Computing toric degenerations of flag varieties

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Motivation: Why toric degenerations?

For toric varieties we have a dictionary between

$$\left\{\begin{array}{c}\text{algebraic and geometric}\\ \text{properties}\\ \text{e.g. smooth, compact}\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{combinatorial}\\ \text{data}\\ \text{e.g. polytope, fan}\end{array}\right\}$$

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$$\begin{aligned} \pi : \mathcal{X} \to \mathbb{A}^1, \text{ s.t} \quad & \pi^{-1}(0) \cong T \quad && \text{toric variety} \\ \text{and} \quad & \pi^{-1}(t) \cong X \quad && \text{for } t \neq 0. \end{aligned}$$

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Flatness preserves (some) algebraic and geometric properties, e.g. dimension, degree, Gromov-width..

$\rightsquigarrow$ can use (parts of) the dictionary for X.

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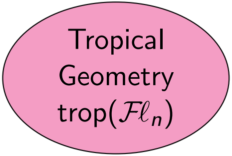
• Can also be realized as SL_n/B, where B is the subgroup of upper triangular matrices with determinant 1. So we can use representation theory of SL_n.

• Consider $U \subset B$ matrices with all diagonal entries being 1. Then SL_n/B and SL_n/U differ only by $(\mathbb{C}^*)^n$. The homogenous coordinate ring $\mathbb{C}[SL_n/U]$ has the structure of a cluster algebra.

$\rightsquigarrow$ lots of additional information to explore different theories

Constructions of toric degenerations



classical Gröbner degeneration [KM16]

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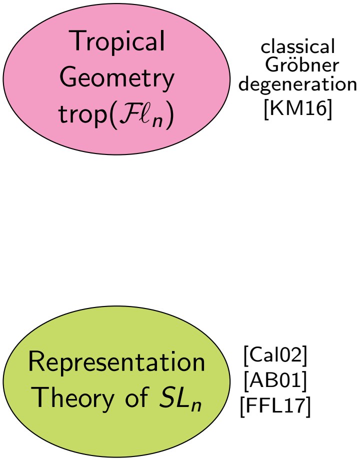

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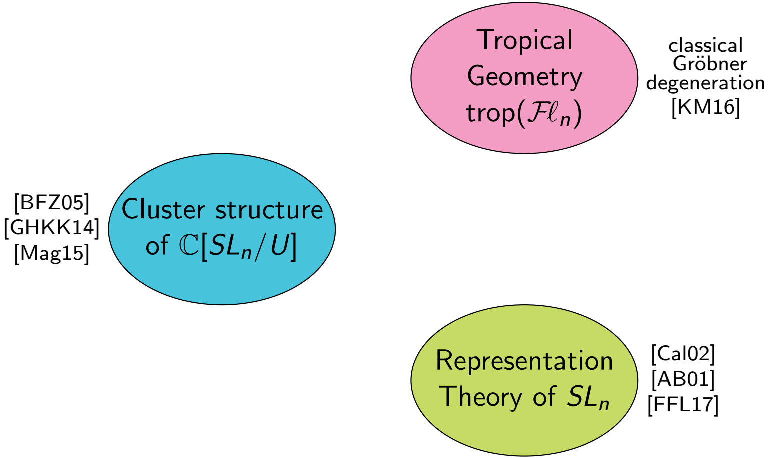

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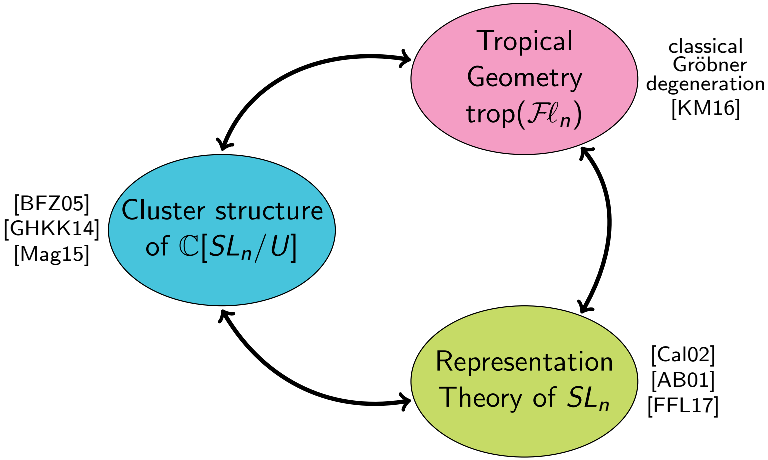

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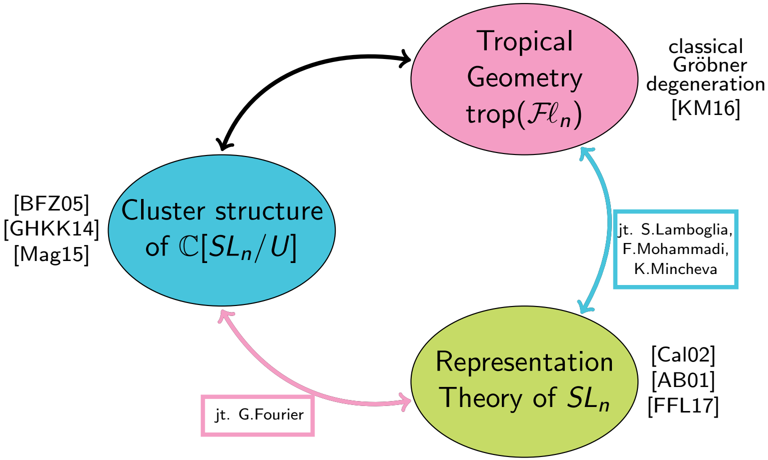

A: Tropical Geometry

Using the Plücker embedding $\mathrm{Gr}(k,n) \hookrightarrow \mathbb{P}^{\binom{n}{k}-1}$ for Grassmannians we fix the embedding

$$\mathcal{F}\ell_n \hookrightarrow \mathrm{Gr}(1,n) \times \cdots \times \mathrm{Gr}(n-1,n) \hookrightarrow \mathbb{P}^{\binom{n}{1}-1} \times \cdots \times \mathbb{P}^{\binom{n}{n-1}-1}.$$

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As a result we obtain an ideal $I_n \subset \mathbb{C}[p_I \mid I \subset \{1,\dots,n\}]$ with $V(I_n) = \mathcal{F}\ell_n$ and I_n is generated by Plücker relations, e.g.

$$I_3 = \langle p_1 p_{23} - p_2 p_{13} + p_3 p_{12} \rangle.$$

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Definition

Let $I \subset \mathbb{C}[x_1,\dots,x_n]$ be an ideal and $f = \sum a_{\mathbf{u}} x^{\mathbf{u}} \in I$. We define with respect to $\mathbf{w} \in \mathbb{R}^n$

- the *initial form of f* as $\mathrm{in}_{\mathbf{w}}(f) = \sum_{\mathbf{w}\cdot\mathbf{u} \text{ minimal}} a_{\mathbf{u}} x^{\mathbf{u}}$, and
- the *initial ideal of I* as $\mathrm{in}_{\mathbf{w}}(I) = \langle \mathrm{in}_{\mathbf{w}}(f) \mid f \in I \rangle$.

A: Tropical Geometry

Example

Take $I_3 \subset \mathbb{C}[p_1, p_2, p_3, p_{12}, p_{13}, p_{23}]$ and $\mathbf{w} = (0, 0, 1, 0, 0, 0) \in \mathbb{R}^6$. Then

$$\text{in}_{\mathbf{w}}(p_1 p_{23} - p_2 p_{13} + p_3 p_{12}) = p_1 p_{23} - p_2 p_{13}.$$

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Let $X = V(I)$ for $I \subset \mathbb{C}[x_1, \ldots, x_n]$ and $\mathbf{w} \in \mathbb{R}^n$ arbitrary. Then we have a flat family $\pi : \mathcal{X} \to \mathbb{A}^1$ with

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If $\text{in}_{\mathbf{w}}(I)$ is binomial and prime, then $V(\text{in}_{\mathbf{w}}(I))$ is a toric variety. Hence, the flat family defines a (Gröbner) toric degeneration of X.

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Definition

The *tropicalized flag variety* is defined as

$$\operatorname{trop}(\mathcal{F}\ell_n) = \{w \in \mathbb{R}^{\binom{n}{1}+\cdots+\binom{n}{n-1}} \mid \operatorname{in}_w(I_n) \text{ contains no monomials}\}.$$

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- It has a fan structure: for $\mathbf{w}, \mathbf{w}'$ in relative interior of a cone C

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- The S_n-action on $\mathcal{F}\ell_n$, for $\sigma \in S_n$ induced by
$$p_{\{i_1,\dots,i_k\}} \mapsto \mathrm{sgn}(\sigma) p_{\{\sigma(i_1),\dots,\sigma(i_k)\}},$$
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Aim: Find (up to symmetry) all *maximal prime* cones $C \subset \mathrm{trop}(\mathcal{F}\ell_n)$, i.e. $\mathrm{in}_C(I_n)$ is binomial and prime.

A: Tropical Geometry

Kaveh-Manon construction:

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NO_C is the polytope associated to the normalization of the toric variety $V(\text{in}_C(I_n))$.

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Theorem (B.-Lamboglia-Mincheva-Mohammadi)

For $\mathcal{F}\ell_4$ there are 78 maximal cones in trop$(\mathcal{F}\ell_4)$ grouped in five $S_4 \times \mathbb{Z}^2$-symmetry classes.

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Orbit	Size	Prime	F-vector of NO_C
1	24	yes	(42, 141, 202, 153, 63, 13)
2	12	yes	(40, 132, 186, 139, 57, 12)
3	12	yes	(42, 141, 202, 153, 63, 13)
4	24	yes	(43, 146, 212, 163, 68, 14)
5	6	no	Not applicable

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Both can be realized as NO-polytopes due to Kaveh, resp. Kiritchenko.

$\rightsquigarrow$ compare to degenerations from trop($\mathcal{F}\ell_n$)

A vs. B

For $\mathcal{F}\ell_4$ up to isomorphism there are four classes of string polytopes and one FFLV polytope. We compare the NO-polytopes from trop($\mathcal{F}\ell_4$) to those using `polymake`:

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Orbit	Combinatorially equivalent polytopes
1	String 2
2	String 1 (Gelfand-Tsetlin)
3	String 3 and FFLV
4	-

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Then $\mu_2(s_0) = \{\underline{p_{13}}, p_3, p_{23}, p_1, p_{12}, p_{123}, p_4, p_{34}, p_{234}\}$.

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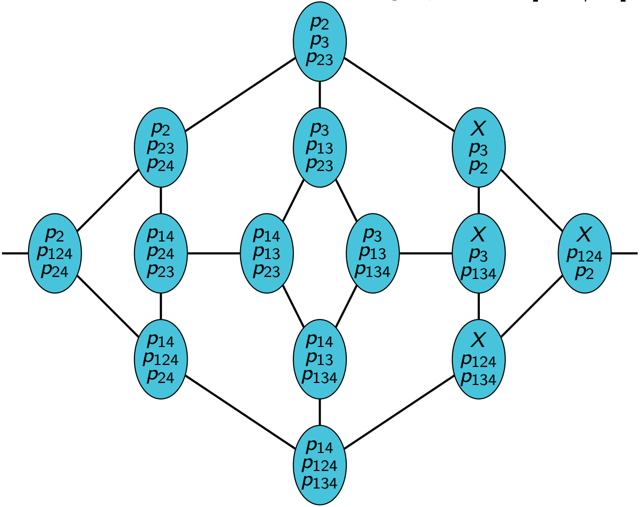

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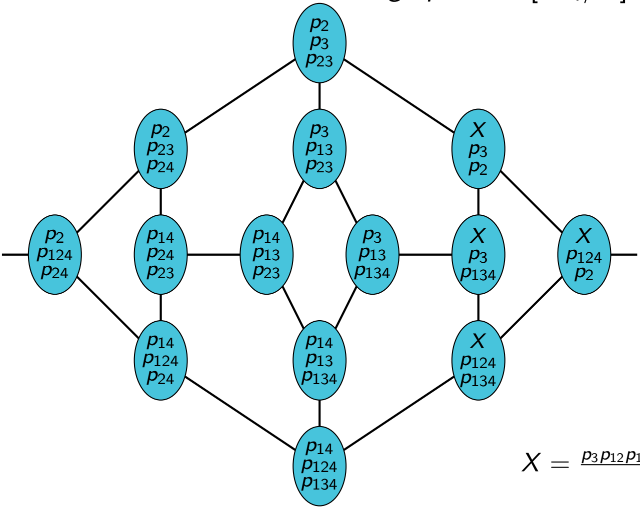

$$X = \frac{p_3 p_{12} p_{134} + p_2 p_{34} p_{123}}{p_{23}}$$

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There is a flat degeneration of $\mathcal{F}\ell_n$ to the toric variety associated to $\Xi_s(\lambda)$.

B vs. C

Question: Does the GHKK-construction specialize to string polytopes?

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Theorem (B.-Fourier)

For every string polytope there exists a unique seed s such that the string polytope is unimodularly equivalent to the polytope $\Xi_s(\lambda)$.

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The string polytopes (resp. FFLV polytope) located in the mutation graph of $\mathbb{C}[SL_4/U]$ up to unimodular (resp. combinat.) equivalence.

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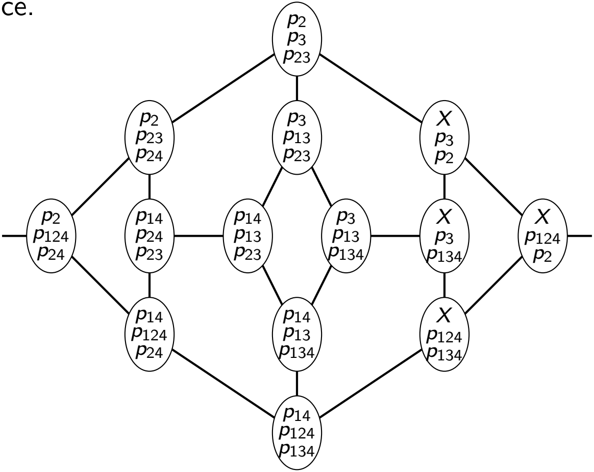

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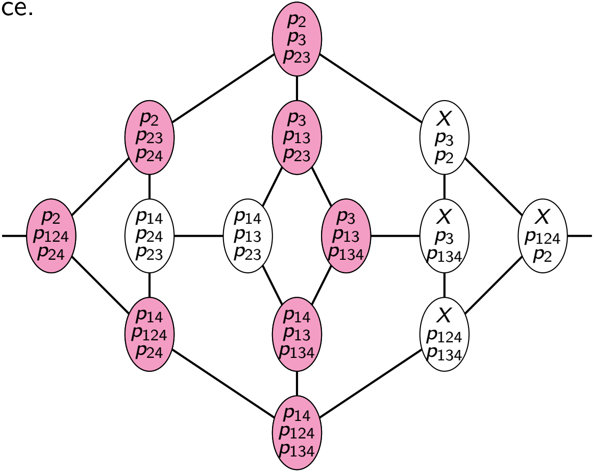

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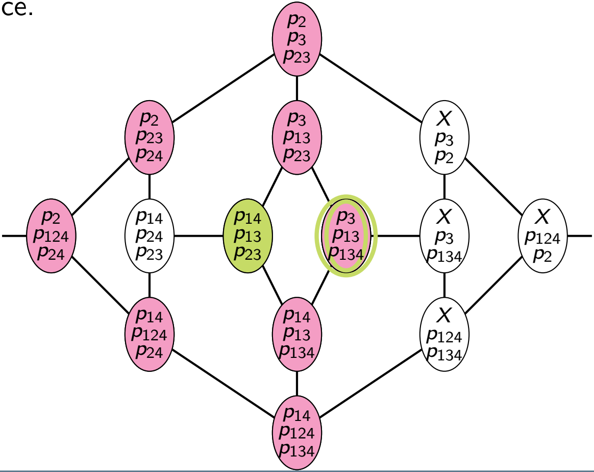

Thank you!

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