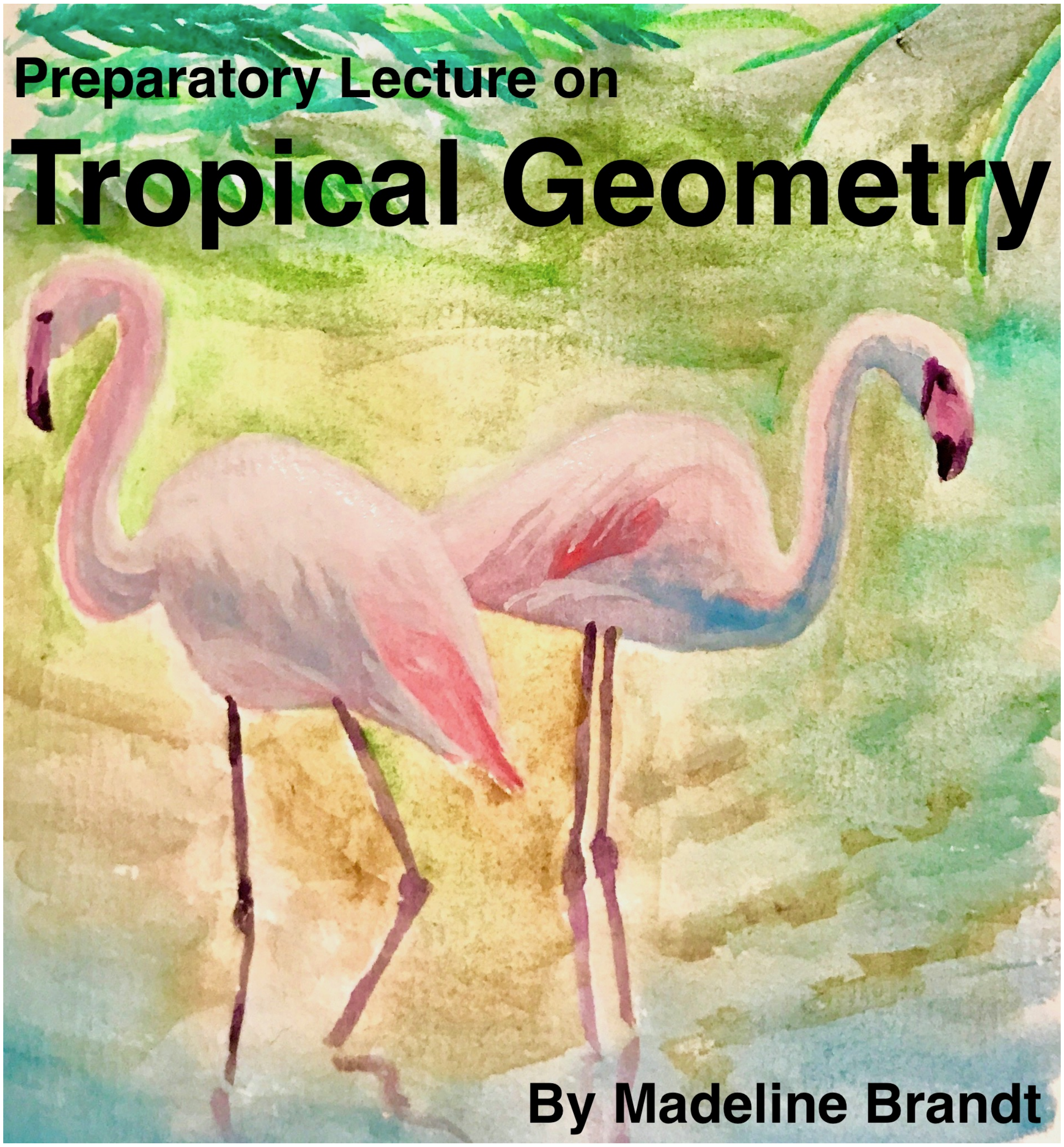

Preparatory Lecture on
Tropical Geometry
By Madeline Brandt

1 Introduction

Tropical geometry is a new subject which creates a bridge between the two islands of algebraic geometry and combinatorics. It has many fascinating connections to other areas as well, as we will learn in this workshop. The aim of this lecture and these notes is to provide the tools necessary to cross these bridges.

First, we set up the necessary definitions for studying tropical objects. We will begin with a discussion of valuations, and then discuss some essentials from polyhedral geometry. Then we will discuss a microcosm of tropical geometry which provides a plethora of examples, namely tropical hypersurfaces. Then, I will give an overview of tropical varieties and the fundamental theorem of tropical geometry. We mostly follow [MS15]

At the end of the notes in Table 1 we summarize the main notation from tropical geometry.

2 Valuations

Tropicalization is a process we apply to varieties over a field which are equipped with a function called a valuation, so we begin here to start the study of tropical geometry.

Definition 2.1. A **valuation** on a field K is a function $\nu : K \to \mathbb{R} \cup \{\infty\}$ satisfying

1. $\nu(a) = \infty$ if and only if $a = 0$,
2. $\nu(ab) = \nu(a) + \nu(b)$,
3. $\nu(a + b) \geq \min\{\nu(a), \nu(b)\}$.

The image of ν is denoted by Γ_{val}, and is called the **value group**. We denote by R the set of all elements with nonnegative valuation. It is a local ring with maximal ideal $\mathfrak{m}$ given by all elements with positive valuation. The quotient ring is denoted by $\Bbbk$ and it is called the **residue field**.

Example 2.2. Every field has a trivial valuation sending $\nu(K^*) = 0$.

Example 2.3. (p-adic valuation) The p-adic valuation on $\mathbb{Q}$ is given by a prime p, and the valuation of a rational number

$$\frac{ap^l}{bp^k}$$

With $p \nmid a, b$ and $(a, b) = 1$ is $l - k$. The local ring R is the localization of $\mathbb{Z}$ at $< p >$, and the residue field is $\mathbb{F}_p$.

Example 2.4. (Puiseaux series) The Puiseaux series are the formal power series with rational exponents and coefficients in $\mathbb{C}$:

$$c(t) = c_1 t^{a_1} + c_2 t^{a_2} + \cdots$$

for a_i an increasing sequence of rational numbers which have a common denominator. The valuation is given by taking $\nu(c) = a_1$. The Puiseaux series are algebraically closed.

A **splitting** of a valuation is a homomorphism $\phi : \Gamma_{\text{val}} \to K^*$ such that $\nu(\phi(w)) = w$. The element $\phi(w)$ is denoted t^w, and t is called a **uniformizer** for K.

Question 1. Show that the residue field of $\mathbb{C}\{\{t\}\}$ is $\mathbb{C}$.

Question 2. What is the residue field of $\mathbb{Q}$ with the p-adic valuation?

3 Crash Course in Polyhedral Geometry

Polyhedral geometry is a deep and beautiful subject in discrete mathematics. Here we just give the basics of what we will need for this workshop. See [Zie95] for more about polyhedral geometry.

Definition 3.1. A set $X \subset \mathbb{R}^n$ is **convex** if for any two points in the set, the line segment between them is also contained in the set. The **convex hull** $\text{conv}(U)$ of a subset $U \subset \mathbb{R}^n$ is the smallest convex set containing U. A **polytope** is a convex set which is expressible as the convex hull of finitely many points.

A **polyhedral cone** C in $\mathbb{R}^n$ is the positive hull of a finite subset of $\mathbb{R}^n$:

$$C = \text{pos}(v_1, \ldots, v_r) := \left\{ \sum_{i=1}^{r} \lambda_i v_i \;\middle|\; \lambda_i \geq 0 \right\}.$$

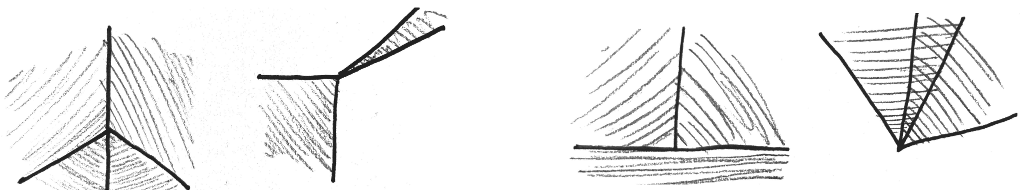
A **polyhedron** is the intersection of finitely many half spaces. Bounded polyhedra are polytopes: these are equivalent ways to define them.

A **face** of a cone C is determined by a linear functional $w \in \mathbb{R}^n$, by selecting the points along which the linear functional is minimized:

$$\text{face}_w(C) = \{x \in C \mid wx \leq wy \text{ for all } y \in C\}.$$

A face which is not contained in any larger proper face is called a **facet**.

Definition 3.2. A **polyhedral fan** is a collection $\mathcal{F}$ of polyhedral cones such that every face of a cone is in the fan, and the intersection of any two cones in the fan is a face of each. For some examples and nonexamples, see Figure 1.



(a) These keep you cool in the tropics.

(b) "I'm not a fan."

Figure 1: Examples and nonexamples of polyhedral fans.

Definition 3.3. A **polyhedral complex** Is a collection Σ of polyhedra such that if $P \in \Sigma$ then every face of P is also in Σ, and if P and Q are polyhedra in Σ then their intersection is either empty or also a face of both P and Q. See Figure 2 for an example.

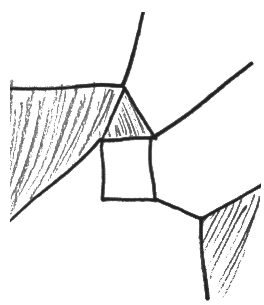


Figure 2: An example of a polyhedral complex.

The **support** $|\Sigma|$ of Σ is the union of all of the faces of Σ.

There is one last thing we will need from polyhedral geometry, and that is the notion of a regular subdivision.

Definition 3.4. Let $v_1, \dots, v_r$ be an ordered list of vectors in $\mathbb{R}^n$. We fix a **weight vector** $w = (w_1, \dots, w_r)$ in $\mathbb{R}^r$ assigning a weight to each vector. Consider the polytope in $\mathbb{R}^{n+1}$ defined by $P = \operatorname{conv}((v_1, w_1), \dots, (v_n, w_n))$. The **regular subdivision** of $v_1, \dots, v_r$ is the polyhedral complex on the points $v_1, \dots, v_r$ whose faces are the faces of P which are "visible from beneath the polytope". More preceisely, the faces σ are the sets for which there exists $c \in \mathbb{R}^n$ with $c \cdot v_i = w_i$ for $i \in \sigma$ and $c \cdot v_i < w_i$ for $i \notin \sigma$.

We will see an example of this shortly, as this comes up frequently.

Question 3. Find all regular subdivisions of $\{(0,0),(0,1),(1,0),(1,1)\}$ and give examples of weight vectors which give these subdivisions.

Question 4. A fan or a polyhedral complex is **pure of dimension d** if every maximal face has the same dimension, d. Give examples of fans which are and are not pure.

Question 5. A pure, d-dimensional polyhedral complex in $\mathbb{R}^n$ is **connected through codimension 1** if for any two d dimensional cells, there is a chain of d-dimensional cells $P = P_1, \dots, P_s = P'$ for which P_i and P_{i+1} share a common facet F_i. Give an example of a polyhedral complex which is not connected through codimension 1 in $\mathbb{R}^2$.

Question 6. Are k-skeleta of polyhedra connected through codimension 1?

4 Tropical Varieties

4.1 Hypersurfaces

Studying hypersurfaces gives us a way to study tropical geometry in a simpler setting. Let K be an arbitrary field, with valuation (possibly the trivial valuation). Consider the ring $K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ of Laurent polynomials over K.

Definition 4.1. Given a Laurent polynomial

$$f = \sum_{u \in \mathbb{Z}^n} c_u x^u,$$

we define its **tropicalization** to be the real valued function on $\mathbb{R}^n$ that is obtained by replacing each c_u by its valuation and preforming all additions and multiplications in the **tropical semiring** $(\mathbb{R}, \oplus, \otimes)$:

$$\operatorname{trop}(f)(w) = \min_{u \in \mathbb{Z}^n} \left(\operatorname{val}(c_u) + u \cdot w\right).$$

Classically, the variety of the Laurent polynomial f is a hypersurface in the algebraic torus $T^n = (K^*)^n$ over the algebraic closure of K. We now define the tropical hypersurface associated to f.

Definition 4.2. The **tropical hypersurface** $\text{trop}(V(f))$ is the set

$$\{w \in \mathbb{R}^n \mid \text{the minimum in } \text{trop}(f)(w) \text{ is achieved at least twice}\}$$

Example 4.3. Here we compute the **tropical line**, which is the classic first example. Let $f = x + y + 1$ in the field $\mathbb{C}\{\{t\}\}$. Then,

$$\begin{aligned} \text{trop}(f)(w) &= \min\left(0 + (1,0)\cdot w, 0 + (0,1)\cdot w, 0\right), \\ &= \min(w_1, w_2, 0). \end{aligned}$$

So, where is this minimum achieved twice? We can break this down in to 3 cases.

1. $w_1 = 0 \leq w_2$: This happens when $w_1 = 0$ and $w_2 \geq 0$. So, this is the ray $\text{pos}(e_2)$.
2. $w_2 = 0 \leq w_1$: This happens when $w_2 = 0$ and $w_1 \geq 0$. So, this is the ray $\text{pos}(e_1)$.
3. $w_1 = w_2 \leq 0$: This adds to our tropical variety the ray $\text{pos}(-1, -1)$.

So, the tropical variety is pictured in Figure 3

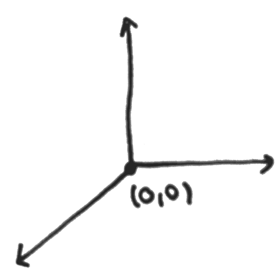


Figure 3: The tropical line studied in Example 4.3.

There is another way to view this set in terms of initial forms.

Definition 4.4. Let $u \mapsto t^u$ be a splitting of the valuation on K. The **initial form** of f with respect to $w \in \mathbb{R}^n$ is

$$\mathrm{in}_w(f) = \sum_{u:\mathrm{val}(c_u)+w\cdot u=\mathrm{trop}(f)(w)} \overline{t^{-\mathrm{val}(c_u)}c_u} x^u.$$

Then, the tropical hypersurface $\mathrm{trop}(V(f))$ is the set of weight vectors $w \in \mathbb{R}^n$ for which the initial form $\mathrm{in}_w(f)$ is not a unit in $\Bbbk[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

Example 4.5. In the previous example, we can compute

$$\mathrm{in}_{(1,0)}(f) = t^0 \cdot 1 \cdot x^0 + t^0 \cdot 1 \cdot x^{(1,0)} = 1 + x$$
$$\mathrm{in}_{(0,1)}(f) = t^0 \cdot 1 \cdot x^0 + t^0 \cdot 1 \cdot x^{(0,1)} = 1 + y$$
$$\mathrm{in}_{(1,0)}(f) = t^0 \cdot 1 \cdot x^{(0,1)} + t^0 \cdot 1 \cdot x^{(1,0)} = y + x$$

I like to think of this as the terms of f which, when tropicalized, achieve the minimum on that part of the tropical variety.

Theorem 4.6 (Kapranov's Theorem)**.** *Let K be an algebraically closed field with a nontrivial valuation. Fix a Laurent polynomial $f = \sum_{u\in\mathbb{Z}^n} c_u x^u$ in $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. The following three sets are the same:*

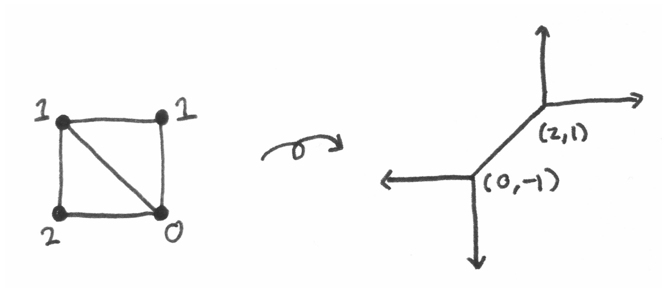
1. *the tropical hypersurface trop$(V(f))$ in $\mathbb{R}^n$,*
2. *the set $\{w \in \mathbb{R}^n \mid in_w(f)$ is not a monomial$\}$,*
3. *the closure in $\mathbb{R}^n$ of $\{(v(y_1), \ldots, v(y_n)) \mid (y_1, \ldots, y_n) \in V(f)\}$.*

Furthermore, if f is irreducible and w is any point in $\Gamma^n_{val} \cap trop(V(f))$, then the set $\{y \in V(f) \mid val(y) = w\}$ is Zariski dense in $V(f)$.

In practice, when you wish to compute a tropical hypersurface, there is a very practical method (this is especially good in $\mathbb{R}^2$).

Proposition 4.7. *Let $f \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be a Laurent polynomial. The tropical hypersurface trop$(V(f))$ is the $(n-1)$-skeleton of the polyhedral complex dual to the regular subdivision of the newton polytope of f induced by the weigths val(c_u) on the lattice points in Newt(f).*

Example 4.8. We now compute $\mathrm{trop}(V(f))$ where $f = 7xy + 5x + 14y + 49$ with the 7-adic valuation. We can compute the tropicalization using the newton polytope as in Proposition 4.7.



Question 7. Pick a univariate polynomial f and compute trop(f). Draw its graph. What do the "corner loci" represent? Factor your polynomial into linear factors over the min-plus semialgebra.

Question 8. In Example 4.8, compute $\mathrm{in}_w(f)$ for one w coming from each one-dimensional polyhedron in $\mathrm{trop}(V(f))$. Pick your favorite point $x \in \mathrm{trop}(V(f))$. Can you find a point x' such that $\nu(x') = x$ and $x' \in V(f)$?

4.2 Fundamental Theorem

The Fundamental Theorem gives some equivalent ways to look at tropical varieties.

Definition 4.9. Let I be an ideal in $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, and let $X = V(I)$ be its variety in the algebraic torus T^n. Then we define the **tropicalization** of X, denoted trop(X), to be:

$$\bigcap_{f \in I} \mathrm{trop}(V(f)).$$

A **tropical variety** is any subset of $\mathbb{R}^n$ of the form trop(X), where X is a subvariety of T^n.

Now we wish to tropicalize arbitrary varieties, not just hypersurfaces. As we know from Gröbner basis theory, arbitrary generating sets for ideals behave badly under taking initial forms. As we saw above in Kapranov's theorem, our tropicalization will reflect the behavior of an ideal under taking initial forms for many different orderings. This should give us the first hint that we will need a special basis for tropicalizaiton. A tropical basis is a generating set for an ideal which behaves nicely with respect to tropicalization.

Definition 4.10. Let I be an ideal in $K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, where K is any field with valuation. A finite generating set T of I is called a **tropical basis** if

$$\mathrm{trop}(V(I)) = \bigcap_{f \in T} \mathrm{trop}(V(f)).$$

Theorem 4.11. *Let K be any valued field. Every ideal in $K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ has a finite tropical basis.*

Definition 4.12. Let $I \subset K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be an ideal. The **initial ideal** $\mathrm{in}_w(I)$ is the ideal in $\Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ generated by the initial forms $\mathrm{in}_w(f)$ for all $f \in I$.

For generic choices of w the initial form $\mathrm{in}_w(f)$ is a unit in $\Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ and the initial ideal $\mathrm{in}_w(I)$ is equal to the whole ring. We wish to study the weight vectors w for which $\mathrm{in}_w(I)$ is a proper ideal in $\Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$.

We now generalize Kapranov's Theorem (Theorem 4.13) to all tropical varieties, and not just hypersurfaces.

Theorem 4.13 (Fundamental Theorem of Tropical Geometry)**.** *Let K be an algebraically closed field with a nontrivial valuation. Fix an ideal I in $K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. The following three sets are the same:*

1. *the tropical variety $trop(V(I))$ in $\mathbb{R}^n$,*
2. *the set $\{w \in \mathbb{R}^n \mid in_w(I) \neq 1\}$,*
3. *the closure in $\mathbb{R}^n$ of $\{(val(y_1), \dots, val(y_n)) \mid (y_1, \dots, y_n) \in V(I)\}$.*

Furthermore, if $V(I)$ is irreducible and w is any point in $\Gamma_{val}^n \cap trop(V(I))$, then the set $\{y \in V(I) \mid val(y) = w\}$ is Zariski dense in $V(I)$.

Question 9. What are some examples of ideals where tropical bases are known?

Question 10. Find an example of an ideal with two generators where the two generators do not form a tropical basis.

5 Additional exercises

Question 11. Find a friend and each write down a polynomial in $\mathbb{C}\{\{t\}\}[x^{\pm 1}, y^{\pm 1}]$. Find the tropical varieties together. Verify the fundamental theorem for these examples.

Question 12. Tropicalize a plane in 3 space.

Question 13. Google search "`structure theorem of tropical geometry`". What kinds of objects can tropical varieties be? What properties do they have?

Question 14. Use software (`gfan`, `polymake`, or `singular`) to compute some examples of tropical varieties:

1. A plane curve
2. The 3×3 determinant of a generic matrix
3. A tropical line in 3 space

symbol	description
$\mathrm{conv}(\star)$	the convex hull of $\star$
Γ_{val}	the value group, the image of ν
$\mathrm{in}_w(\star)$	the initial form of the laurent polynomial $\star$ with respect to w
K	The big field, with valuation ν
$\Bbbk$	The (little) residue field $\mathsf{R}/\mathfrak{m}$
$\mathfrak{m}$	The maximal ideal of the local ring R
$\mathrm{pos}(\star)$	the positive hull of $\star$
$\mathrm{trop}(V(\star))$	the tropical variety of the Laurent polynomial or ideal $\star$
R	Elements of K with non-negative valuation
ν	a valuation

Table 1: main notation from tropical geometry

References

[MS15] D. Maclagan and B. Sturmfels, **Introduction to tropical geometry:**, Graduate Studies in Mathematics, American Mathematical Society, 2015.

[Zie95] Günter M. Ziegler, **Lectures on polytopes**, Graduate Texts in Mathematics, vol. 152, Springer-Verlag, New York, 1995.