The Caldero–Chapoton formula as a dimer partition function

joint work with İlke Çanakçı and Alastair King

Matthew Pressland

Universität Stuttgart

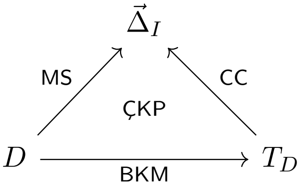
Tropical Geometry Meets Representation Theory, Universität zu Köln

Setting

- Fix integers $1 \leq k \leq n$. We study the Grassmannian G_k^n of k-subspaces of $\mathbb{C}^n$, and the coordinate ring $\mathbb{C}[\hat{G}_k^n]$ of its affine cone.
- The 'standard' generators of $\mathbb{C}[\hat{G}_k^n]$ are Plücker coordinates Δ_I for $I \in \binom{n}{k} = \{I \subseteq \{1, \ldots, n\} : |I| = k\}$.
- By work of Scott, $\mathbb{C}[\hat{G}_k^n]$ has a cluster algebra structure, in which all Δ_I are cluster variables.
- This cluster algebra is categorified by Jensen, King and Su: more details to follow.
- One way of connecting the cluster algebra and its categorification is via dimer models, certain bipartite 'graphs' drawn in a disk: again, more details to follow.
- The dimer models help to translate between combinatorics and representation theory—this will be a theme.

Twisted Plücker coordinates

- For each $I \in \binom{n}{k}$, there is a cluster monomial $\vec{\Delta}_I \in \mathbb{C}[\hat{G}^n_k]$; a *twisted Plücker coordinate*.
- A dimer model D determines a 'cluster' of Plücker coordinates, in which we can express $\vec{\Delta}_I$ as a Laurent polynomial, computable in two ways.
- Marsh and Scott compute this Laurent polynomial combinatorially from D—this expresses $\vec{\Delta}_I$ as a 'dimer partition function'.
- Alternatively, the Caldero–Chapoton cluster character computes the Laurent polynomial homologically from a 'maximal rigid' object T_D in the JKS cluster category.
- The relationship between D and T_D is explained by work of Baur, King and Marsh.



Dimer models

- Take a disc with marked points $1, \ldots, n$ on its boundary.
- A dimer D is a bipartite graph in the interior of the disc, together with n 'half-edges' connecting black nodes to the marked points on the boundary.
- Require that zig-zag paths (turn right at black nodes, left and white nodes) connect i to $i-k$ modulo n—the collection of these paths is a 'Postnikov diagram', and is equivalent data to D.
- Labelling each tile to the right of $i \rightsquigarrow i-k$ by i yields a set $\mathcal{C}(D) \subseteq \binom{n}{k}$ of labels, and a cluster $\{\Delta_I : I \in \mathcal{C}(D)\}$ of $\mathbb{C}[\hat{G}^n_k]$.
- Get algebra A_D by taking quiver dual to graph (vertices in faces, arrows across edges with the black node on the left), and relations $p^+_\alpha = p^-_\alpha$ whenever there are paths p^+_α and p^-_α completing an arrow α to a cycle around a black $(+)$ and white $(-)$ node.

Example

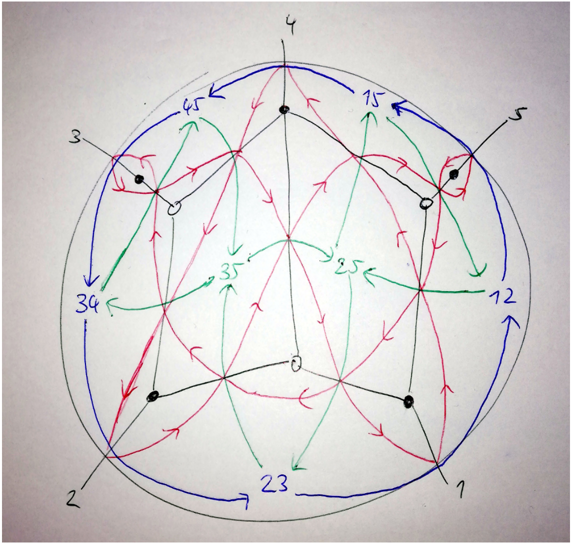


Figure: A dimer model for $n = 5$, $k = 2$.

The JKS category

- The algebra A_D is free of finite rank over a central subalgebra $Z \cong \mathbb{C}[[t]]$.
- Let e be the sum of vertex idempotents at the boundary tiles, and $B = eAe$; this algebra is also free of finite rank over Z.

Theorem (Jensen–King–Su)

The category $\mathrm{CM}(B)$ of Cohen–Macaulay B-modules (those free of finite rank over Z) categorifies the cluster algebra $\mathbb{C}[\hat{G}_k^n]$. In particular, there is a bijection between rigid objects of $\mathrm{CM}(B)$ (up to isomorphism) and cluster monomials.

Theorem (Baur–King–Marsh)

The algebra B depends only on k and n (and not on D) up to isomorphism. The B-module $T_D := eA_D$ is a maximal rigid object in $\mathrm{CM}(B)$, and $\mathrm{End}_B(T_D)^{\mathrm{op}} \cong A_D$.

Finding $\vec{\Delta}_I$ in $\mathrm{CM}(B)$

- Since $\vec{\Delta}_I$ is a cluster monomial, it has a corresponding rigid object in $\mathrm{CM}(B)$, which we want to find.
- Let M_I be the rigid (indecomposable) object corresponding to the Plücker coordinate Δ_I, and P_i that corresponding to the Plücker coordinate $\Delta_{\{i,\cdots,i+k-1\}}$.
- All of these modules can be explicitly described, and the P_i are the indecomposable projective B-modules.

Proposition (Geiß–Leclerc–Schröer, Çanakçı–King–P)

Let $I \in \binom{n}{k}$. Then there is a 'canonical' exact sequence

$$0 \longrightarrow \Omega M_I \longrightarrow \bigoplus_{i\in I} P_i \longrightarrow M_I \longrightarrow 0,$$

determining ΩM_I up to isomorphism. The module ΩM_I corresponds to $\vec{\Delta}_I$ under the bijection in the JKS theorem.

The CC formula

- Fix a dimer model D, with corresponding maximal rigid object $T_D \in \mathrm{CM}(B)$, and set of Plücker labels $\mathcal{C}(D)$.
- Let $F = \mathrm{Hom}_B(T_D, -)$ and $G = \mathrm{Ext}^1_B(T_D, -)$; both are functors $\mathrm{CM}(B) \to \mathrm{mod}\, A_D$.
- Then the Caldero–Chapoton map (which gives the JKS bijection) is

$$\mathrm{CC}(X) = \sum_{N \leq GX} \Delta^{\pi(FX) - \pi(N)}$$

- Here $\pi(FX) - \pi(N) \in \mathbb{Z}^{\mathcal{C}(D)}$ is a vector computed from projective resolutions of the A_D-modules FX and N, and we use the notation

$$\Delta^x = \sum_{I \in \mathcal{C}(D)} \Delta_I^{x_I}$$

  given such a vector x.
- In particular,

$$\vec{\Delta}_I = \sum_{N \leq G\Omega M_I} \Delta^{\pi(FM_I) - \pi(N)}$$

The Marsh–Scott formula

- A perfect matching μ of D is a set of edges of D (including half-edges) such that every node of D is incident with exactly one edge of μ.
- Since D has exactly k more black nodes than white, any perfect matching must include exactly k half-edges, and the so the boundary marked points adjacent to these half-edges form a set $I(\mu) \subseteq \binom{n}{k}$.
- The Marsh–Scott formula for $\vec{\Delta}_I$ is then

$$\vec{\Delta}_I = \sum_{\mu : I(\mu) = I} \Delta^{wt(\mu)} \qquad \left(\text{cf. CC: } \vec{\Delta}_I = \sum_{N \leq G\Omega M_I} \Delta^{\pi(FM_I) - \pi(N)}\right)$$

for a vector $wt(\mu) \in \mathbb{Z}^{\mathcal{C}(D)}$ computed combinatorially from μ.

Theorem (Çanakçı–King–P: 'MS=CC')

The CC and Marsh–Scott formulae are 'the same', in the sense that there is a bijection between $\{\mu : I(\mu) = I\}$ and $\{N \leq G\Omega M_I\}$ with the property that $wt(\mu) = \pi(FM_I) - \pi(N)$ when N and μ correspond.

Perfect matching modules

- We sketch the bijection. Let μ be a perfect matching of D.
- Define an A_D-module $\hat{N}_\mu$ by placing a copy of $\mathbb{C}[[t]]$ at each vertex, and having arrows act by multiplication with t if they are dual to edges in μ, and by the identity otherwise.
- Applying F to the exact sequence defining ΩM_I gives an exact sequence

$$F\left(\bigoplus_{i\in I} P_i\right) \xrightarrow{f} FM_I \xrightarrow{g} G\Omega M_I \longrightarrow 0$$

Theorem (Çanakçı–King–P)

The submodules of FM_I containing $\operatorname{im} f$ are precisely the $\hat{N}_\mu$ with $I(\mu) = I$. Setting $N_\mu := g\hat{N}_\mu$, the assignment $\mu \mapsto N_\mu$ is a bijection $\{\mu : I(\mu) = I\} \xrightarrow{\sim} \{N \leq G\Omega M_I\}$, and we have $wt(\mu) = \pi(FM_I) - \pi(N_\mu)$.

- The final part of the theorem is proved by constructing an explicit projective resolution of $\hat{N}_\mu$ from μ.