Cyclic Sieving Phenomenon of Plane Partitions and Cluster Duality of Grassmannian

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Joint work with Jiuzu Hong and Linhui Shen

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Throughout this talk, let a, b, c be three positive integers and let $n := a + b$.

Cyclic Sieving Phenomenon

Definition

Let S be a finite set. Let g be a permutation on S that is of order m. Let $F(q)$ be a polynomial in q. We say that the triple $(S, g, F(q))$ exhibits the *cyclic sieving phenomenon (CSP)* if the fixed point set cardinality $\#S^{g^d}$ is equal to the polynomial evaluation $F(\zeta^d)$ for all $d \geq 0$ where ζ is a primitive mth root of unity.

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Example

Let $[n] := \{1, \ldots, n\}$ and let $\binom{[n]}{k}$ be the set of k-element subsets of $[n]$. Consider the cyclic shift $R : i \mapsto i + 1 \mod n$ on $[n]$ and the induced action on $\binom{[n]}{k}$. It is known that the triple $\left(\binom{[n]}{k}, R, \begin{bmatrix} n \\ k \end{bmatrix}_q\right)$ exhibits CSP, where $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the quantum binomial coefficient.

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Although the definition of CSP seems very combinatorial, many proofs of known CSP involve quite a bit of geometric representation theory. Please see Sagan's survey [Sag11] for more detailed examples.

Plane Partitions

Definition

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Remark. Think of a plane partition as a 3d Young diagram.

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- Denote the collection of $a \times b$ plane partitions with entries no bigger than some $c > 0$ by $P(a, b, c)$.

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- For any triple (a, b, c), define

$$M_{a,b,c}(q) := \sum_{\pi \in P(a,b,c)} q^{|\pi|}.$$

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- In [Rob16], Roby defined a *toggling operation* η on a plane partition π by changing each entry from bottom to top in each column and from left to right across all columns according to

$$\pi'_{i,j} = \min\{\pi_{i-1,j}, \pi_{i,j-1}\} + \max\{\pi_{i+1,j}, \pi_{i,j+1}\} - \pi_{i,j}.$$

Plane Partitions

Example

Here's an example of $\eta(\pi)$ for some plane partition $\pi \in P(2,3,6)$.

	6	6	6	
6	3	2	2	0
6	3	1	0	0
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Theorem (Hong-Shen-W.)

The toggling operation η has order $n = a + b$, and the triple $(P(a,b,c), \eta, M_{a,b,c}(q))$ exhibits CSP.

Decorated Grassmannian

Definition

The *decorated Grassmannian* is defined to be

$$\mathscr{G}r_a(n) := \mathrm{SL}_a \Big\backslash \mathrm{Mat}_{a,n}^{\text{full rank}} \qquad \mathscr{G}r_a^{\times}(n) := \mathrm{SL}_a \backslash \mathrm{Mat}_{a,n}^{\times}$$

where superscript $\times$ indicates an additional consecutive general position condition.

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where superscript $\times$ indicates an additional consecutive general position condition.

- Elements of $\mathscr{G}r_a(n)$ can be represented by the a-fold exterior product α of the row vectors of a matrix in $\mathrm{Mat}_{a,n}$.
- $\mathcal{O}(\mathscr{G}r_a(n))$ is generated by Plücker coordinates Δ_g for any $g \in \bigwedge^a \mathbb{C}^n$, which is defined by

$$\Delta_g(\alpha) := \langle g, \alpha \rangle .$$

Decorated Grassmannian

- There is a $\mathbb{G}_m$-action on $\mathscr{G}r_a(n)$ defined by $t.\alpha := t\alpha$; with respect to this $\mathbb{G}_m$-action

$$\mathcal{O}\left(\mathscr{G}r_a(n)\right) = \bigoplus_{c>0} \mathcal{O}\left(\mathscr{G}r_a(n)\right)_c, \quad \mathcal{O}\left(\mathscr{G}r_a(n)\right)_c = V_{c\omega_a},$$

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  where $V_{c\omega_a}$ is a representation of GL_n.
- There is a boundary divisor $D = \bigcup_i D_i$ such that $\mathscr{G}r_a^\times(n) = \mathscr{G}r_a(n) \setminus D$.
- Define a *twisted cyclic rotation*

$$C_a := \begin{pmatrix} 0 & (-1)^{a-1} \\ \mathrm{Id}_{n-1} & 0 \end{pmatrix} \in \mathrm{GL}_n$$

  which acts on $\mathrm{Mat}_{a,n}^{\text{full rank}}$ by matrix multiplication on the right. This action descends to an action of C_a on $\mathscr{G}r_a(n)$ and induces an action of C_a on $\mathcal{O}\left(\mathscr{G}r_a(n)\right)$ that is compatible with the GL_n-action.

Decorated Grassmannian

- Consider the maximal torus $T \subset \mathrm{GL}_n$ consisting of invertible diagonal matrices, which acts on $\mathrm{Mat}_{a,n}^{\text{full rank}}$ by matrix multiplication on the right. This action descends to an action of T on $\mathscr{G}r_a(n)$ and induces an action of T on $\mathcal{O}(\mathscr{G}r_a(n))$ that is compatible with the GL_n-action. Thus by using such T-action we can further decompose $\mathcal{O}(\mathscr{G}r_a(n))_c \cong V_{c\omega_a}$ into weight spaces

$$V_{c\omega_a} = \bigoplus_{\mu} V_{c\omega_a}(\mu).$$

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$$V_{c\omega_a} = \bigoplus_{\mu} V_{c\omega_a}(\mu).$$
- A result of Scott [Sco06] can be generalized to show that $\mathcal{O}\left(\mathscr{G}r_a^{\times}(n)\right) \cong \mathrm{up}\left(\mathscr{A}_{a,n}\right)$ for some cluster variety $\mathscr{A}_{a,n}$.

Decorated Configuration Space

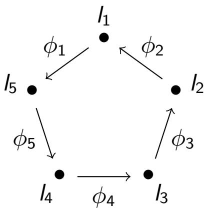
Motivated by an idea of Goncharov, we define decorated configuration space as follows.

Definition

The decorated configuration space $\mathscr{C}onf_n^\times(a)$ is defined to be

$$\mathrm{GL}_a \Big\backslash \left\{ \left(\phi_i : l_i \overset{\cong}{\to} l_{i-1}, l_i \subset \mathbb{C}^a \right)_{i=1}^{n} \right\}$$

with an additional consecutive general position condition.



Decorated Configuration Space

- By composing all the ϕ_i in a decorated configuration we obtain its monodromy; its *twisted monodromy* $P : \mathscr{C}onf_n^\times(a) \to \mathbb{G}_m$ is deifned to be $(-1)^{a-1}$ multiple of the monodromy.

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- For each i define $\vartheta_i : \mathscr{C}onf_n^\times(a) \to \mathbb{A}^1$ to be the number such that

$$\phi_{i-a+1}\left(v_{i-a+1}\right) - \vartheta_i v_{i-a} \in \mathrm{Span}\left\{l_{i-a+2}, \ldots, l_i\right\};$$

then define the *potential function*

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- There is a *cyclic rotation* R acting on $\mathscr{C}onf_n^{\times}(a)$ defined by

$$[\phi_1, l_1, \phi_2, l_2, \ldots, \phi_n, l_n] \mapsto [\phi_n, l_n, \phi_1, l_1, \ldots, \phi_{n-1}, l_{n-1}].$$

Decorated Configuration Space

- By fixing a volume form ω on $\mathbb{C}^a$, for each i we define a regular function $M_i : \mathscr{C}onf_n^\times(a) \to \mathbb{G}_m$ by

$$M_i := \frac{\omega\left(\phi_{i-a+1}\left(v_{i-a+1}\right) \wedge \cdots \wedge \phi_i\left(v_i\right)\right)}{\omega\left(v_{i-a+1} \wedge \cdots \wedge v_i\right)}.$$

  This gives rise to a map

$$M : \mathscr{C}onf_n^\times(a) \to T^\vee$$

  where $T^\vee$ is the torus dual to the maximal torus $T \subset \mathrm{GL}_n$.

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  where $T^\vee$ is the torus dual to the maximal torus $T \subset \mathrm{GL}_n$.
- We prove that $\mathcal{O}\left(\mathscr{C}onf_n^\times(a)\right) \cong \mathrm{up}\left(\mathscr{X}_{a,n}\right)$ for some cluster variety $\mathscr{X}_{a,n}$.

Summary of the Duality between Decorated Spaces

$$\left\{\begin{array}{l}\text{decorated Grassmannian } \mathscr{G}r_a^{\times}(n)\\ \text{a } \mathbb{G}_m\text{-action on } \mathscr{G}r_a(n)\\ \text{boundary divisors } D=\bigcup_i D_i\\ \text{twisted cyclic rotation } C_a\\ \text{an action by } T\subset \mathrm{GL}_n \text{ on } \mathscr{G}r_a(n)\end{array}\right\}$$

$$\nwarrow\!\!\searrow$$

$$\left\{\begin{array}{l}\text{decorated configuration space } \mathscr{C}onf_n^{\times}(a)\\ \text{twisted monodromy } P:\mathscr{C}onf_n^{\times}(a)\to \mathbb{G}_m\\ \text{potential function } \mathcal{W}=\sum_i \vartheta_i\\ \text{cyclic rotation } R\\ \text{a projection } \mathscr{C}onf_n^{\times}(a)\to T^{\vee}\end{array}\right\}$$

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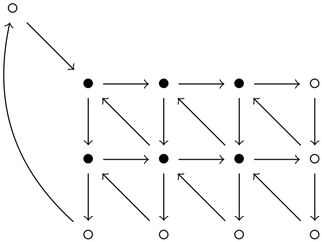
Cluster Structures on the Decorated Spaces

Up to codimension 2, $\mathscr{G}r_a^\times(n) \cong \mathscr{A}_{a,n}$ and $\mathscr{C}onf_n^\times(a) \cong \mathscr{X}_{a,n}$, where $(\mathscr{A}_{a,n}, \mathscr{X}_{a,n})$ is the cluster ensemble associated to some quiver $Q_{a,n}$. Below is what $Q_{3,7}$ looks like.



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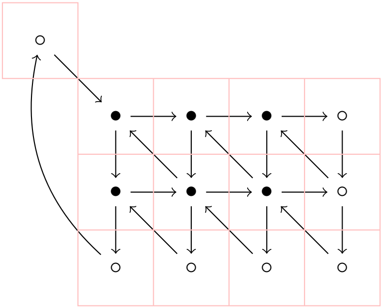

Set of Tropical Points

Definition

Given a positive space X (i.e., an algebraic variety with a semifield of rational functions $P(X)$) and a semifield S, the set of S-points of X is defined to be

$$X(S) := \mathrm{Hom}_{\text{semifield}}\left(P(X), S\right).$$

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Example

Consider an algebraic torus T with $P(T)$ defined to be the semifield generated by its characters inside the field of rational functions. Then for any semifield S, the set of S-points $T(S)$ can be identified with $X_*(T) \otimes_{\mathbb{Z}} S$ where $X_*(T)$ denotes the cocharacter lattice of T.

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Cluster varieties (both type $\mathscr{A}$ and type $\mathscr{X}$) are known to be positive spaces, and an important set of tropical points in our story is $\mathscr{X}\left(\mathbb{Z}^t\right)$, where $\mathbb{Z}^t$ is the semifield of tropical integers $\left(\mathbb{Z}^t, \min, +\right)$.

Cluster Duality

Fock and Goncharov conjectured the following statement in [FG09].

Conjecture (Fock-Goncharov Cluster Duality)

For a quiver Q, $\mathrm{up}(\mathscr{A}_Q)$ *admits a canonical basis parametrized by* $\mathscr{X}_Q(\mathbb{Z}^t)$ *and* $\mathrm{up}(\mathscr{X}_Q)$ *admits a canonical basis parametrized by* $\mathscr{A}_Q(\mathbb{Z}^t)$*.*

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Gross, Hacking, Keel, and Kontsevich gave a sufficient condition for the Fock-Goncharov cluster duality conjecture in [GHKK14], which can be reformulated as follows.

Theorem (Gross-Hacking-Keel-Kontsevich)

The full Fock-Goncharov cluster duality holds for the cluster ensemble $(\mathscr{A}_Q, \mathscr{X}_Q)$ *if the following two conditions are satisfied:*

- *a cluster Donaldson-Thomas transformation (defined by Goncharov and Shen in [GS18]) exists on* $\mathscr{X}_Q^{\mathrm{uf}}$*;*
- *the canonical map* $p : \mathscr{A}_Q \to \mathscr{X}_Q^{\mathrm{uf}}$ *is surjective.*

Cluster Duality of Grassmannian

- In the case of the cluster ensemble $(\mathscr{A}_{a,n}, \mathscr{X}_{a,n})$, the cluster variety $\mathscr{X}^{\mathrm{uf}}_{a,n} \cong \mathrm{Conf}^{\times}_{n}(a)$, which is the configuration space of lines without isomorphisms between them, and the cluster Donaldson-Thomas transformation was constructed in [Wen18].

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- The surjectivity of the p map follows from surjectivity of π and the following commutative diagram.

$$\begin{array}{ccc} \mathscr{G}r^{\times}_{a}(n) & \overset{\cong}{\dashrightarrow} & \mathscr{A}_{a,n} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle p} \\ \mathrm{Conf}^{\times}_{n}(a) & \underset{\cong}{\dashrightarrow} & \mathscr{X}_{a,n} \end{array}$$

Here π is defined by taking the configuration of the spans of the column vectors of a matrix representative in $\mathrm{Mat}^{\times}_{a,n}$.

Cluster Duality of Grassmannian

Theorem (Hong-Shen-W.)

The Fock-Goncharov cluster duality holds on the cluster ensemble $(\mathscr{A}_{a,n}, \mathscr{X}_{a,n}) \cong (\mathscr{G}r_a^\times(n), \mathscr{C}onf_n^\times(a))$. *In particular,*

$$\mathcal{O}\left(\mathscr{G}r_a^\times(n)\right) = \bigoplus_{q \in \mathscr{C}onf_n^\times(a)(\mathbb{Z}^t)} \theta_q,$$

$$\mathcal{O}\left(\mathscr{C}onf_n^\times(a)\right) = \bigoplus_{p \in \mathscr{G}r_a^\times(n)(\mathbb{Z}^t)} \vartheta_p.$$

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Remark. The regular functions ϑ_i we defined on $\mathscr{C}onf_n^\times(a)$ are precisely the basis vectors corresponding to the basic lamination of the frozen vertices of $Q_{a,n}$.

Gelfand-Zetlin Coordinates

- Using the quiver $Q_{a,n}$, we get a coordinate system $\{x_{0,0}\} \cup \{x_{i,j}\}_{1 \le i \le a}^{1 \le j \le b}$ on $\mathscr{C}onf_n^{\times}(a)\left(\mathbb{Z}^t\right) \cong \mathbb{Z}^{ab+1}$.

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$$l_{i,j} := \sum_{i\leq k, j\leq l} x_{k,l}.$$

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	$x_{2,1}$	$x_{2,2}$	$x_{2,3}$	$x_{2,4}$
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- A result of Gross, Hacking, Keel, and Kontsevich on partial compactification [GHKK14] implies that a basis vector θ_q can be extended to a regular function after adding the boundary divisor $D = \bigcup_i D_i$ if and only if $\mathcal{W}^t(q) \geq 0$.

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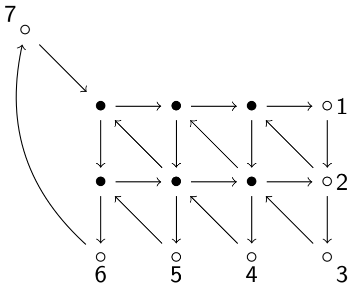
Theorem (Hong-Shen-W.)

Define $\Theta(a, b, c) := \left\{\theta_q \;\middle|\; \mathcal{W}^t(q) \geq 0, P^t(q) = c\right\}$. Then $\Theta(a, b, c)$ is a basis of the irreducible representation $V_{c\omega_a} \cong \mathcal{O}\left(\mathscr{G}r_a(n)\right)_c$, and there is a natural bijection between $\Theta(a, b, c)$ and plane partitions $P(a, b, c)$.

Gelfand-Zetlin Coordinates

In [GSV10], Gekhtman, Shapiro, and Vainshtein observed that the rotation R on $\mathscr{C}onf_n^{\times}(a)$ is in fact a cluster transformation which can be realized by a sequence of mutations in the order similar to the toggling operation η.



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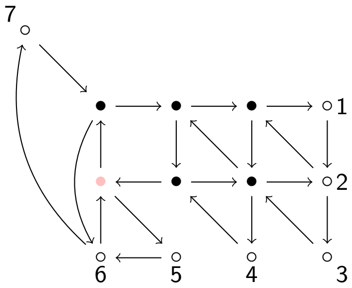

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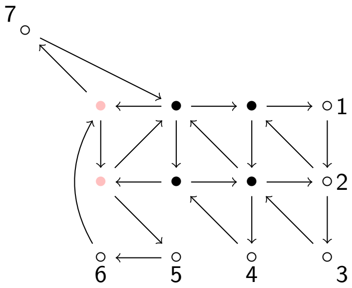

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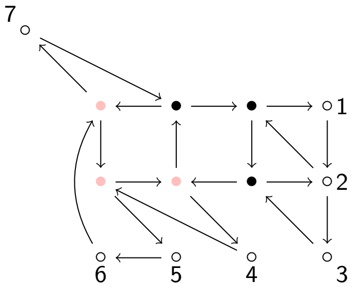

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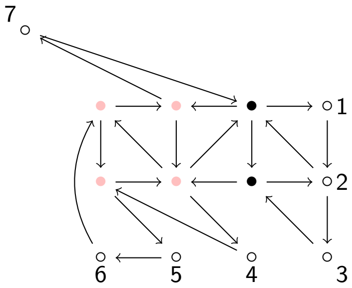

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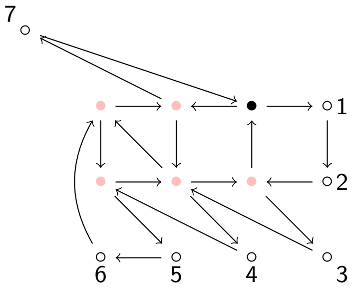

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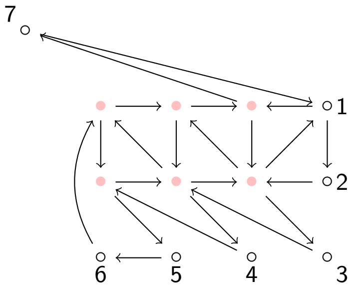

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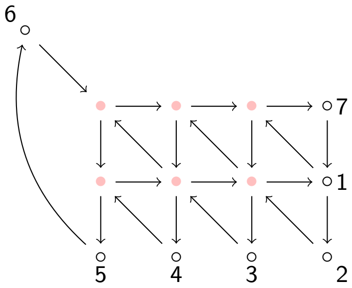

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- Following the cluster duality of Grassmannian we prove that $C_a.\theta_q = \theta_{R(q)}$. Therefore we obtain the following theorem.

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Theorem (Hong-Shen-W.)

The action of the twisted cyclic rotation C_a on the basis $\Theta(a, b, c)$ is given by $C_a.\theta_\pi = \theta_{\eta(\pi)}$. In particular, this implies that η is of order n.

Gelfand-Zetlin Coordinates

- In fact, the Gelfand-Zetlin coordinates $(l_{i,j})_{1\le i\le a}^{1\le j\le b}$ can be expanded into a Gelfand-Zetlin pattern for $V_{c\omega_a}$ by adding a triangle with entries c on the left and a triangle with entries 0 at the bottom.

$$
\begin{array}{ccccccc}
c & c & c & l_{1,1} & l_{1,2} & l_{1,3} & l_{1,4} \\
 & c & c & l_{2,1} & l_{2,2} & l_{2,3} & l_{2,4} \\
 & & c & l_{3,1} & l_{3,2} & l_{3,3} & l_{3,4} \\
 & & & 0 & 0 & 0 & 0 \\
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 & & & & & 0 & 0 \\
 & & & & & & 0
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 & & c & l_{3,1} & l_{3,2} & l_{3,3} & l_{3,4} \\
 & & & 0 & 0 & 0 & 0 \\
 & & & & 0 & 0 & 0 \\
 & & & & & 0 & 0 \\
 & & & & & & 0
\end{array}
$$

- By computation we show that the tropicalization $M_i^t(q)$ can be computed as $d_i - d_{i-1}$ using the triangle above, where d_i is the sum of the entries along the ith diagonal (counting from the right).

Gelfand-Zetlin Coordinates

Following cluster duality of Grassmannian we further prove the following statement.

Theorem (Hong-Shen-W.)

The tropicalization $M^t : \mathscr{C}onf_n^{\times}(a)\left(\mathbb{Z}^t\right) \to T^{\vee}\left(\mathbb{Z}^t\right) \cong X^*(T)$ *gives the weight of* θ_q *under the action of the maximal torus* $T \subset \mathrm{GL}_n$*. In particular, the basis* $\Theta(a, b, c)$ *of* $V_{c\omega_a}$ *is compatible with the weight decomposition* $V_{c\omega_a} = \bigoplus_{\mu} V_{c\omega_a}(\mu)$.

Proof of CSP of Plane Partitions

- $\#\left\{\pi \in P(a,b,c) \mid \eta^d(\pi) = \pi\right\} = \mathrm{Tr}_{V_{c\omega_a}} C_a^d.$

Proof of CSP of Plane Partitions

- $\#\left\{\pi \in P(a,b,c) \mid \eta^d(\pi) = \pi\right\} = \mathrm{Tr}_{V_{c\omega_a}} C_a^d$.
- The characteristic polynomial of C_a is

$$\det\left(\lambda \mathrm{Id}_n - C_a\right) = \lambda^n - (-1)^{a-1},$$

which has n distinct roots $\zeta^{-\frac{a-1}{2}}, \zeta^{-\frac{a-1}{2}}\zeta, \dots, \zeta^{-\frac{a-1}{2}}\zeta^{n-1}$ (ζ is a primitive nth root of unity). Therefore C_a is conjugate to

$$D = \mathrm{Diag}\left(\zeta^{-\frac{a-1}{2}}\zeta^{n-1}, \zeta^{-\frac{a-1}{2}}\zeta^{n-2}, \dots, \zeta^{-\frac{a-1}{2}}\right),$$

which implies that

$$\mathrm{Tr}_{V_{c\omega_a}} C_a^d = \mathrm{Tr}_{V_{c\omega_a}} D^d.$$

Proof CSP of Plane Partitions

- The character formula tells us that

$$\mathrm{Tr}_{V_\lambda} \mathrm{Diag}\left(pq^{n-1}, pq^{n-2}, \ldots, p\right) = p^{\langle \omega_n, \lambda \rangle} \sum_{\mu} \dim V_\lambda(\mu) q^{\langle \rho, \mu \rangle},$$

where $\rho = (n-1, n-2, \ldots, 1, 0)$; therefore by setting $q := \zeta^d$, we have

$$\mathrm{Tr}_{V_{c\omega_a}} D^d = q^{-\frac{a(a-1)c}{2}} \sum_{\mu} \dim V_{c\omega_a}(\mu) q^{\langle \rho, \mu \rangle}.$$

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$$\mathrm{Tr}_{V_{c\omega_a}} D^d = q^{-\frac{a(a-1)c}{2}} \sum_{\mu} \dim V_{c\omega_a}(\mu) q^{\langle \rho, \mu \rangle}.$$

- But from the Gelfand-Zetlin pattern we also know that

$$\dim V_{c\omega_a}(\mu) = \# \left\{ \pi \in P(a,b,c) \;\middle|\; M^t(\pi) = \mu \right\};$$

therefore

$$\dim V_{c\omega_a}(\mu) q^{\langle \rho, \mu \rangle} = \sum_{M^t(\pi) = \mu} q^{\left\langle \rho, M^t(\pi) \right\rangle}.$$

Proof of CSP of Plane Partitions

- By simple computation one can see that $\langle \rho, M^t(\pi) \rangle$ is just the sum of all entries in the Gelfand-Zetlin pattern, which is equal to $\frac{a(a-1)c}{2} + |\pi|$.

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- Now plug everything in, we get that for $q = \zeta^d$,

$$\begin{aligned}\mathrm{Tr}_{V_{c\omega_a}} C_a^d =& q^{-\frac{a(a-1)c}{2}} \sum_{\pi \in P(a,b,c)} q^{\langle \rho, M^t(\pi) \rangle} \\ =& q^{-\frac{a(a-1)c}{2}} \sum_{\pi \in P(a,b,c)} q^{\frac{a(a-1)c}{2}} q^{|\pi|} \\ =& \sum_{\pi \in P(a,b,c)} q^{|\pi|},\end{aligned}$$

which finishes the proof of our theorem.

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