Spectral Flow of Paths of Self–Adjoint Fredholm Operators
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First we discuss some difficulties with the currently available definitions of spectral flow (SF). Then we use the Cayley transform to study the topology of the space \(\mathcal{C}_{\text{sa}}(H)\) of (generally unbounded) self–adjoint Fredholm operators in a fixed complex separable Hilbert space \(H\) and give two different (but equivalent) rigorous definitions of SF as a homotopy invariant for continuous paths in \(\mathcal{C}_{\text{sa}}(H)\). Our study is based on the gap (= projection or graph norm) topology. As examples, we consider families of operators of Dirac type on a compact manifold \(M\) with boundary, acting on sections of a fixed Hermitian vector bundle \(E\) with domains defined by varying global well–posed boundary conditions. Such families are continuous families in \(\mathcal{C}_{\text{sa}}(L^2(M; E))\) if the coefficients of the Dirac operators and the boundary conditions vary continuously. No additional assumptions are required.

This is mainly a report on the topology of the space \(\mathcal{C}_{\text{sa}}\) of (generally unbounded) densely defined self–adjoint Fredholm operators in a complex separable Hilbert space \(H\) and the definition and homotopy invariance of SF for continuous paths of operators of this kind (relative to fixed endpoints).

SF has been well investigated for norm–continuous curves in the space \(\mathcal{C}_{\text{sa}}\) of bounded self–adjoint Fredholm operators (see [6], [7] for a rigorous and comprehensive treatment). A special feature of the bounded case is that the topology of \(\mathcal{C}_{\text{sa}}\) is well known: it has three connected components made up by the contractible spaces \(\mathcal{C}_{\text{sa}}^\pm\) of essentially positive, respectively essentially negative operators, and their complement \(\mathcal{C}_{\text{sa}}^*\) which is a classifying space of the topological functor \(K^1\). In particular we have \(\pi_1(\mathcal{C}_{\text{sa}}) = [S^1, \mathcal{C}_{\text{sa}}^*] \cong K^1(S^1) = \mathbb{Z}\) with the isomorphism given by SF.

Heuristically, SF is just the net number of eigenvalues (counting multiplicities) which pass through zero in the positive direction from the start of the path to its end. Once the first homotopy group is established and the homotopy invariance of SF is proved, the preceding intuitive definition of SF suffices, also in defining SF for (possibly non–periodic) paths because we always may deform the path into a generic situation.

One of the principal aims of our study is to specify minimal conditions under which the usual assertions about SF are true: that is, under the assumption that we have a path in \(\mathcal{C}_{\text{sa}}\) which is continuous in the gap metric. Note that in the previous applications the paths consist of differential operators (Dirac operators on closed manifolds or manifolds with boundary) which are neither bounded in \(L^2\) nor, in general, describable by operator norm continuous paths in \(\mathcal{C}_{\text{sa}}\).

We refer to [8] for a full length presentation of our results.

We define the convergence in the space \(\mathcal{C}_{\text{sa}}\) of (generally unbounded) self–adjoint operators in \(H\) by the gap metric, i.e. the convergence of the orthogonal projections onto the graphs of the operators. The gap metric is (uniformly) equivalent to the operator metric of the resolvents.

Of course, some care is needed when dealing with sequences and curves of unbounded opera-
tors as the following example may illustrate. It is a variant of an example due to B. Fuglede (noted in [9] and presented in [10]).

Let \( \{e_k\}_{k \in \mathbb{N}} \) be a complete orthonormal system for \( H \). Consider the multiplication operator \( T_0 : \mathcal{D}(T_0) \rightarrow H \) defined by \( e_k \mapsto ke_k \) with \( \mathcal{D}(T_0) = \{ \sum_k a_k e_k \mid \sum_k k^2 |a_k|^2 < +\infty \} \).

Then \( T_0, T_n \in \mathcal{CF}^a \), where \( T_n := T_0 - 2nP_n \) with \( P_n \) the orthogonal projection onto the line through \( e_n \) for \( n \in \mathbb{N} \). For the resolvents we have \( \| (T_n + iI)^{-1} - (T_0 + iI)^{-1} \| = \left| \frac{1}{1-n} - \frac{1}{1+n} \right| \rightarrow 0 \). This gives

**Proposition 1.**

(a) The sequence \( \{T_n\}_{n \in \mathbb{N}} \) converges to \( T_0 \) in \( \mathcal{CF}^a \) (in the gap metric).

(b) The piecewise linear path of linear interpolations \((1-t)T_n + iT_{n+1}\) belongs to \( \mathcal{CF}^a \) and has rapidly oscillating spectrum near \( 0 \) with principal value of \( SF \) equal to \( 0 \) (note: this path does not connect \( T_1 \) to \( T_0 \)).

(c) However, the linear path from \( T_0 \) to \( T_1 \) is continuous and has \( SF \) equal to \(-1\).

Clearly, the problem with the preceding example is that the path of linear interpolations does not converge to \( T_0 \) even though the corners \( T_n \) converge to \( T_0 \).

To us, the only safe strategy of dealing with \( SF \) is to use the methods of \( B^a \) operators. We shall discuss four different approaches of quite different range and value: (i) deformation into \( F^a \) by a density argument - non viable; (ii) Riesz transformation into \( F^a \) - of limited value; (iii) Cayley transformation into a subspace of the group \( \mathcal{U} \) of unitary operators - of universal value; (iv) piecewise reduction to continuous curves of finite range operators - also of universal value.

(i): On the space \( B^a \) of bounded self–adjoint operators the topologies defined by the gap metric and by the operator norm are equivalent, but the metrics are not uniformly equivalent. By spectral resolutions we prove

**Proposition 2.** The space \( B^a \) is dense in \( \mathcal{C}^a \) in the gap metric.

Actually, we can prove that \( F^a \) is dense in \( \mathcal{CF}^a \). However, these two spaces have completely different topology. E.g., the second space is connected (see Theorem 5 below). So, in general we cannot deform a continuous path in \( \mathcal{CF}^a \) into a continuous path in \( F^a \) even if the endpoints are in \( F^a \).

(ii): Next we recall that the Riesz map \( T \mapsto F_T := (1 + T^2)^{-1/2} \) is a bijection from \( \mathcal{C}^a \) onto the subset of \( B^a \) of all \( T \) with \( \|T\| \leq 1 \) and \( T \pm I \) both injective. It contracts the spectrum in a continuous way. In [9] it was shown that we can reduce our problem to the bounded case by the Riesz transform in a special case:

**Theorem 3.** Fix a \( T \in \mathcal{CF}^a \) so that we have a continuous translation \( \tau_T : B^a \rightarrow \mathcal{CF}^a \) defined by \( C \mapsto T + C \). Then the combined map \( F \circ \tau_T : B^a \rightarrow F^a \) is continuous, and \( SF \) is well defined and homotopy invariant for families of the form \( \{T + C(t)\}_{t \in [0,1]} \) is any continuous path in \( B^a \).

So, for families of Dirac operators we are on safer grounds, if we fix one self–adjoint \( L^2 \)–extension of a Dirac operator and vary only the connection (physically speaking, the background field). On a closed manifold this means fixing the principal symbol of the Dirac operators to be considered; on a manifold with boundary, additionally, this means fixing a well-posed boundary condition. This covers most of the classical cases considered in the 70s and 80s, but not families with varying domain.

By the example underlying Proposition 1, the Riesz map is not continuous on the full space \( \mathcal{C}^a \) and even not on \( \mathcal{CF}^a \). Actually, we have \( \|F_{T_n} e_n - F_{T_0} e_n\| = \left( \frac{2n}{\sqrt{1+n^2}} \right) \rightarrow 2 \). So the approach of the preceding Theorem of establishing \( SF \) by reducing to the bounded case by the Riesz map is not viable in general.

One way to get around the problems was shown in [10] by just defining the metric in \( \mathcal{C}^a \) as the one which makes \( F \) into a homeomorphism and then establishing this Riesz continuity for curves of relatively bounded perturbations of a fixed (unbounded) operator. A priori, this still means that the domains remain fixed. However, for suf-
The Cayley transform

Theorem 4. The Cayley transform \( \kappa \) induces a homomorphism \( \kappa \) of \( \mathcal{C}^\omega \) onto

\[ \mathcal{U}_\kappa := \{ U \in \mathcal{U}(H) \mid U - I \text{ is injective} \} \]

by \( T \mapsto \kappa(T) = (T-i)(T+i)^{-1} \) which maps \( \mathcal{C}^\omega \) onto \( \mathcal{U}_\kappa := \mathcal{U} \cap \mathcal{U}_\kappa \), where

\[ \mathcal{U} := \{ U \in \mathcal{U} \mid U + I \text{ Fredholm} \} \]

It is well known (e.g., [5]) that the natural inclusions \( \mathcal{U}_\infty \hookrightarrow \mathcal{U}_\kappa \hookrightarrow \mathcal{U} \) are homotopy equivalences, but we do not know whether \( \mathcal{U}_\kappa \) and \( \mathcal{U} \) have the same homotopy type. Here the subscript to the right, resp. left indicates a property of \( U - (\pm 1) \), at the right, resp. left side of the spectral circle, i.e. \( U - I \) of finite range, compact or injective, respectively \( U + I \) Fredholm.

In particular, \( \mathcal{U} \) is a classifying space for \( K^1 \) with the isomorphism \( \pi_1(\mathcal{U}) \cong \mathbb{Z} \) given by the winding number. By combination with \( \kappa \), this gives a rigorous definition of homotopy invariant SF for gap–continuous paths in \( \mathcal{C}^\omega \) which coincides with the established SF for norm–continuous paths in \( \mathcal{F}^\omega \).

The Cayley picture gives us some additional information by the canonical spectral correspondences and by spectral deformation:

Theorem 5. The set \( \mathcal{C}^\omega \) is path--connected with respect to the gap metric and open in \( \mathcal{C}^\omega \).

Moreover, its Cayley image \( \mathcal{U}_\kappa \) is dense in \( \mathcal{U} \).

The Cayley picture gives us some additional information (the “corners” of our Proposition 1) indicates the topological intricacies of the space \( \mathcal{U} \): The spectrum of \( U_n := \kappa(T_n) \) consists of discrete eigenvalues which all are lying in the lower half plane except one in the upper half plane with a corresponding hole in the lower half plane sequence, plus the accumulation point 1 where \( U_n - I \) is injective, but not invertible. The same is true for \( U_0 := \kappa(T_0) \), but now having all eigenvalues in the lower half plane. By Proposition 1.a and Theorem 4, the sequence \( \{U_n\}_{n \in \mathbb{N}} \) converges to \( U_0 \) in \( \mathcal{U}_\kappa \). So, the eigenvalues of the sequence flip between the upper half plane and the lower half plane close to +1 without actually crossing +1. This phenomenon rules out the possibility of retracting \( \mathcal{U}_\kappa \) onto the more intelligible space of all \( U \in \mathcal{U} \) with \( U + I \) truly invertible (that is the image of \( \mathcal{F}^\omega \) in \( \mathcal{U}_\kappa \)).

(iv): There is another way of looking at continuous curves of self–adjoint Fredholm operators which more closely resembles what is done in the bounded self–adjoint setting. We show that one can (continuously) isolate the spectra of the unbounded self–adjoint Fredholm operators in an open interval about 0. This is quite appealing from an operator algebra point of view: it is surprising that this can be done without the Riesz map being continuous!

Here, our main result is

Proposition 6. (a) Let \( a < b \) be real numbers. Then the set \( \Omega_{a,b} := \{ T \in \mathcal{C}^\omega \mid a, b \notin \text{spec } T \} \) is open in the gap topology and the map \( \Omega_{a,b} \to \mathcal{B}^\omega \), \( T \mapsto 1_{[a,b]}(T) \) is continuous.

(b) Fix \( T_0 \in \mathcal{C}^\omega \). Then there is a positive number \( a \) and an open neighborhood \( \mathcal{N} \subset \mathcal{C}^\omega \) of \( T_0 \) in the gap topology such that the map \( \mathcal{N} \to \mathcal{B}^\omega \), \( T \mapsto 1_{[-a,a]}(T) \) is continuous and finite–rank projection–valued, and hence \( T \mapsto T 1_{[-a,a]}(T) \) is also continuous.

(c) If \( -a \leq c < d \leq a \) are points so that \( c, d \notin \text{spec } T \) for all \( T \in \mathcal{N} \) then the map \( T \mapsto 1_{[c,d]}(T) \) is continuous on \( \mathcal{N} \) and has finite rank on \( \mathcal{N} \). Of course, on any connected subset of \( \mathcal{N} \) this rank is constant.

We proceed exactly as in [7, p. 462] and obtain SF which by definition is homotopy invariant and coincides precisely with previously defined SF.
An Example: Paths of Dirac Operators on Manifolds with Boundary. Now we consider a fixed compact Riemannian manifold $M$ with boundary $\Sigma$ and a fixed Hermitian vector bundle over $M$. We assume that $M$ has no closed connected component. Let $\{D_s\}_{s \in X}$, $X$ a metric space, be a family of linear symmetric elliptic differential operators of first order acting on sections of $E$.

We shall specify under which conditions curves of self-adjoint $L^2$–extensions become continuous curves in $C^\infty(L^2(M; E))$ in the gap topology. We make two assumptions:

**Assumptions 7.** (1) For each $s \in X$, the operator $D_s$ takes the form

$$D_s[u] = \sigma_s(y, \tau)(\frac{\partial}{\partial \tau} + A_{s, \tau} + B_{s, \tau})$$

in a bi-collar $U = \Xi \times [-\varepsilon, \varepsilon]$ of any hypersurface $\Xi \subset M \setminus \Sigma$, and a similar form in a collar of $\Sigma$, where $\sigma_s(\cdot, \tau), A_{s, \tau}, B_{s, \tau} : C^\infty(\Xi_\tau; E|_{\Xi_\tau}) \to C^\infty(\Xi_\tau; E|_{\Xi_\tau})$ are a unitary bundle morphism; a symmetric elliptic differential operator of first order; and a skew–symmetric bundle morphism, respectively, with $\sigma_s(\cdot, \tau)^2 = -I$, $\sigma_s(\cdot, \tau)A_{s, \tau} = -A_{s, \tau}\sigma_s(\cdot, \tau)$, and $\sigma_s(\cdot, \tau)B_{s, \tau} = B_{s, \tau}\sigma_s(\cdot, \tau)$.

Here $\tau$ denotes the normal variable and $\Xi_\tau$ a hypersurface parallel to $\Xi$ in a distance $\tau$.

(2) In each local chart, the coefficients of $D_s$ depend continuously on $s$.

Condition (1) is satisfied for all operators of Dirac type. From (1) it follows that all $D_s$ satisfy the (weak) Unique Continuation Property, i.e., each $u \in L^2(M; E)$ with supp $u \subset M \setminus \Sigma$ and $D_s u = 0$ vanishes identically on all of $M$, [6].

Let $P_{a,s}$ denote the Calderón projection of $L^2(\Sigma; E|_{\Sigma})$ onto the Cauchy data spaces (the traces at $\Sigma$ of the kernel of $D_s$). It differs from the Atiyah–Patodi–Singer projection on the non–negative eigenspace of $A_{s,0}$ by a smoothing operator, [12]. We consider the Fredholm Grassmannian $Gr^a(D_s) := \{ P \text{ pseudodifferential projection with } P^* = P, P = \sigma_s|\Sigma(I - P)(\sigma_s|\Sigma)^* \text{ and } PP_{+s} = \text{ran } P \text{ Fredholm} \}$.

By the method of the invertible double and explicit calculation of the resolvents (see [6, Chapters 9, 19], [13]) we obtain

**Theorem 8.** Let $\{P_t\}_{t \in Y}$, $Y$ a metric space, be a norm–continuous path of orthogonal projections in $L^2(\Sigma; E|_{\Sigma})$. Let $P_t \in \bigcap_{t \in X} Gr^a(D_s)$, $t \in Y$. Then $X \times Y \ni (s, t) \mapsto (D_s)_{P_t} \in C^\infty(L^2(M; E))$ is continuous. Here $(D_s)_{P_t}$ denotes the $L^2$–extension of $D_s$ with domain defined by the vanishing of $P_t$ on the traces at $\Sigma$.

Note: we do not assume that the metric structures of $M$ and $E$ are product near $\Sigma$; nor that the tangential symmetric and skew–symmetric operator components $A_{s, \tau}, B_{s, \tau}$ are independent of the normal variable near $\Sigma$; nor that the principal symbol of the operator family $\{D_s\}$ is fixed.

**REFERENCES**