection with the simple roots $\alpha_1, \dots, \alpha_r$. There is a length function on $\mathcal{W}$, where $w \in \mathcal{W}$ is sent to the length of one of its so called reduced word decompositions in the simple reflections. The longest word $\omega_0 \in \mathcal{W}$ always has length N equal to the number of positive roots. We let $R(\omega_0)$ be the set of reduced word decompositions of ω_0. Each element $\mathbf{i} \in R(\omega_0)$, $\mathbf{i} = (i_1, \dots, i_N)$ corresponds to a decomposition $\omega_0 = s_{i_1} \circ \dots \circ s_{i_N}$.

A reduced decomposition $\mathbf{i} \in R(\omega_0)$ corresponds to a sequence of raising operators $\mathbf{e}_{\mathbf{i}} = (e_{i_1}, \dots, e_{i_N})$. Note that this sequence could have repetitions. We may view each $\mathbf{e}_{\mathbf{i}}$ as a string of nilpotent operators on the coordinate ring $\mathbf{k}[U_+]$ of the unipotent subgroup corresponding to $\mathfrak{n}_+$. Accordingly, we let $\mathfrak{v}_{\mathbf{i}} : \mathbf{k}[U_+] \setminus \{0\} \to \mathbb{Z}^N$ be the corresponding string valuation.

The variety $U_+ \times T$ is naturally a dense open subset of $U_- \backslash\backslash G \cong G/U$, so we obtain string valuations $\mathfrak{v}_{\mathbf{i}} : \mathbf{k}[G/U] \to \mathbb{Z}^{N+h}$, where $h = dim(\mathfrak{h})$. These induce string valuations on each projective coordinate ring of the flag varieties G/P. Likewise, $U_+ \times T \times T$ can be realized as a dense, open subset of $P_3(G)$, yielding a string valuation $\mathfrak{v}_{\mathbf{i},3} : \mathbf{k}[P_3(G)] \to \mathbb{Z}^{N+2h}$. We note that the valuations on $\mathbf{k}[G/U]$ and $\mathbf{k}[P_3(G)]$ are T, resp. T^3-homogeneous. For the following we refer the reader to [Man16] and [BZ01b].

Theorem 11.3. *For each string parameter* $\mathbf{i}$ *we have the following:*

(1) *There are polyhedral cones* $P_{\mathbf{i}} \subset \mathbb{R}^{N+h}$, $P_{\mathbf{i},3} \subset \mathbb{R}^{N+2h}$ *so that* $S(\mathbf{k}[G/U], \mathfrak{v}_{\mathbf{i}}) = \mathbb{Z}^{N+h} \cap P_{\mathbf{i}}$ *and* $S(\mathbf{k}[P_3(G)], \mathfrak{v}_{\mathbf{i},3}) = \mathbb{Z}^{N+2h} \cap P_{\mathbf{i},3}$,

(2) *Each value semigroup* $S(\mathbf{k}[G/U], \mathfrak{v}_{\mathbf{i}}), S(\mathbf{k}[P_3(G)], \mathfrak{v}_{\mathbf{i},3})$ *is saturated and therefore finitely generated,*

(3) $\mathbf{k}[G/U]$ *and* $\mathbf{k}[P_3(G)]$ *possess finite Khovanskii bases with respect to these valuations, accordingly there are maximal dimensional cones* $C_{\mathbf{i}}$ *and* $C_{\mathbf{i},3}$ *of valuations on these algebras.*

The emergence of prime cones is due to the main results in [KMb]. One can find explicit inequalities for the string cones $P_{\mathbf{i}}$ and $P_{\mathbf{i},3}$ in [Man16] and [BZ01b]. Kaveh [Kav15] has realized these valuations on G/U via Bott-Samuelson resolutions. A particual choice yields a semigroup which is linearly isomorphic to the Gel'fand-Zetlin patterns from Example 9.5. The Berenstein-Zelevinsky triangles emerge from a particular choice of string for the variety $P_3(SL_r)$.

Remark 11.4. The construction of string valuations is part of the larger story of *birational sequences* developed in [FFL].

Remark 11.5. The idea of using the string parameters to degenerate the coordinate algebra of a G-algebra goes back to Alexeev and Brion [AB04] and Caldero [Cal02]. In these cases the valuations obtained are adapted to Lusztig's dual canonical basis [Lus90].

12. The Plücker algebra $R_{3,n}$

There have been several developments in the tropical geometry of Grassmannian varieties lately. Rietsch and Williams [RW] have explained how to obtain many Newton-Okounkov bodies from their cluster structure. Each of the value semigroups attached to their construction is finitely generated, it follows from what we've seen in Section 10 that there is an associated full dimensional prime cone in the tropicalizations corresponding to the associated Khovanskii bases. The Rietsch-Williams construction is studied in the $m = 2$ and $m = 3, n = 6$ cases in [BFF+].

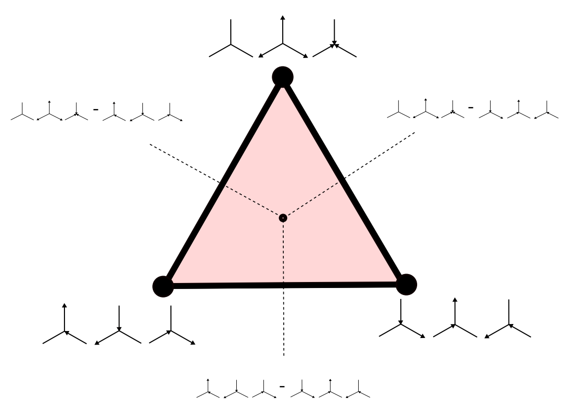
The Plücker algebra $R_{m,n}$ can be realized as the ring of invariants in $\mathbf{k}[M_{m\times n}]$ with respect to the left action of SL_m. We may think of each column of $M_{m\times n}$ has a copy of $\mathbb{A}^m$ with its standard action by SL_m. Accordingly, the coordinate ring of $\mathbf{k}[\mathbb{A}^m]$ has isotypical decomposition $\bigoplus_{N\geq 0} V(N\omega_1)$ with multiplication given by Cartan multiplication: $\mathbf{k}[\mathbb{A}^m] \subset \mathbf{k}[SL_m/U]$. It follows that $R_{m,n}$ is a (multi)graded subalgebra of $\mathbf{k}[P_n(SL_m)]$.

We can use the construction from Section 3 associated to a tree $\mathcal{T}$ on $P_n(G)$. We obtain, for each trivalent tree $\mathcal{T}$ a cone of valuations $[\Delta^\vee]^{E(\mathcal{T})}$ with associated graded algebra the coordinate ring of a quotient $T^{E(\mathcal{T})}\backslash\backslash[\prod_{v\in V(\mathcal{T})} P_3(G)]$. Here we think of each copy of $P_3(G)$ as being associated to a vertex $v \in V(\mathcal{T})$, with each of its three actions by T associated to an edge leaving v. When two tori are associated to the same edge (ie, when two vertices are connected by the same edge), we quotient by a certain diagonal subtorus $T \subset T^2$. To complete this construction to obtain valuation cones of maximal dimension, it suffices to find T^3-homogeneous full rank valuations on $\mathbf{k}[P_3(G)]$. These are of course provided by the string valuations from Section 11. For the following see [Man16].

Theorem 12.1. *For any space $P_n(G)$, we get a full rank valuation with finite Khovanskii basis, a toric degeneration, and a maximal cone of valuations associated to any choice of the following data:*

(1) a trivalent tree $\mathcal{T}$ with n leaves,
abstract
(2) a choice of T^3-homogeneous valuation at each vertex of $\mathcal{T}$, in particular the string valuations suffice.

We can apply this theorem to the Plücker algebra $R_{m,n} \subset \mathbf{k}[P_n(SL_m)]$ to obtain many degenerations and prime cones, however the Khovanskii bases of the resulting can be wildly different. For $G = SL_3$ the situation is more controllable, and it is possible to realize $\mathbf{k}[P_3(SL_3)]$ as the quotient of a polynomial ring by a hypersurface; this yields three distinct full dimensional prime cones. The intial forms of these cones are illustrated in the figure. Two of these prime cones are extracted from string valuations, unlike the third. The non-string prime cone comes from a construction related to the longest root of sl_3 and the Wess-Zumino-Novikov-Witten model of conformal field theory, see [MZ14],[Man13] for more details.



Any maximal rank valuation with one of these three prime cones may be placed at the vertices of a tree $\mathcal{T}$ and used to induce a full rank valuation on $R_{3,n}$. The Khovanskii bases for these valuations have a nice description, they are in bijection with subtrees $\mathcal{T}' \subset \mathcal{T}$ with two properties: any leaf of $\mathcal{T}'$ is also a leaf of $\mathcal{T}$, and for any pair of leaves $\ell_1, \ell_2 \in L(\mathcal{T}')$, the number of trivalent vertices on the unique path from ℓ_1 to ℓ_2 is odd. This can be shown to recover the Gel'fand-Zetlin patterns when $\mathcal{T}$ is a "caterpillar" tree. The actual members of the Khovanskii bases of these valuations are always invariants in a tensor product only involving a 3-divisible number of copies of $V(\omega_1)$, and there are no more than $\sum_{i=0}^{\lfloor \frac{n}{3} \rfloor} \binom{n}{3i}$ of them.

Remark 12.2. The material in this section has numerous ties to the cluster algebra structure on the coordinate ring of $P_n(G)$. This topic has been extremely active. Gross, Hacking, Keel, and Kontsevich [Le16] prove that the Cluster structure on $\mathbf{k}[P_3(SL_r)]$ has a number of nice properties which allow a cluster solution to the associated branching problem to emerge without mentioning representation theory. They also prove a toric degeneration result which is very much in line with what appears in Section 8. More work on these varieties and the deep relationship between their cluster structure, tropical geometry, and associated representation theory appears in the papers of Le [Le16], Goncharov and Shen [GS15], and Magee [Mag].

Lecture 3

In this lecture I will apply the techniques of the previous two lectures to several classes of interesting varieties associated to a reductive group G. We will see how combinatorial and polyhedral objects from representation theory emerge naturally from the tropical geometry of these examples.

13. A motivating example III: Outerspace and character varieties

In this Section we follow [Man14]. The construction of $\mathcal{X}(F_g, G)$ given above benefits from a quiver interpretation. Let Q_g be the quiver with precisely one vertex and g edges. Instead of placing a Hom-space on each edge of Q_g, let's put in a copy of G, thought of as an $G \times G$-variety. We'll assign G to the unique vertex of Q_g, and have it act in the "quiver fashion" on the edge space G^g; this means that each incoming arrow gets a right action and each outgoing arrow gets a left action. Notice this just means that we act on each copy of G in the edge space by the adjoint action. When we take the affine GIT quotient by the vertex action (with respect to the trivial character), we recover $\mathcal{X}(F_g, G)$.

Now, what if we do this exact same construction, only with a larger quiver? We define $G(Q) = G^{V(G)}$ and $M(Q) = G^{E(Q)}$. the copy of $G \subset G^{V(G)}$ at a vertex $v \in V(Q)$ acts on copies of $G \subset G^{E(Q)}$ associated to the edges which contain it; on the right for incoming edges and the left for outgoing edges. We let $\mathcal{X}(Q, G) = G(Q)\backslash\backslash M(Q)$. Unlike the case with actual quiver varieties, it turns out the answer only depends on the first Betti number of the chosen quiver Q, in particular directions of arrows do not matter. For the following see [Man14] and [FL13].

Theorem 13.1. *[M; Florentino, Lawton] Let Q be a connected quiver with $\beta_1(Q) = g$, then $\mathcal{X}(Q, G) \cong \mathcal{X}(F_g, G)$.*

Sketch of proof. Since $G//G \cong pt$ (right action) and $[G \times G]//G \cong G$ ((right, left) action) as G, respectively $G \times G$ spaces, we need only consider the case where every vertex of Q has valence at least 3. Now one proves that if an edge e joins two distinct vertices of valences $r+1, s+1$, then $[G \times G]\backslash\backslash[G^r \times G \times G^s] \cong G\backslash\backslash[G^{r+s}]$, where the action of the first G is on the right of the G^r and the left of G, and the action of the second G is on the right of G and the left of G^s, and the action of G on G^{r+s} is on the left. This effectively "collapses" any edge with boundary having more than 1 vertex. □

Remark 13.2. Other character varieties can be described by attaching cells to Q and insisting that the product of any entries from the boundary of a cell is the identity.

Now we specialize to the case $G = SL_2$, (we will say more about general connected reductive G in a moment). Recall the isotypical decomposition of the coordinate ring $\mathbf{k}[SL_2]$:

$$\mathbf{k}[SL_2] = \bigoplus_{n \geq 0} End(V(n)). \tag{29}$$

Recall that $V(n) = Sym^n(\mathbf{k}^2)$, and that $SL_2 \times SL_2$ acts on each isotypical component of $\mathbf{k}[SL_2]$ irreducibly. Following the quiver construction above, each Q gives an inclusion of coordinate algebras:

$$\pi_Q^* : \mathbf{k}[\mathcal{X}(F_g, SL_2)] \to \mathbf{k}[SL_2^{E(Q)}] \tag{30}$$

Each copy of SL_2 comes equipped with the valuation $v : \mathbf{k}[SL_2] \setminus \{0\} \to \mathbb{Z}$ which assigns $f \in End(V(n))$ the number n. There is an associated compactification of $SL_2 \subset X \subset \mathbb{P}^4 = Proj(\mathbf{k}[a, b, c, d, t])$ which is cut out by the equation $ad - bc - t^2 = 0$. Let $\hat{X}$ be the affine cone of this compactification. The valuation v is $-deg_D$ for the divisor $D \cong \mathbb{P}^1 \times \mathbb{P}^1$ defined by $t = 0$. Note that everything in sight carries an action of $SL_2 \times SL_2$. As a consequence, there is a cone of valuations on $\mathcal{X}(F_g, SL_2)$ which we can identify with $C_Q = \mathbb{R}_{\geq 0}^{E(Q)}$; one copy of $\mathbb{R}_{\geq 0}$ for each copy of SL_2. We can imagine a point $q \in C_Q$ as a metric on the quiver Q (possibly with some 0 edges).

A surjective map of quivers $\pi : Q \to Q'$ obtained by contracting edges in Q corresponds to a canonical inclusion $\pi^* : C_{Q'} \to C_Q$, obtained by sending $q \in C_{Q'}$ to the function $\pi^*(q)[e] = q(\pi(e))$ if e is not contracted, and $\pi^*(q)[e] = 0$ if e is contracted. In this way, all the faces of C_Q are identified with $C_{Q'}$ for some $\pi : Q \to Q'$. A quiver Q gives a cone C_Q of maximal dimension ($= 3g - 3$) if Q is trivalent with $\beta_1(Q) = g$. We note that two quivers give the same cone if one is obtained from the other by reversing a few arrows.

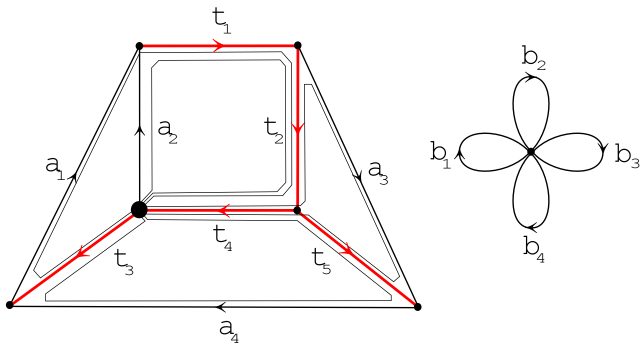
The variety $\mathcal{X}(F_g, G)$ has a large group of algebraic automorphisms coming from the outer automorphism group $Out(F_g)$ of the free group. For any $\rho \in \mathcal{X}(F_g, G)$ and $\tau \in Out(F_g)$ we obtain a new representation $\tau \circ \rho$ of F_g into G. For a fixed quiver Q we obtain a new cone of valuations on $\mathbf{k}[\mathcal{X}(F_g, SL_2)]$ by precomposing each $v \in C_Q$ with $\tau^* : \mathbf{k}[\mathcal{X}(F_g, SL_2)] \to \mathbf{k}[\mathcal{X}(F_g, SL_2)]$. In order to take this action into account, we introduce markings of quivers (see [CV86], [Man14]). Fix the quiver Q_g of with $\beta_1(Q_g) = g$ and $|V(Q_g)| = 1$. Loosely speaking a marking $\pi : Q_g \to Q$ is a topological map on the associated $1-$complexes of these quivers which induces a homotopy equivalence (the marking itself is taken up to homotopy). The end of it is that each marking induces a specific isomorphism between $\mathcal{X}(F_g, G)$ and the quotient $\mathcal{X}(Q, G)$; the group $Out(F_g)$ then changes acts on these isomorphisms by changing the marking.

In [CV86], Culler and Vogtmann introduce a (very large) simplicial space O_g called *outer space* in order to study $Out(F_g)$. This space is built with a simplex of dimension $3g - 4$ for every quiver marking (up to reversal of arrows). Let $\hat{O}_g$ be the space obtained by "coning" over O_g; that is, introduce a simplicial cone (of dimension $3g - 3$) instead of a simplex for each marking. The following is in [Man14].

Theorem 13.3. *There is a $1 - 1$ map $\Phi_g : \hat{O}_g \to \mathcal{X}(F_g, SL_2)^{an}$, realizing every (possibly scaled) point of outerspace as a valuation on $\mathbf{k}[\mathcal{X}(F_g, SL_2)]$. In particular, the cell in $\hat{O}_g$ corresponding to a marked quiver Q is mapped bijectively to C_Q.*

From now on we identify $\hat{O}_g$ with its image in $\mathcal{X}(F_g, SL_2)$.

In order to figure out how the complex $\hat{O}_g$ maps into the tropical varieties of $\mathcal{X}(F_g, SL_2)$ we need to understand how the valuations in $\hat{O}_g$ are computed on regular functions of the character variety. Character varieties are named as such essentially because their coordinate functions are characters of the representations which make up their points. For any $\rho \in Hom(F_g, SL_2)$ and word $w \in F_g$ we can take the trace $\tau_w(\rho)$ of the matrix $\rho(w) \in SL_2$; this is invariant under equivalence of representations, so it passes to a regular function $\tau_w : \mathcal{X}(F_g, SL_2) \to \mathbb{A}^1$. The τ_w are called the traceword functions on $\mathcal{X}(F_g, SL_2)$. It can be shown that they span $\mathbf{k}[\mathcal{X}(F_g, SL_2)]$. We are interested in computing $v_q(\tau_w)$ for some point in the valuation cone $q \in C_Q$ associated to a marked quiver Q.



The marking $m : Q_g \to Q$ tells us exactly how to associate $w \in \pi_1(Q_g)$ with a path $m(w) \in Q$; it turns out there is always a "minimal" choice in a certain sense. Using $m(w)$ we can compute the weight $\omega_w(e)$ of w at each edge $e \in E(Q)$, this is the number of times $m(w)$ passes through e. To compute $v_q(\tau_w)$ we "dot" the weights with the metric value $q(e)$ at each edge of Q:

Proposition 13.4.

$$v_q(\tau_w) = \sum_{e \in E(Q)} \omega_w(e) q(e). \tag{31}$$

Remark 13.5. This rule is the *non*-tree analogue of the computation of the value of a valuation in a cell of the space of phylogenetic trees $C_\mathcal{T}$ on a Plücker monomial in the Plücker algebra $R_{2,n}$.

The following theorem wraps up the properties of the valuations $\hat{O}_g$, see [Man14]:

Theorem 13.6.

(1) *For each marked quiver Q there is a set of traceword functions τ_w which form an adapted basis for each $v_q, q \in C_Q$.*

(2) *For each marked quiver Q the (finite) set of traceword functions with weight ≤ 2 on each edge of Q is a Khovanskii basis for every $v_q, q \in C_Q$.*

Remark 13.7. We have set up a comparison between the complex $\hat{O}_g$ and the space of phylogenetic trees on n leaves. The latter can be realized as an actual tropical variety of the Plücker algebra, can anything like this be proved for outer space? A partial answer comes from considering *spanned* quivers; these are quivers with $\beta_1 = g$ along with a choice of spanning tree $\mathcal{T} \subset Q$, and a labelling of the edges $E(Q) \setminus E(\mathcal{T})$ by numbers $1, \ldots, g$. We obtain an associated polyhedral complex S_g by gluing together the cones C_Q associated to spanned quivers along maps which preserve the labelled edges. Using Theorem 13.6, we can embed S_g into the non-negative points of the tropical tropical variety associated to the set of traceword functions which have weight ≤ 2 along the labelled edges. It is unknown if this set accounts for the whole (non-negative part of the) tropical variety.

If Q is trivalent and $q \in C_Q$ is chosen from the interior, the associated graded algebra $gr_q(\mathbf{k}[\mathcal{X}(F_g, SL_2)])$ is an affine semigroup algebra $\mathbf{k}[S_Q]$. Let $\mathcal{L}_Q \subset \mathbb{Z}^{E(Q)}$ be the lattice of integer points $w : E(Q) \to \mathbb{Z}$ with the property that $w(e)+w(f)+w(g) \in 2\mathbb{Z}$ whenever e, f, g share a vertex. Let $P_Q \subset \mathbb{R}^{E(Q)}$ be the cone defined by the conditions $w(e) \geq 0 \ \forall e \in E(Q)$, and the inequalities which ensure that $w(e), w(f), w(g)$ are the sides of a triangle whenever e, f, g share a vertex. In short, we ask that the Clebsch-Gorden rule for SL_2 tensor product decomposition hold at each vertex of Q. The semigroup S_Q is $P_Q \cap \mathcal{L}_Q$.

The normal toric variety $Spec(\mathbf{k}[S_Q])$ can be obtained in a pleasant way using the construction $\mathcal{X}(Q, -)$ associated to the quiver Q. The singular matrices $SL_2^c \subset M_{2,2}$ carry an $SL_2 \times SL_2$ action, so we may feed it into the quiver construction used in Theorem 13.1 to obtain $\mathcal{X}(Q, SL_2^c)$; this is isomorphic to $Spec(\mathbf{k}[S_Q])$.

The cone C_Q can also be obtained by taking degrees along the divisors in a certain combinatorial normal crossings compactification of $\mathcal{X}(F_g, SL_2)$; this is obtained by making the construction $\mathcal{X}(Q, X)$, where $SL_2 \subset X$ is the compactification corresponding to the highest weight valuation $v : \mathbf{k}[SL_2] \to \mathbb{Z}$ above. This is a projective GIT quotient by $SL_2^{V(Q)}$ taken with respect to the outer tensor product of line bundles $\boxtimes_{e \in E(Q)} \mathcal{O}(1)$ on the product $X^{E(Q)}$. The divisors at infinity in this compactification are also obtained with the quiver construction. Each irreducible component is obtained by replacing the X on one edge of Q with a copy of the $SL_2 \times SL_2$ divisor $D \subset X$.

14. $\mathcal{X}(F_g, G)$ FOR GENERAL CONNECTED REDUCTIVE G

We briefly summarize some results from [Man16].

The approach to the tropical geometry of $\mathcal{X}(F_g, SL_2)$ is to first equate this variety with $\mathcal{X}(Q, SL_2)$ for a marked quiver Q, and then use this relaxed context to port the $SL_2 \times SL_2$-invariant tropical geometry of SL_2 onto each edge of the quiver Q. This works for a general connected reductive group G to a point. Recall that this group also has distinguished cone of valuations isomorphic to the dual Weyl chamber $\Delta^\vee$. Associated to this cone we also have a $G \times G$ compactification $G \subset X$ by a combinatorial normal crossings divisor obtained by adding one divisor for each simple positive root at infinity (this is the *wonderful* compactification if G is the adjoint form of a simple Lie algebra $\mathfrak{g}$). We will list the benefits and drawbacks of this construction.

(1) Any point in the cone $C_{Q,G} = (\Delta^\vee)^{E(Q)}$ defines a valuation on $\mathbf{k}[\mathcal{X}(F_g, G)]$ with finite Khovanskii basis,

(2) The associated graded algebra of a point taken from the interior of $C_{Q,G}$ is the coordinate ring of $\mathbf{k}[\mathcal{X}(Q, G^c)]$, where G^c is the *horospherical contraction* of G; in particular a new action by $T^{E(Q)}$ is gained by passing to the associated graded algebra.

(3) There is an associated compactification $\mathcal{X}(Q, X)$ with combinatorial normal crossings divisors obtained by applying projective GIT.

(4) Unfortunately, the space $\mathcal{X}(Q, G^c)$ is not toric, so the cone $C_{Q,G}$ does not define a full dimensional cone of valuations on $\mathcal{X}(F_g, G)$.

In order to complete the cones $C_{Q,G}$ to full dimension we will take a closer look at the space $\mathcal{X}(Q, G^c)$. Recall that the contraction G^c is isomorphic to the GIT quotient $T\backslash\backslash[G\backslash\backslash U \times U//G]$ as a $G \times G$ variety. This implies that we can construct $\mathcal{X}(Q, G^c)$ in a different way by exchanging the G quotients at the vertices for T quotients on the edges. First we take a product of the spaces $P_3(G)^{V(G)}$, which as the structure of a $(T \times T)^{E(Q)}$ variety (one action for every edge leaving a vertex). Let T_Q be the image of the embedding $T^{E(Q)} \to (T \times T)^{E(Q)}$, where $T \subset T \times T$ acts with characters given by weights on the left (incoming direction) and dual weights on the right (outgoing direction). We have:

$$\mathcal{X}(Q, G^c) \cong T_Q\backslash\backslash P_3(G)^{E(Q)}. \tag{32}$$

Now we can use T^3-homogeneous valuations on each copy of $\mathbf{k}[P_3(G)]$ to obtain a toric associated graded algebra of $\mathcal{X}(Q, G^c)$; and therefore also of $\mathcal{X}(F_g, G)$. We summarize the necessary choices below, see [Man16].

Theorem 14.1. *For each choice of the following data:*

(1) a marked quiver Q,

(2) a T^3 homogeneous maximal rank valuation $\mathfrak{w}_v$ on $P_3(G)$ with finite Khovanskii basis for each vertex $v \in V(Q)$ (e.g. a string valuation),

there is

(1) a maximal dimension cone $C_{Q,G} \subset C_{Q,\mathfrak{w},G}$ of valuations on $\mathbf{k}[\mathcal{X}(F_g, G)]$ with a common adapted basis and finite Khovanskii basis,

(2) a toric degeneration of $\mathcal{X}(F_g, G)$ to an affine toric variety $\mathbf{k}[S_{Q,\mathfrak{w},G}]$. This toric variety is normal if all the $\mathfrak{w}_v$ are string valuations,

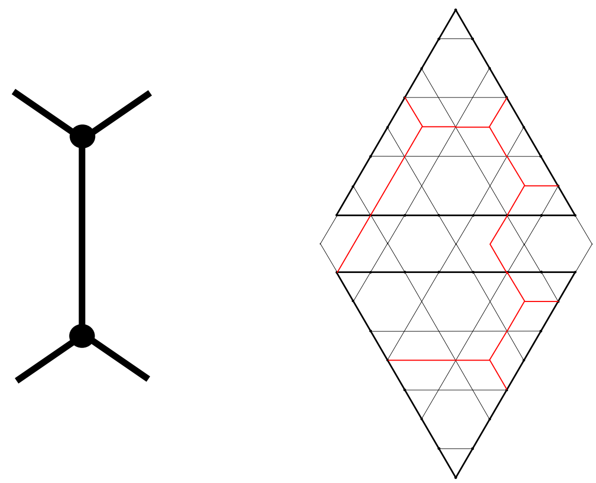
(3) a compactification $\mathcal{X}(Q, \mathfrak{w}, G)$ of $\mathcal{X}(F_g, G)$ by a combinatorial normal crossings divisor.

For $G = SL_r$ we can choose the string valuation at each vertex which gives the Berenstein-Zelevinsky triangles when applied to $\mathbf{k}[P_3(G)]$. The resulting value semigroup is composed of the Berenstein-Zelevinsky quilts. Berenstein Zelevinsky quilts are made by stitching together Berenstein Zelevinsky triangles along shared edges over the quiver Q.

Remark 14.2. The compactifications $\mathcal{X}(Q, \mathfrak{w}, G)$ associated to choices of string valuations should be interesting spaces.

15. QUIVER VARIETIES

Now we turn our attention to one more type of variety built from a quiver Q: a quiver variety. Recall that an edge $e \in Q$ has a distinguished tail $\delta_1(e)$ and a head



$\delta_2(e)$. We define the following group and space associated to a quiver Q with no leaves equipped with a rank function $\mathbf{r} : V(Q) \to \mathbb{Z}_{\geq 0}$:

$$G(Q, \mathbf{r}) = \prod_{v \in V(Q)} GL_{\mathbf{r}(v)}, \quad M(Q, \mathbf{r}) = \prod_{e \in E(Q)} M_{\mathbf{r}(\delta_1(e)) \times r(\delta_2(e))}. \tag{33}$$

The action of $G(Q, \mathbf{r})$ on $M(Q, \mathbf{r})$ is given by having $GL_{\mathbf{r}(\delta_1(e))}$ act on the left of $M_{\mathbf{r}(\delta_1(e)) \times \mathbf{r}(\delta_2(e))}$ and $GL_{\mathbf{r}(\delta_2(e))}$ act on the right. We choose an additional piece of information: an assignment of character $det^{\mathbf{s}(v)} : GL_{\mathbf{r}(v)} \to \mathbb{G}_m$ to each vertex; these are assembled into a character $\chi_{\mathbf{s}} : G(Q, r) \to \mathbb{G}_m$. The quiver variety $\mathcal{X}(Q, \mathbf{r}, \mathbf{s})$ is defined to be the GIT quotient:

$$\mathcal{X}(Q, \mathbf{r}, \mathbf{s}) = G(Q, \mathbf{r}) \backslash\backslash_{\chi_{\mathbf{s}}} M(Q, \mathbf{r}). \tag{34}$$

The variety $\mathcal{X}(Q, \mathbf{r}, \mathbf{s})$ is obtained by taking Proj of the graded ring $R_{Q,\mathbf{r},\mathbf{s}}$ whose N-th graded component is the space of invariants $R_{Q,\mathbf{r},\mathbf{s}}(N) = [\mathbb{C}\chi_{\mathbf{s}}^N \otimes \mathbf{k}[M(Q, \mathbf{r})]]^{G(Q,\mathbf{r})}$.

Now we recall the isotypical decomposition of $M_{m \times n}$:

$$\mathbf{k}[M_{m \times n}] = \bigoplus_{\lambda \in \Pi_+(min(m,n))} V(\lambda)^* \otimes V(\lambda). \tag{35}$$

Here λ is a dominant weight of the general linear group attached to the smaller of m, n, which can likewise be regarded as a dominant weight of the larger of m, n. Our goal is to construct degenerations of $R_{Q,\mathbf{r},\mathbf{s}}(N)$ to affine semigroup algebras by finding valuations on this ring, as we have done with other spaces. In analogy with the case of character varieties $\mathcal{X}(F_g, G)$, notice that we can start by replacing each space of matrices $M_{m \times n}$ in the description of $\mathcal{X}(Q, \mathbf{r}, \mathbf{s})$ with its horospherical contraction $M^c_{m,n}$. As this contraction turns the dominant weight decomposition of $\mathbf{k}[M_{m\times n}]$ into a grading, the result is a certain subalgebra of the torus quotient of $GL_m/U_m \times GL_n/U_n$:

$$\mathbf{k}[M^c_{m\times n}] \subset \mathbf{k}[GL_m/U_m \times GL_n/U_n]^{T_{m,n}}. \tag{36}$$

Also in analogy with the character variety case, this reduces the problem to understanding a certain torus quotient. We fix a character $det^{\mathbf{s}(v)} : GL_{\mathbf{r}(v)} \to \mathbb{G}_m$ and consider the following projective GIT quotient:

$$P_{\mathbf{s}(v),n_v}(GL_{\mathbf{r}(v)}) = GL_{\mathbf{r}(v)} \backslash\backslash_{\mathbf{s}(v)} [GL_{\mathbf{r}(v)}/U_{\mathbf{r}(v)} \times \ldots \times GL_{\mathbf{r}(v)}/U_{\mathbf{r}(v)}]. \tag{37}$$

Here n_v is the number of edges containing the vertex v in Q. This quotient slightly differs from $P_n(G)$, but it's close enough that we can make use of the methods we've built for such problems. We can also realize $P_{\mathbf{s}(v),n_v}(GL_{\mathbf{r}(v)})$ as a quotient with respect to the trivial character of $GL_{\mathbf{r}(v)}$ by making use of a "shifting trick:' '

$$P_{\mathbf{s}(v),n_v}(GL_{\mathbf{r}(v)}) = GL_{\mathbf{r}(v)} \backslash\backslash_0 [\mathbb{G}_m(\mathbf{s}(v)) \times GL_{\mathbf{r}(v)}/U_{\mathbf{r}(v)} \times \ldots \times GL_{\mathbf{r}(v)}/U_{\mathbf{r}(v)}]. \tag{38}$$

Here $GL_{\mathbf{r}(v)}$ acts on $\mathbb{G}_m(\mathbf{s}(v)) \cong \mathbb{G}_m$ through the character $det^{-\mathbf{s}(v)}$. The coordinate ring $\mathbf{k}[P_{\mathbf{s}(v),n_v}(GL_{\mathbf{r}(v)})]$ can now be naturally realized as a T^{n_v+1}-homogeneous subalgebra of $\mathbf{k}[P_{n_v+1}(GL_{\mathbf{r}(v)})]$. This means that we can use all of our branching and string valuation techniques on $\mathbf{k}[P_{\mathbf{s}(v),n_v}(GL_{\mathbf{r}(v)})]$, because these all respect the torus action.

Lemma 15.1. *For every choice of trivalent tree $\mathcal{T}$ with n_v+1 leaves and choice of strings $\mathbf{i}_w$ at the vertices of $\mathcal{T}$, we obtain a T^{n_v+1}-homogeneous toric degeneration of $\mathbf{k}[P_{\mathbf{s}(v),n_v}(GL_{\mathbf{r}(v)})]$.*

Now we act as in the character variety case and sew this degenerated spaces together into a degeneration of $\mathcal{X}(Q,\mathbf{r},\mathbf{s})$. This relies on realizing the "horospherical contraction" of $\mathcal{X}(Q,\mathbf{r},\mathbf{s})$ (where we contract each edge space $M_{\mathbf{r}(\delta_1(e))\times\mathbf{r}(\delta_2(e))}$) as a torus quotient of the spaces $P_{\mathbf{s}(v),n_v}(GL_{\mathbf{r}(v)})$, and works in the same way as the character variety case.

Theorem 15.2. *For every choice of the following data:*

(1) trivalent trees $\mathcal{T}_v$ with n_v leaves,

(2) strings $\mathbf{i}(w)$ at each vertex w of a $\mathcal{T}_v$,

we obtain

(1) a degeneration of $\mathcal{X}(Q,\mathbf{r},\mathbf{s})$ to a projective toric variety,

(2) a maximal dimension cone of valuations on $R_{Q,\mathbf{r},\mathbf{s}}$ which all share a common finite Khovanskii basis.

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University of Kentucky
E-mail address: `Christopher.Manon@uky.edu`